Perturbation theory for Hamiltonian operator matrices and Riccati equations

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Contents

Chapter 1

Introduction

In this thesis we show the existence and obtain representations of solutions of the algebraic Riccati equation

$$A^*X + XA + XQ_1X - Q_2 = 0 \tag{1.1}$$

where the coefficients A, Q_1, Q_2 and the solution X are linear operators on a Hilbert space, which are unbounded in general, and Q_1, Q_2 are selfadjoint. The existence of solutions is a major problem because Riccati equations are quadratic operator equations and the involved operators do not commute in general. Our approach uses the well-known relation between solutions of (1.1) and invariant graph subspaces of the associated Hamiltonian operator matrix

$$T = \begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix}. \tag{1.2}$$

To obtain a description of the invariant subspaces of T, we introduce the concept of finitely determining l^2-decompositions and apply perturbation theory to prove their existence for Hamiltonian operators.

In Theorem 4.4.1 we show the existence of infinitely many selfadjoint solutions of the Riccati equation for the case that Q_1 and Q_2 are unbounded and nonnegative. The known existence results from control theory (see e.g. [14]) and by Langer, Ran and van de Rotten [31] and Bubák, van der Mee and Ran [10] only apply to the case of bounded Q_1, Q_2 and only yield a nonnegative and a nonpositive solution. For bounded Q_1, Q_2 we derive characterisations of all bounded solutions of (1.1), see Theorems 4.4.4 and 4.4.5. Similar characterisations were obtained by Kuiper and Zwart [29] for Riesz-spectral Hamiltonians and by Curtain, Iftime and Zwart [13] under the assumption of the existence of a bounded, boundedly invertible solution of (1.1). Our notion of finitely determining l^2-decompositions is more general than that of Riesz-spectral operators, and we prove the existence of bounded, boundedly invertible solutions for the case that Q_1 and Q_2 are uniformly positive.

The Riccati equation (1.1) and the associated Hamiltonian operator play a key role in the theory of linear quadratic optimal control, see e.g. the monographs of Curtain and Zwart [14], Lasiecka and Triggiani [34], and Lancaster and Rodman [30]. Besides that, Riccati equations of the type (1.1) are also important in areas such as total least squares techniques (cf. [30]) and inverse problems involving Neumann-to-Dirichlet maps, see [8].

Before describing the results of this thesis in greater detail, we sketch the relation between the theory of optimal control and the Riccati equation, see also [14] and Section 5.2. A *control system* is a linear system of the form

$$\begin{aligned}\dot{z}(t) &= Az(t) + Bu(t), \qquad z(0) = z_0,\\ y(t) &= Cz(t).\end{aligned}$$

The state z of the system evolves in time subject to a parameter u, the control, and determines the output y. The state, control and output are functions in respective Hilbert spaces Z, U and Y. For systems described by ordinary differential equations these spaces are usually finite-dimensional and A, B, C are matrices.

By contrast, systems governed by partial differential equations lead to an infinite-dimensional Hilbert space of states, A becomes the generator of a strongly continuous semigroup, and B and C are typically bounded operators. In this case, the control system has a unique so-called mild solution $z \in C^0([0,\infty[\,, Z)$ for every $z_0 \in Z$ and $u \in L^2([0,\infty[\,, U)$, see [14].

The problem of *linear quadratic optimal control* on the infinite-time horizon is then the following: For given initial state z_0 minimise the *cost functional*

$$J(z_0, u) = \int_0^\infty \left(\|y(t)\|^2 + \|u(t)\|^2\right) dt \tag{1.3}$$

among all controls $u \in L^2([0,\infty[\,, U)$. Essentially, this amounts to bringing the output back to the stationary point $y = 0$. The first term in (1.3) measures how fast this is achieved, while the second term accounts for how much effort is needed.

The Riccati equation is connected to the problem of optimal control as follows: For a bounded selfadjoint operator X we compute

$$\begin{aligned}\frac{d}{dt}(Xz|z) &= (Az + Bu|Xz) + (Xz|Az + Bu) \qquad (1.4)\\ &= (Az|Xz) + (Xz|Az) + \|u + B^*Xz\|^2 - \|B^*Xz\|^2 - \|u\|^2\\ &= \big((A^*X + XA - XBB^*X + C^*C)z\big|z\big) + \|u + B^*Xz\|^2 - \|Cz\|^2 - \|u\|^2.\end{aligned}$$

So if X is a bounded nonnegative solution of the Riccati equation

$$A^*X + XA - XBB^*X + C^*C = 0, \tag{1.5}$$

then, integrating (1.4), we obtain

$$\begin{aligned} J(z_0,u) &= \int_0^\infty \left(\|Cz\|^2 + \|u\|^2\right) dt \\ &\le \sup_{t_1\ge 0}\left(\int_0^{t_1} \left(\|Cz\|^2 + \|u\|^2\right) dt + (Xz(t_1)|z(t_1))\right) \\ &= \int_0^\infty \|u + B^*Xz\|^2\, dt + (Xz_0|z_0). \end{aligned}$$

For the case of feedback control $u_{\mathrm{fb}} = -B^*Xz$, this yields $J(z_0, u_{\mathrm{fb}}) \le (Xz_0|z_0)$. In particular, for every z_0 there exists a control u such that $J(z_0, u)$ is finite; the system is said to be *optimisable*. In control theory the order of arguments is now reversed: An orthogonal projection method is used to show that if the system is optimisable, then there exists a minimal nonnegative solution X_+ of (1.5) and the problem of optimal control has a solution given by feedback control using X_+; see [29, §6] and Theorem 5.2.2.

Our approach of solving the Riccati equation uses the well-known relation to invariant graph subspaces of the associated Hamiltonian operator matrix and its symmetry with respect to two indefinite inner products. For the brief discussion here, we assume for simplicity that all operators are bounded. For unbounded operators, the relations continue to hold formally but are much more subtle to formulate, see Sections 4.2 and 4.3 for more details. In particular, there are several non-equivalent notions of solutions of the Riccati equation in the unbounded case.

Consider an operator X whose graph

$$\Gamma(X) = \left\{ \begin{pmatrix} u \\ Xu \end{pmatrix} \,\middle|\, u \in H \right\}$$

is invariant under T, i.e., for every $u \in H$ there exists $v \in H$ such that

$$\begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix} \begin{pmatrix} u \\ Xu \end{pmatrix} = \begin{pmatrix} Au + Q_1Xu \\ Q_2u - A^*Xu \end{pmatrix} = \begin{pmatrix} v \\ Xv \end{pmatrix}.$$

Inserting the expression for v from the first component into the second one, we obtain

$$Q_2u - A^*Xu = X(Au + Q_1Xu) = XAu + XQ_1Xu \quad \text{for all} \quad u \in H;$$

X is a solution of (1.1). Obviously the other implication also holds: If X is a solution of (1.1), then $\Gamma(X)$ is T-invariant; we have a one-to-one correspondence between solutions of the Riccati equation and graph subspaces invariant under the Hamiltonian.

Note that the Hamiltonian corresponding to the Riccati equation (1.5) from the problem of optimal control is

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}.$$

Because of the minus signs in the off-diagonal entries, a general Hamiltonian is sometimes denoted by

$$\begin{pmatrix} A & -D \\ -Q & -A^* \end{pmatrix},$$

for example in [29] and [31]. Our sign convention in (1.2) was also used by Azizov, Dijksma and Gridneva [4] and appears to be more natural in view of the J_2-accretivity of the Hamiltonian discussed next.

Connected to both the Hamiltonian operator matrix and to graph subspaces are two indefinite inner products on $H \times H$ defined by

$$\langle x|y\rangle = (J_1 x|y), \qquad [x,y] = (J_2 x|y)$$

where $(\cdot|\cdot)$ is the standard scalar product on $H \times H$ and

$$J_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix};$$

the pairs $(H \times H, \langle\cdot|\cdot\rangle)$ and $(H \times H, [\cdot|\cdot])$ are Krein spaces. We then have

$$\begin{aligned} \Big\langle T \begin{pmatrix} u \\ v \end{pmatrix} \Big| \begin{pmatrix} x \\ y \end{pmatrix} \Big\rangle &= i(Au + Q_1 v|y) - i(Q_2 u - A^* v|x) \\ &= i(u|A^* y - Q_2 x) - i(v| - Q_1 y - Ax) = -\Big\langle \begin{pmatrix} u \\ v \end{pmatrix} \Big| T \begin{pmatrix} x \\ y \end{pmatrix} \Big\rangle, \end{aligned}$$

and hence T is skew-symmetric with respect to $\langle\cdot|\cdot\rangle$ or simply J_1-skew-symmetric. Moreover, from

$$\Big\langle \begin{pmatrix} u \\ Xu \end{pmatrix} \Big| \begin{pmatrix} u \\ Xu \end{pmatrix} \Big\rangle = i(u|Xu) - i(Xu|u)$$

it follows that X is symmetric if and only if $\langle x|x\rangle = 0$ for all $x \in \Gamma(X)$; the graph $\Gamma(X)$ is so-called J_1-neutral. For the inner product $[\cdot|\cdot]$ we have

$$\begin{aligned} \operatorname{Re}\Big[T \begin{pmatrix} u \\ v \end{pmatrix} \Big| \begin{pmatrix} u \\ v \end{pmatrix} \Big] &= \operatorname{Re}\big((Au|v) + (Q_1 v|v) + (Q_2 u|u) - (v|Au) \big) \\ &= (Q_1 v|v) + (Q_2 u|u). \end{aligned}$$

So if Q_1 and Q_2 are nonnegative, then $\operatorname{Re}[Tx|x] \geq 0$ for all $x \in H \times H$ and T is called J_2-accretive. Furthermore, for symmetric X we find

$$\Big[\begin{pmatrix} u \\ Xu \end{pmatrix} \Big| \begin{pmatrix} u \\ Xu \end{pmatrix} \Big] = 2(Xu|u);$$

hence X is nonnegative if and only if $[x|x] \geq 0$ for all $x \in \Gamma(X)$; the graph is J_2-nonnegative. In fact, we will use the J_1-skew-symmetry and J_2-accretivity of the Hamiltonian to obtain J_1-neutral as well as J_2-nonnegative and J_2-nonpositive invariant subspaces.

In the finite-dimensional case, the method of solving Riccati equations using invariant subspaces of T is well known in control theory. It goes back to Potter [41] in 1966, who considered diagonalisable Hamiltonians and gave an explicit formula for every possible solution X of (1.1) in terms of eigenvectors of T. He also obtained conditions such that X is symmetric or nonnegative. The case of generalised eigenvectors of T was then studied by Mårtensson [38] in 1971. A comprehensive account of the theory may be found in the monograph of Lancaster and Rodman [30].

The connection of J_1 to the Hamiltonian is also well known: It was used for example by Potter [41], Lancaster and Rodman [30], Kuiper and Zwart [29], and Langer, Ran and van de Rotten [31]. By contrast, the relation of J_2 to the Hamiltonian was first exploited by Langer, Ran and Temme [32] in 1997, followed by Langer, Ran and van de Rotten [31] in 2001, Azizov, Dijksma and Gridneva [4] in 2003, and Bubák, van der Mee and Ran [10] in 2005. The equivalences between properties of an operator X and its graph $\Gamma(X)$ with respect to J_1 and J_2 have been studied by Dijksma and de Snoo [16] and Langer, Ran and van de Rotten [31].

The correspondence between solutions of Riccati equations and invariant graph subspaces holds for general block operator matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Invariant subspaces of dichotomous block operator matrices have been used to prove the existence of bounded solutions of Riccati equations by Langer and Tretter [33] and Ran and van der Mee [42]. Kostrykin, Makarov and Motovilov [26] obtained a characterisation of all (possibly unbounded) solutions of the Riccati equation associated with a bounded selfadjoint block operator matrix.

A different method to prove the existence and also uniqueness of solutions of Riccati equations uses fixed point iterations, see e.g. Motovilov [39] and Adamjan, Langer and Tretter [1].

In the following we give a more detailed description of the main results of this thesis including remarks on the actual state of research.

One major problem in our approach of solving the Riccati equation is the existence of invariant subspaces of the Hamiltonian. For a normal operator the spectral theorem yields a complete description of all invariant subspaces of the operator. However, Hamiltonian operators are non-normal in general, and the existence of invariant subspaces has only been proved for certain classes of non-normal operators, e.g. spectral operators [18, 20], Riesz-spectral operators [14, 29] and dichotomous operators [7, 31, 42].

In Chapter 2 we introduce the concept of a finitely determining l^2-decomposition for an operator T. It yields a large class of invariant subspaces and amounts to an l^2-decomposition of the Hilbert space into a sequence of finite-dimensional

T-invariant subspaces V_k such that the restrictions $T|_{V_k}$ determine the properties of the whole operator T. If the spectra of the restrictions are pairwise disjoint, we say that the l^2-decomposition is finitely spectral.

The notion of a finitely determining l^2-decomposition is equivalent to the existence of a Riesz basis with parentheses of Jordan chains with the additional property that each Jordan chain is completely contained inside some parenthesis. Riesz bases of this kind are frequently used in the literature, for example by Markus [36] and Tretter [47]. Sometimes the term "Riesz basis with parentheses of root vectors" can be found: While strictly speaking this is a more general notion (see Example 2.3.12), the operators in question usually have a Riesz basis with parentheses of Jordan chains of the above kind.

Since for an operator T with a finitely determining l^2-decomposition the spectrum of a restriction $T|_{V_k}$ may be any finite subset of $\mathbb{C}$, cf. Example 2.3.5, the class of such operators generalises Riesz-spectral operators, for which each V_k is one-dimensional, and spectral operators with compact resolvent, for which each $T|_{V_k}$ has one eigenvalue only. It also allows for non-dichotomous operators, cf. Corollary 2.4.9 and Example 5.1.1. The relations of finitely determining l^2-decompositions to other classes of non-normal operators including the above ones are summarised in Theorem 2.3.17.

In Section 2.4 we show the existence of so-called compatible T-invariant subspaces generated by the choice of an invariant subspace in each V_k. In particular, for every subset of the point spectrum we obtain an associated compatible subspace; these associated subspaces naturally generalise spectral subspaces for the class of operators with a finitely determining l^2-decomposition.

Finitely determining l^2-decompositions are then applied to symmetric and accretive operators in Krein spaces. In Theorem 2.5.16 we consider a J-symmetric operator T with a finitely spectral l^2-decomposition and no eigenvalues on the imaginary axis. We show the symmetry of the point spectrum $\sigma_p(T)$ with respect to the real axis and that the compatible subspaces associated with a partition of $\sigma_p(T)$ which separates conjugate points are hypermaximal neutral; i.e., the subspaces coincide with their J-orthogonal complement. In Proposition 2.6.6 we show that for a J-accretive operator the compatible subspaces associated with the right and left half-plane are J-nonnegative and J-nonpositive, respectively. The corresponding result for J-accretive dichotomous operators was obtained by Langer, Ran and van de Rotten [31] and Langer and Tretter [33]. For a J-skew-symmetric dichotomous operator the hypermaximal neutrality of the spectral subspaces associated with the right and left half-plane, respectively, was shown in [31].

In Chapter 3 we use an approach due to Markus and Matsaev [37] to prove the existence of finitely spectral l^2-decompositions for non-normal operators. We consider an operator $T = G + S$ where G is normal with compact resolvent and S is p-subordinate to G with $0 \leq p < 1$. As an example of p-subordinate perturbations, an ordinary differential operator of order k with bounded coefficient functions on a compact interval is k/n-subordinate to an nth order differential operator; if the coefficients are L^2-functions, it is $(k+1)/n$-subordinate, see Propositions 3.2.15 and 3.2.16.

The first perturbation result, Proposition 3.4.1 and Theorem 3.4.4, is a reformulation of [36, Theorem 6.12]: If the eigenvalues of G lie on a finite number of rays from the origin and the density of the eigenvalues has an appropriate asymptotic behaviour depending on p, then T has a compact resolvent, almost all of its eigenvalues lie inside parabolas surrounding the rays, and T admits a finitely spectral l^2-decomposition. In Theorem 3.4.7 we make the stronger assumption that the spectrum of G has sequences of gaps on the rays, whose size depends on p. This allows us to control the multiplicities of the eigenvalues of T and, under an additional assumption, to show that T is a spectral operator. This additional assumption is satisfied for example if almost all eigenvalues of G are simple, which reestablishes results due to Kato [24, Theorem V.4.15a], Dunford and Schwartz [20, Theorem XIX.2.7], and Clark [11]. Moreover, the assumption also holds in cases where the eigenvalues of G have multiplicity greater than one, provided we have a priori knowledge about the separation of the eigenvalues of T; this is an important ingredient in the proof of Theorem 4.4.5.

As an application of the perturbation results, we obtain finitely spectral l^2-decompositions for a class of diagonally dominant block operator matrices (Proposition 3.4.5) and for ordinary differential operators on a compact interval with bounded as well as unbounded coefficient functions, see Section 3.5. The existence of a Riesz basis (possibly with parentheses) of root vectors is well known for differential operators with bounded coefficients and regular boundary conditions [11], [20, Theorem XIX.4.16], [43]. Unbounded coefficients are treated in [44].

In Chapter 4 we apply the results of the previous two chapters to Hamiltonian operator matrices to obtain solutions of Riccati equations. We first derive results about the symmetry and separation of the spectrum of the Hamiltonian with respect to the imaginary axis (Corollary 4.1.3, Proposition 4.1.6) and conditions on the Hamiltonian implying that all neutral invariant subspaces are graph subspaces (Propositions 4.2.5, 4.2.6). Similar conditions were considered by Langer, Ran and van de Rotten [31]. For the case that A, Q_1, Q_2 and X are all unbounded, we introduce the concept of a core solution of the Riccati equation, which implies that a variant of (1.1) holds on a core of X. Unbounded solutions were also considered in [31] for bounded Q_1, Q_2 and by Kostrykin, Makarov and Motovilov [26] for the Riccati equation associated with a bounded selfadjoint block operator matrix.

The main theorems of this thesis are then established in Section 4.4. In Theorem 4.4.1 we consider a Hamiltonian such that A is normal with compact resolvent, the eigenvalues of A lie on finitely many rays from the origin, Q_1, Q_2 are nonnegative and p-subordinate to A, and the density of the spectrum of A has an appropriate asymptotic behaviour depending on p. We show that the Hamiltonian has a finitely spectral l^2-decomposition which is then used to prove the existence of infinitely many selfadjoint core solutions of (1.1), among them a nonnegative solution X_+ and a nonpositive solution X_-. In Theorem 4.4.4 we consider bounded, not necessarily nonnegative operators Q_1, Q_2 and derive a characterisation of all bounded solutions of (1.1) in terms of invariant subspaces compatible with the l^2-decomposition. In Theorem 4.4.5 we assume that Q_1, Q_2 are bounded and uniformly positive, A is skew-adjoint, and almost all of its eigenvalues are simple and sufficiently separated. We then obtain the existence of infinitely many bounded, boundedly invertible solutions and show that every bounded solution has the representation

$$X = X_+P + X_-(I - P)$$

with some projection P. Moreover, every bounded selfadjoint solution is also boundedly invertible and satisfies

$$X_- \leq X \leq X_+ \quad \text{and} \quad X_-^{-1} \leq X^{-1} \leq X_+^{-1}.$$

For dichotomous Hamiltonian operators with bounded nonnegative Q_1, Q_2, the existence of a selfadjoint nonnegative and a selfadjoint nonpositive solution was obtained by Langer, Ran and van de Rotten [31]. The two solutions were shown to be bounded and boundedly invertible, respectively, for the case that $-A$ is maximal uniformly sectorial, which implies that the spectrum of A is contained in a sector in the right half-plane strictly separated from the imaginary axis. A similar result was proved by Bubák, van der Mee and Ran [10] for a Hamiltonian which is exponentially dichotomous with Q_1 compact.

For a Riesz-spectral Hamiltonian, Kuiper and Zwart [29, Theorem 5.6] obtained a representation of all bounded solutions of the Riccati equation in terms of eigenvectors of the Hamiltonian. Under the assumption that all eigenvalues of T are simple, the authors gave conditions such that T is Riesz-spectral. Theorem 4.4.4 applies to the more general class of Hamiltonians with a finitely spectral l^2-decomposition and requires no assumption on the eigenvalue multiplicities.

For the Riccati equation from optimal control, i.e. $Q_1 = -BB^*$, $Q_2 = -C^*C$, the representation $X = X_+P + X_-(I - P)$ was obtained by Curtain, Iftime and Zwart [13] for all bounded selfadjoint solutions under the assumption that there exists a bounded, boundedly invertible, negative solution of the Riccati equation. On the other hand, they did not have to assume that the operators Q_1, Q_2 are uniformly positive. In the finite-dimensional case, the above representation was derived by Willems [51] in 1971.

In Chapter 5 we first consider examples in which finitely spectral l^2-decompositions and solutions of the Riccati equation can be calculated explicitly. The examples illustrate phenomena such as unbounded solutions, non-selfadjoint solutions, solutions depending on a continuous parameter, and Hamiltonians with Jordan chains of arbitrary length. Then we consider two non-trivial Riccati equations: Example 5.1.6 features unbounded differential operators Q_1, Q_2, whereas in Example 5.1.7 Q_1 and

Q_2 are bounded multiplication operators, and bounded, boundedly invertible solutions are obtained.

Finally we apply our theory to the problem of optimal control. In Theorem 5.2.3 we assume that A is normal with compact resolvent and B, C are bounded. We show the existence of infinitely many selfadjoint core solutions of the Riccati equation and obtain a representation of all bounded solutions in terms of compatible invariant subspaces of the Hamiltonian. The theorem is applied to the two-dimensional heat and the one-dimensional wave equation with distributed control. In Example 5.2.7 we consider the heat equation with an unbounded control operator B and also prove the existence of solutions of the associated Riccati equation in this case.

Preliminaries

Throughout this thesis, the term *operator* will denote a (generally unbounded) linear operator. For an introduction to the theory of unbounded linear operators we refer to the books of Davies [15], Dunford and Schwartz [19, Chapter XII], Gohberg, Goldberg and Kaashoek [21], and Kato [24]. Here, we only recall and fix notions and notations which are not always present in textbooks or occasionally differ among them.

Let V be a Banach space. We say that a subset $U \subset V$ is a *subspace* of V if it is a linear subspace in the algebraic sense, not necessarily closed with respect to the topology[1]. For a *linear operator* from a Banach space V into another Banach space W, i.e., a linear mapping $T : \mathcal{D}(T) \to W$ with domain of definition $\mathcal{D}(T) \subset V$, we use the notation $T(V \to W)$. The range of T is denoted by $\mathcal{R}(T)$, the kernel by $\ker T$. For injective T, the inverse $T^{-1}(W \to V)$ is an operator with $\mathcal{D}(T^{-1}) = \mathcal{R}(T)$ and $\mathcal{R}(T^{-1}) = \mathcal{D}(T)$.

A subspace $U \subset V$ is called *T-invariant* if $x \in U \cap \mathcal{D}(T)$ implies $Tx \in U$. We say that a subspace $D \subset \mathcal{D}(T)$ is a *core* for T if for every $x \in \mathcal{D}(T)$ there exists a sequence $(x_k)_{k\in\mathbb{N}}$ in D such that $x_k \to x$ and $Tx_k \to Tx$.

For an operator $T(V \to V)$ on a Banach space V, we define the *resolvent set* $\varrho(T)$ to be the set of those $z \in \mathbb{C}$ for which $T - z : \mathcal{D}(T) \to V$ is bijective with bounded inverse. Then $\varrho(T) \neq \varnothing$ implies that T is closed. On the other hand, if T is closed and $T - z : \mathcal{D}(T) \to V$ is bijective, then $z \in \varrho(T)$ by the closed graph theorem.

For $\lambda \in \mathbb{C}$, the *root subspace* $\mathcal{L}(\lambda)$ of T is defined by the formula

$$\mathcal{L}(\lambda) = \bigcup_{k\in\mathbb{N}} \ker(T - \lambda)^k. \tag{1.6}$$

In particular $\mathcal{L}(\lambda) \neq \{0\}$ if and only if λ is an eigenvalue of T. The non-zero elements of $\mathcal{L}(\lambda)$ are called *root vectors*. A finite sequence $(x_1, \dots, x_n)$ of non-zero vectors in

[1] Another term used in this situation is *linear (sub)manifold*.

$\mathcal{L}(\lambda)$ is called a *Jordan chain* if

$$(T-\lambda)x_1 = 0 \quad \text{and} \quad (T-\lambda)x_k = x_{k-1} \quad \text{for} \quad k = 2, \ldots, n. \tag{1.7}$$

The Jordan chain is said to be *generated by* x_n, and the elements $x_2, \ldots, x_n$ are called *generalised eigenvectors*. Note that a Jordan chain need not be maximal. In particular, every non-zero element x of a root subspace is contained in a Jordan chain, the Jordan chain generated by x.

Suppose $\sigma \subset \sigma(T)$ is a compact isolated component of the spectrum of T. Let Γ be the positively oriented piecewise regular boundary[2] of a bounded open set $U \subset \mathbb{C}$ with $\sigma \subset U$ and $\sigma(T) \setminus \sigma \subset \mathbb{C} \setminus \overline{U}$. Then the operator

$$P = \frac{i}{2\pi} \int_\Gamma (T-\lambda)^{-1}\, d\lambda \tag{1.8}$$

is a projection, $\mathcal{R}(P)$ and $\ker P$ are T-invariant, $\mathcal{R}(P) \subset \mathcal{D}(T)$, $T|_{\mathcal{R}(P)}$ is bounded, and

$$\sigma(T|_{\mathcal{R}(P)}) = \sigma, \qquad \sigma(T|_{\ker P}) = \sigma(T) \setminus \sigma\,.$$

P does not depend on the particular choice of Γ and is called the *Riesz projection* associated with the component σ of the spectrum; for a proof see [15, Theorem 1.5.4], [21, Theorem XV.2.1], or [24, Theorem III.6.17].

If $(T-z_0)^{-1}$ is compact for some $z_0 \in \varrho(T)$, we say that T is an operator with *compact resolvent*. In this case, $(T-z)^{-1}$ is compact for all $z \in \varrho(T)$, $\sigma(T)$ is a discrete set and every $\lambda \in \sigma(T)$ is an eigenvalue with $\dim \mathcal{L}(\lambda) < \infty$, see [24, Theorem III.6.29]. If P_λ is the Riesz projection associated with $\{\lambda\}$, then $\mathcal{R}(P_\lambda) = \mathcal{L}(\lambda)$.

Let H be a Hilbert space with scalar product $(\cdot|\cdot)$ and T a densely defined operator on H. The *adjoint operator* $T^*(H \to H)$ is defined by

$$\mathcal{D}(T^*) = \bigl\{ y \in H \,\big|\, \mathcal{D}(T) \ni x \mapsto (Tx|y) \text{ is bounded} \bigr\},$$
$$(Tx|y) = (x|T^*y) \quad \text{for all} \quad x \in \mathcal{D}(T),\, y \in \mathcal{D}(T^*).$$

We have $z \in \varrho(T) \Leftrightarrow \bar{z} \in \varrho(T^*)$ and $((T-z)^{-1})^* = (T^*-\bar{z})^{-1}$ for $z \in \varrho(T)$. In particular, T has a compact resolvent if and only if T^* has one.

An operator T on a Hilbert space is called *Hermitian* if $(Tx|y) = (x|Ty)$ for all $x, y \in \mathcal{D}(T)$. A densely defined operator T is Hermitian if and only if $T \subset T^*$; it is said to be *symmetric* in this case. The operator is called *selfadjoint (skew-adjoint)* if $T = T^*$ ($T = -T^*$) and *normal* if it is closed and satisfies $TT^* = T^*T$. If T is normal with compact resolvent, then there exists an orthonormal basis of H consisting of eigenvectors of T, see [24, §III.3.8].

[2]That is, $\Gamma = \partial U$ is a finite union of simply closed curves. Each curve γ is piecewise continuously differentiable with $\gamma'(t) \neq 0$ always and oriented in such a way that U lies left of γ.

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Chapter 2

Operators with determining l^2-decompositions

The spectral theorem provides a complete description of all properties of a normal operator. For example it yields the existence of invariant subspaces and a formula for the resolvent. For non-normal operators, tools similar to the spectral measure only exist for certain classes, for example spectral operators [18, 20] and Riesz-spectral operators [14, 29].

In order to obtain invariant subspaces of non-normal operators, we introduce the concepts of finitely determining and spectral l^2-decompositions for operators. They are a generalisation of Riesz-spectral operators and spectral operators with compact resolvent and equivalent to the existence of a Riesz basis with parentheses of Jordan chains where each Jordan chain lies inside some parenthesis.

In the first two sections we present results about l^2-decompositions of Banach and Hilbert spaces. In Section 2.3, finitely determining and spectral l^2-decompositions are defined, formulas for the spectrum and the resolvent are proved, and the relation to other classes of non-normal operators is investigated. Invariant and spectral subspaces are treated in Section 2.4. In the last two sections we apply the theory to symmetric and accretive operators in Krein spaces.

2.1 l^2-decompositions of Banach spaces

In this and the next section we study the well-known concept of an l^2-decomposition of a Banach or Hilbert space into a sequence of subspaces and the relation of l^2-decompositions to Riesz bases. The presentation unifies material from the monographs of Gohberg and Krein [22, Chapter VI], Singer [46, §15], and Markus [36, pages 25–27]. The term "l^2-decomposition" is used in [46], other notions are "basis of subspaces equivalent to an orthogonal one" [22] and "Riesz basis of subspaces" [50]. An l^2-decomposition into finite-dimensional subspaces is equivalent to an unconditional or Riesz basis with parentheses after choosing a basis in each of the subspaces, see Proposition 2.2.12.

Although later we will always deal with countable l^2-decompositions of Hilbert spaces, the general case of Banach spaces and decompositions of arbitrary cardinality is considered first. We study expansions in terms of the l^2-decomposition and investigate how an l^2-decomposition of the entire space gives rise to l^2-decompositions of certain subspaces. To start with, we recall some facts about bases in Banach spaces, see also the books of Singer [45] or Davies [15, Chapter 3].

Definition 2.1.1 Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in a complex Banach space V. We say that $(x_k)_{k\in\mathbb{N}}$ is

(i) *finitely linearly independent* if $(x_0, \dots, x_n)$ is linearly independent for every $n \in \mathbb{N}$;

(ii) *complete* if $\operatorname{span}\{x_k \mid k \in \mathbb{N}\} \subset V$ is dense;

(iii) a *basis* if every $x \in V$ has a unique representation

$$x = \sum_{k=0}^{\infty} \alpha_k x_k \quad \text{with} \quad \alpha_k \in \mathbb{C}; \tag{2.1}$$

(iv) an *unconditional basis* if it is a basis and the convergence in (2.1) is unconditional.

A sequence $(x_k)_{k\in\mathbb{N}}$ in a Hilbert space H is called a *Riesz basis* if there exists an isomorphism $T : H \to H$ such that $(Tx_k)_{k\in\mathbb{N}}$ is an orthonormal basis of H. ⌟

Every basis is finitely linearly independent and complete. However, not every finitely linearly independent complete sequence is also a basis. In a Hilbert space the notions of unconditional and Riesz bases are equivalent, up to a normalisation of the basis. For this and other equivalent conditions for a sequence to be a Riesz basis, see Bari [6], Gohberg and Krein [22, §VI.2], and Proposition 2.2.10.

We recall some facts about direct sums of subspaces. By a subspace of a Banach space V we understand a linear subspace in the algebraic sense, i.e., it need not be topologically closed. For a finite system $U_1, \dots, U_n \subset V$ of subspaces, the sum $U_1 + \dots + U_n$ is called *algebraic direct*, denoted

$$U_1 \dotplus \dots \dotplus U_n\,,$$

if $x_1 + \dots + x_n = 0$ with $x_j \in U_j$ implies $x_1 = \dots = x_n = 0$. The corresponding projections $P_j : U_1 \dotplus \dots \dotplus U_n \to U_j$ are not necessarily bounded and we shall use the term *algebraic projection* in this context. The sum is called *topological direct*, denoted

$$U_1 \oplus \dots \oplus U_n\,,$$

if it is algebraic direct and the algebraic projections $P_1, \dots, P_n$ are bounded. In this case, the sum is closed (and thus a Banach space) if and only if every U_j is closed. The notion "*projection* on a Banach space V" will always refer to a bounded operator $P: V \to V$ satisfying $P^2 = P$; such a projection gives rise to the topological direct sum $V = \ker P \oplus \mathcal{R}(P)$.

Let $(V_\lambda)_{\lambda\in\Lambda}$ be a family of subspaces of a Banach space V with Λ an arbitrary index set. We will denote by

$$\sum_{\lambda\in\Lambda} V_\lambda = \left\{ x_{\lambda_1} + \cdots + x_{\lambda_n} \,\middle|\, n \in \mathbb{N}, x_{\lambda_j} \in V_{\lambda_j} \right\}$$

the *sum* of the family $(V_\lambda)_{\lambda\in\Lambda}$ in the algebraic sense. There is an obvious generalisation of algebraic direct sums to the case of infinitely many subspaces:

Definition 2.1.2 The family $(V_\lambda)_{\lambda\in\Lambda}$ of subspaces of a Banach space V is called *finitely linearly independent* if

$$x_{\lambda_1} + \cdots + x_{\lambda_n} = 0\,,\ x_{\lambda_j} \in V_{\lambda_j} \quad \Rightarrow \quad x_{\lambda_1} = \cdots = x_{\lambda_n} = 0$$

for every finite subset $\{\lambda_1, \dots, \lambda_n\} \subset \Lambda$. ⌟

Lemma 2.1.3 *For a family $(V_\lambda)_{\lambda\in\Lambda}$ of subspaces of a Banach space V, the following properties are equivalent:*

(i) *$(V_\lambda)_{\lambda\in\Lambda}$ is finitely linearly independent.*

(ii) *Every $x \in \sum_{\lambda\in\Lambda} V_\lambda$ has a unique representation $x = \sum_{\lambda\in\Lambda} x_\lambda$ with $x_\lambda \in V_\lambda$ and almost all x_λ zero.*

(iii) *There is a family of algebraic projections $(P_\lambda)_{\lambda\in\Lambda}$ corresponding to $(V_\lambda)_{\lambda\in\Lambda}$ with domain $\mathcal{D}(P_\lambda) = \sum_{\mu\in\Lambda} V_\mu$, range $\mathcal{R}(P_\lambda) = V_\lambda$, and the property that $P_\mu P_\lambda = 0$ whenever $\mu \neq \lambda$.*

Proof. The implication (i)⇒(ii) is clear; for (ii)⇒(iii) define $P_\lambda(\sum_\mu x_\mu) = x_\lambda$ for each $\lambda \in \Lambda$.

(iii)⇒(i): From $\mathcal{R}(P_\lambda) = V_\lambda$ and $P_\mu P_\lambda = 0$ for $\mu \neq \lambda$ we obtain $P_\lambda x_\mu = 0$ for $x_\mu \in V_\mu$ and $\mu \neq \lambda$. Hence $x_{\lambda_1} + \cdots + x_{\lambda_n} = 0$ implies $x_{\lambda_j} = P_{\lambda_j}(x_{\lambda_1} + \cdots + x_{\lambda_n}) = 0$. □

Because of the uniqueness of the expansion $x = \sum_{\lambda\in\Lambda} x_\lambda$, we call the sum of the finitely linearly independent family $(V_\lambda)_{\lambda\in\Lambda}$ *algebraic direct* and use the notation

$$\sum_{\lambda\in\Lambda}^{\dotplus} V_\lambda\,.$$

Definition 2.1.4 We say that a family $(V_\lambda)_{\lambda\in\Lambda}$ of closed subspaces of a Banach space V forms an l^2-*decomposition* of V if

(i) the sum $\sum_{\lambda\in\Lambda} V_\lambda \subset V$ is dense and

(ii) there exists $c \geq 1$ such that

$$c^{-1}\sum_{\lambda\in F}\|x_\lambda\|^2 \leq \Big\|\sum_{\lambda\in F} x_\lambda\Big\|^2 \leq c\sum_{\lambda\in F}\|x_\lambda\|^2 \tag{2.2}$$

for all finite subsets $F \subset \Lambda$ and $x_\lambda \in V_\lambda$.

If we want to explicitly specify the value of c, we shall speak of a *decomposition with constant c.* ⌟

From (2.2) it follows that if a family $(V_\lambda)_{\lambda\in\Lambda}$ forms an l^2-decomposition then it is finitely linearly independent. The corresponding algebraic projections P_λ onto V_λ are densely defined. As the next lemma shows, they are even bounded and can thus be extended to the entire space V.

Lemma 2.1.5 *Let the family $(V_\lambda)_{\lambda\in\Lambda}$ form an l^2-decomposition of a Banach space V. Then we have:*

(i) For every subset $J \subset \Lambda$ there is a projection $P_J : V \to V$ with $P_J|_{V_\lambda} = I_{V_\lambda}$ for $\lambda \in J$, $P_J|_{V_\lambda} = 0$ for $\lambda \notin J$, and $\|P_J\| \leq c$.

(ii) For $\lambda \in \Lambda$ let P_λ be the projection corresponding to the subset $\{\lambda\} \subset \Lambda$. Then $\mathcal{R}(P_\lambda) = V_\lambda$. Moreover, $P_\lambda x = 0$ for all λ implies $x = 0$.

(iii) For every $x \in V$, if $J = \{\lambda \in \Lambda \mid P_\lambda x \neq 0\}$ then $P_J x = x$.

Proof. (i): Since $(V_\lambda)_{\lambda\in\Lambda}$ is finitely linearly independent, we may consider the algebraic projection $\widetilde{P}_J$ with domain $\mathcal{D}(\widetilde{P}_J) = \sum^{+}_{\lambda\in\Lambda} V_\lambda$ defined by

$$\widetilde{P}_J x_\lambda = \begin{cases} x_\lambda & \text{if } x_\lambda \in V_\lambda,\ \lambda \in J,\\ 0 & \text{if } x_\lambda \in V_\lambda,\ \lambda \notin J.\end{cases}$$

An arbitrary $x \in \mathcal{D}(\widetilde{P}_J)$ is of the form $x = \sum_{\lambda\in F} x_\lambda$, $x_\lambda \in V_\lambda$, with some finite $F \subset \Lambda$, and (2.2) yields

$$\big\|\widetilde{P}_J x\big\|^2 = \Big\|\sum_{\lambda\in F\cap J} x_\lambda\Big\|^2 \leq c\sum_{\lambda\in F\cap J}\|x_\lambda\|^2 \leq c\sum_{\lambda\in F}\|x_\lambda\|^2 \leq c^2\|x\|^2.$$

Hence, the densely defined operator $\widetilde{P}_J$ has a bounded linear extension $P_J \in L(V)$ with $\|P_J\| \leq c$. The identity $P_J^2 = P_J$ holds on the dense subspace $\mathcal{D}(\widetilde{P}_J)$ and thus on V; so P_J is a projection.

(ii): By (i), P_λ is the bounded extension of $\widetilde{P}_{\{\lambda\}}$ with $\mathcal{R}(\widetilde{P}_{\{\lambda\}}) = V_\lambda$. Since V_λ is closed, this implies $\mathcal{R}(P_\lambda) = V_\lambda$. Now let $x \in V$ with $P_\lambda x = 0$ for all λ, $\varepsilon > 0$, and $y \in \sum_{\lambda\in\Lambda} V_\lambda$ such that $\|x - y\| < \varepsilon$. Then $y = \sum_{\lambda\in F} y_\lambda$, $y_\lambda \in V_\lambda$, for some finite $F = \{\lambda_1, \dots, \lambda_n\}$. We have $P_F = P_{\lambda_1} + \cdots + P_{\lambda_n}$ since this relation holds on the dense subspace $\sum_{\lambda\in\Lambda} V_\lambda$. Therefore $y = P_F y$, $P_F x = 0$, and

$$\|y\| = \|P_F y\| \le \|P_F x\| + \|P_F\|\|x - y\| \le c\,\|x - y\|.$$

This implies

$$\|x\| \le \|x - y\| + \|y\| \le (1 + c)\|x - y\| < (1 + c)\varepsilon$$

and we conclude $x = 0$.

(iii): First observe that $P_\lambda P_J = P_\lambda$ for $\lambda \in J$ and $P_\lambda P_J = 0$ for $\lambda \notin J$ since these relations hold on $\sum^{+}_{\lambda\in\Lambda} V_\lambda$. Hence $P_\lambda(P_J x - x) = 0$ for all λ, and using (ii) we obtain $P_J x - x = 0$. □

Proposition 2.1.6 *Let the family $(V_\lambda)_{\lambda\in\Lambda}$ form an l^2-decomposition of a Banach space V.*

(i) Let P_λ be the projection onto V_λ defined in the previous lemma. Then for every $x \in V$ the relation

$$c^{-1} \sum_{\lambda\in\Lambda} \|P_\lambda x\|^2 \le \|x\|^2 \le c \sum_{\lambda\in\Lambda} \|P_\lambda x\|^2 \tag{2.3}$$

holds; in particular $P_\lambda x \ne 0$ for at most countably many λ.

(ii) If $x_\lambda \in V_\lambda$ with $\sum_{\lambda\in\Lambda} \|x_\lambda\|^2 < \infty$, then the series $\sum_{\lambda\in\Lambda} x_\lambda$ converges unconditionally.

(iii) Every $x \in V$ has a unique expansion

$$x = \sum_{\lambda\in\Lambda} x_\lambda \quad \textit{with} \quad x_\lambda \in V_\lambda\,; \tag{2.4}$$

its members are given by $x_\lambda = P_\lambda x$.

Because of the uniqueness of the expansion $x = \sum_{\lambda\in\Lambda} x_\lambda$, we use the notation

$$V = \bigoplus_{\lambda\in\Lambda}^{2} V_\lambda \tag{2.5}$$

for an l^2-decomposition. In terms of this expansion, the projections P_J defined above are of the form

$$P_J : \sum_{\lambda\in\Lambda} x_\lambda \longmapsto \sum_{\lambda\in J} x_\lambda\,.$$

Moreover, (2.3) shows that the original norm on V is equivalent to the l^2-type norm $(\sum_{\lambda\in\Lambda} \|P_\lambda x\|^2)^{1/2}$, hence the notion "$l^2$-decomposition".

Proof of the proposition. For every $x \in V$ we first show that $P_\lambda x \neq 0$ for at most countably many λ. Consider a finite subset $F \subset \Lambda$. For $x \in \sum^{+}_{\lambda\in\Lambda} V_\lambda$, i.e. $x = \sum_{\lambda\in F_0} x_\lambda$, $x_\lambda \in V_\lambda$, for some finite $F_0 \subset \Lambda$, we know from (2.2) that

$$\sum_{\lambda\in F} \|P_\lambda x\|^2 = \sum_{\lambda\in F\cap F_0} \|x_\lambda\|^2 \le \sum_{\lambda\in F_0} \|x_\lambda\|^2 \le c\Big\| \sum_{\lambda\in F_0} x_\lambda\Big\|^2,$$

i.e. $\sum_{\lambda\in F} \|P_\lambda x\|^2 \le c\|x\|^2$. By continuity, this relation is valid for all $x \in V$. For every $n \ge 1$ it follows that $\|P_\lambda x\| \ge n^{-1}$ holds for at most finitely many λ; hence $P_\lambda x \neq 0$ for at most countably many λ.

Now we want to prove the expansion (2.4). Let $(\lambda_j)_{j\in\mathbb{N}}$ be an enumeration of

$$J = \{\lambda \in \Lambda \,|\, P_\lambda x \neq 0\}$$

and consider $\varepsilon > 0$. We know that $x = \lim_{n\to\infty} y_n$ where $(y_n)_{n\in\mathbb{N}}$ is a sequence in $\sum_{\lambda\in\Lambda} V_\lambda$. With the help of the previous lemma we have $x = P_J x = \lim_{n\to\infty} P_J y_n$. Hence, there exists $y \in \sum_{\lambda\in J} V_\lambda$ with $\|x - y\| \le \varepsilon$ and $y = \sum_{j=0}^{n_0} y_j$, $y_j \in V_{\lambda_j}$, for some n_0. For every $n \ge n_0$ we obtain

$$\begin{aligned}\Big\|\sum_{j=0}^{n} P_{\lambda_j} x - x\Big\| &\le \Big\|\sum_{j=0}^{n} P_{\lambda_j}(x-y)\Big\| + \Big\|\sum_{j=0}^{n} P_{\lambda_j} y - x\Big\| \\ &\le \bigg(\Big\|\sum_{j=0}^{n} P_{\lambda_j}\Big\| + 1\bigg)\|x-y\| \le (c+1)\varepsilon.\end{aligned}$$

Therefore $\sum_{j=0}^{n} P_{\lambda_j} x$ converges to x as n tends to infinity. Since the enumeration of J was arbitrary, the convergence is even unconditional. The inequality (2.3) now follows from (2.2) if we set $x_\lambda = P_\lambda x$, $F = \{\lambda_1, \dots, \lambda_n\}$, and then take the limit $n \to \infty$. Finally, given any expansion $x = \sum_\lambda x_\lambda$, $x_\lambda \in V_\lambda$, we have $x_\lambda = P_\lambda x$; thus the uniqueness of the expansion.

Only (ii) remains to be shown. The assumption $\sum_{\lambda\in\Lambda} \|x_\lambda\|^2 < \infty$ implies that the set $J = \{\lambda \in \Lambda \,|\, x_\lambda \neq 0\}$ is at most countable. Choosing an enumeration of J, we obtain

$$\Big\|\sum_{j=n_1}^{n_2} x_{\lambda_j}\Big\|^2 \le c \sum_{j=n_1}^{n_2} \|x_{\lambda_j}\|^2;$$

hence $(\sum_{j=0}^{n} x_{\lambda_j})_{n\in\mathbb{N}}$ is a Cauchy sequence. Therefore we have a converging series $x = \sum_{j=0}^{\infty} x_{\lambda_j}$, and as we have seen in the previous paragraph, this expansion is unique and unconditional. □

Remark 2.1.7 The family $(V_k)_{k=1,\dots,n}$ of closed subspaces forms an l^2-decomposition if and only if we have the topological direct sum

$$V = V_1 \oplus \cdots \oplus V_n\,.$$

Indeed for $\Lambda = \{1,\dots,n\}$ finite, (2.2) just means that on $V_1 \dotplus \cdots \dotplus V_n$ the original norm $\|\cdot\|$ of V is equivalent to the norm

$$\|x_1 + \cdots + x_n\|_2 = \sqrt{\|x_1\|^2 + \cdots + \|x_n\|^2}\,, \quad x_j \in V_j\,;$$

and this is the case if and only if the sum $V_1 \dotplus \cdots \dotplus V_n$ is topological direct. Since $V_1 \oplus \cdots \oplus V_n$ is closed, it is dense if and only if it is equal to V.

If $P_1,\dots,P_n$ are the projections corresponding to the topological direct sum, the constant in (2.2) can be chosen as

$$c = \|P_1\|^2 + \cdots + \|P_n\|^2.$$

This follows from the fact that if $x = x_1 + \cdots + x_n$ with $x_j \in V_j$, then

$$\sum_{j=1}^n \|x_j\|^2 = \sum_{j=1}^n \|P_j x\|^2 \le \sum_{j=1}^n \|P_j\|^2 \cdot \|x\|^2 \quad \text{and}$$

$$\|x\|^2 \le \left(\sum_{j=1}^n \|x_j\|\right)^2 \le n \sum_{j=1}^n \|x_j\|^2 \le \sum_{j=1}^n \|P_j\|^2 \cdot \sum_{j=1}^n \|x_j\|^2.$$

For the Hilbert space case, a sharper constant will be obtained in Lemma 2.2.6. ⌟

Now we turn to the question of how an existing l^2-decomposition $V = \bigoplus^2_{\lambda\in\Lambda} V_\lambda$ gives rise to other decompositions. Let $U_\lambda \subset V_\lambda$ be closed subspaces. As we can restrict the relation (2.2) to the subspaces U_λ, we clearly obtain the l^2-decomposition

$$\overline{\sum_{\lambda\in\Lambda} U_\lambda} = \bigoplus_{\lambda\in\Lambda}^2 U_\lambda\,. \tag{2.6}$$

In particular, if $J \subset \Lambda$ and we have $U_\lambda = V_\lambda$ for $\lambda \in J$ and $U_\lambda = \{0\}$ otherwise, we shall write

$$\bigoplus_{\lambda\in J}^2 V_\lambda\,.$$

For the projection P_J associated with the subset J, this yields

$$\mathcal{R}(P_J) = \bigoplus_{\lambda\in J}^2 V_\lambda\,, \qquad \ker P_J = \bigoplus_{\lambda\in\Lambda\setminus J}^2 V_\lambda\,,$$

and we get the topological direct sum

$$V = \bigoplus_{\lambda\in J}^2 V_\lambda \oplus \bigoplus_{\lambda\in\Lambda\setminus J}^2 V_\lambda\,. \tag{2.7}$$

So we have split the l^2-decomposition into two parts with every V_λ entirely belonging to one part. Alternatively, we may split each subspace V_λ itself:

Proposition 2.1.8 *Suppose that for the l^2-decomposition $V = \bigoplus^2_{\lambda\in\Lambda} V_\lambda$ we have $V_\lambda = U_\lambda \oplus W_\lambda$. Then the sum*

$$\bigoplus_{\lambda\in\Lambda}^2 U_\lambda \dotplus \bigoplus_{\lambda\in\Lambda}^2 W_\lambda \subset V \tag{2.8}$$

is algebraic direct and dense.

Proof. Let $x \in \bigoplus^2_\lambda U_\lambda \cap \bigoplus^2_\lambda W_\lambda$. We thus have the expansions $x = \sum_\lambda u_\lambda$ with $u_\lambda \in U_\lambda$ and $x = \sum_\lambda w_\lambda$ with $w_\lambda \in W_\lambda$. As both are also expansions with respect to $\bigoplus^2_\lambda V_\lambda$, they must be identical, $u_\lambda = w_\lambda$. Since $U_\lambda \cap W_\lambda = \{0\}$, this implies $u_\lambda = 0$; hence $x = 0$. Moreover, the sum $\bigoplus^2_\lambda U_\lambda \dotplus \bigoplus^2_\lambda W_\lambda$ is dense since it contains every subspace V_λ. □

Remark 2.1.9 The sum (2.8) is not topological direct in general, see 5.1.1 as an example of such a situation. In fact, (2.8) is topological direct if and only if the projections $U_\lambda \oplus W_\lambda \to U_\lambda$ are uniformly bounded in $\lambda \in \Lambda$, and this is the case if and only if the system $(U_\lambda, W_\lambda)_{\lambda\in\Lambda}$ forms an l^2-decomposition; compare Lemma 2.1.10 and Remark 2.1.7. ⌟

The decomposition (2.7) can be generalised: Suppose Λ is written as a disjoint union $\Lambda = \dot\bigcup_{\gamma\in\Gamma} J_\gamma$. Then the closed subspaces $\bigoplus^2_{\lambda\in J_\gamma} V_\lambda$ constitute an l^2-decomposition of V,

$$V = \bigoplus_{\gamma\in\Gamma}^2 \Bigl(\bigoplus_{\lambda\in J_\gamma}^2 V_\lambda \Bigr) ; \tag{2.9}$$

we omit the simple proof. The next lemma analyses the reversed situation:

Lemma 2.1.10 *Let $V = \bigoplus^2_{\lambda\in\Lambda} W_\lambda$ be an l^2-decomposition with constant c_0. Let $W_\lambda = \bigoplus^2_{\mu\in J_\lambda} V_{\lambda\mu}$ be l^2-decompositions for all $\lambda \in \Lambda$ with common constant c_1. Then the family $(V_{\lambda\mu})_{\lambda\in\Lambda,\, \mu\in J_\lambda}$ forms an l^2-decomposition of V with constant c_0c_1.*

Proof. Since $\sum_{\lambda\in\Lambda} W_\lambda$ is dense in V and for every $\lambda \in \Lambda$ the subspace $\sum_{\mu\in J_\lambda} V_{\lambda\mu}$ is dense in W_λ, we see that $\sum_{\lambda\in\Lambda,\mu\in J_\lambda} V_{\lambda\mu}$ is dense in V. Consider $F \subset \Lambda$ finite, $F_\lambda \subset J_\lambda$ finite for each $\lambda \in F$, and $x_{\lambda\mu} \in V_{\lambda\mu}$. Then

$$\Bigl\| \sum_{\substack{\lambda\in F\\ \mu\in F_\lambda}} x_{\lambda\mu} \Bigr\|^2 \le c_0 \sum_{\lambda\in F} \Bigl\| \sum_{\mu\in F_\lambda} x_{\lambda\mu} \Bigr\|^2 \le c_0 \sum_{\lambda\in F} c_1 \sum_{\mu\in F_\lambda} \|x_{\lambda\mu}\|^2 = c_0c_1 \sum_{\substack{\lambda\in F\\ \mu\in F_\lambda}} \|x_{\lambda\mu}\|^2$$

and similarly $\| \sum_{\lambda\in F,\mu\in F_\lambda} x_{\lambda\mu}\|^2 \ge c_0^{-1}c_1^{-1} \sum_{\lambda\in F,\mu\in F_\lambda} \|x_{\lambda\mu}\|^2$. □

Note that in the previous lemma the existence of the common constant c_1 is guaranteed if $|J_\lambda| = 1$ for almost all λ, that is, if only finitely many subspaces W_λ are decomposed.

2.2 l^2-decompositions of Hilbert spaces

In this section we focus on countable l^2-decompositions of separable Hilbert spaces. Following again Gohberg and Krein [22], Markus [36], and Singer [46], we obtain several equivalent conditions for a sequence of closed subspaces to form an l^2-decomposition and also relations to Riesz bases.

The following observation shows that it is often natural to consider l^2-decompositions of a Hilbert space:

Remark 2.2.1 Let $V = \bigoplus^2_{\lambda\in\Lambda} V_\lambda$ be an l^2-decomposition of a Banach space such that each V_λ is isomorphic to a Hilbert space H_λ. Then V is isomorphic to the Hilbert space orthogonal sum $\bigoplus_{\lambda\in\Lambda} H_\lambda$ by (2.3). This isomorphism induces a scalar product on V giving it the structure of a Hilbert space with an orthogonal decomposition $V = \bigoplus_{\lambda\in\Lambda} V_\lambda$. An example of such a situation is the case where all V_λ are finite-dimensional. ⌟

Up to an isomorphism, an l^2-decomposition of a separable Hilbert space is completely determined by the dimensions of its constituting subspaces:

Proposition 2.2.2 *Let $H = \bigoplus^2_{k\in\mathbb{N}} V_k$ be an l^2-decomposition of a separable Hilbert space and $(W_k)_{k\in\mathbb{N}}$ a sequence of closed subspaces of H. Then $(W_k)_{k\in\mathbb{N}}$ forms an l^2-decomposition of H with $\dim V_k = \dim W_k$ if and only if there is an isomorphism $T : H \to H$ with $T(V_k) = W_k$.*

Proof. ($\Rightarrow$): Since the subspaces V_k and W_k are both closed and of the same Hilbert space dimension (either finite or countable since H is separable) there exist isometric isomorphisms $T_k : V_k \to W_k$. Define $\widetilde{T}$ with $\mathcal{D}(\widetilde{T}) = \sum^{\dot{+}}_{k\in\mathbb{N}} V_k$ and $\widetilde{T}|_{V_k} = T_k$. Let c_V and c_W be the constants of the decompositions $\bigoplus^2_k V_k$ and $\bigoplus^2_k W_k$, respectively. For $x = \sum_{k=0}^n x_k \in \sum^{\dot{+}}_k V_k$ we have

$$\|\widetilde{T}x\|^2 = \Big\| \sum_{k=0}^n T_k x_k \Big\|^2 \le c_W \sum_{k=0}^n \|T_k x_k\|^2 = c_W \sum_{k=0}^n \|x_k\|^2 \le c_W c_V \|x\|^2$$

and similarly $\|\widetilde{T}x\|^2 \ge c_W^{-1} c_V^{-1} \|x\|^2$. Thus $\widetilde{T}$ extends to an isomorphism T of H with the desired property.

($\Leftarrow$): Since the subspace $\sum_k V_k$ is dense in H and T is an isomorphism, $\sum_k W_k = T(\sum_k V_k) \subset H$ is dense as well. Now, for $k = 1, \dots, n$, let $y_k \in W_k$ and $y_k = Tx_k$.

Then

$$\Big\|\sum_{k=0}^{n} y_k\Big\|^2 = \Big\|T\sum_{k=0}^{n} x_k\Big\|^2 \le \|T\|^2 c_V \sum_{k=0}^{n} \|x_k\|^2 = c_V\|T\|^2 \sum_{k=0}^{n} \|T^{-1}y_k\|^2$$
$$\le c_V\|T\|^2\|T^{-1}\|^2 \sum_{k=0}^{n} \|y_k\|^2$$

and

$$\Big\|\sum_{k=0}^{n} y_k\Big\|^2 \ge \frac{1}{\|T^{-1}\|^2}\Big\|T^{-1}\sum_{k=0}^{n} y_k\Big\|^2 \ge \frac{c_V^{-1}}{\|T^{-1}\|^2}\sum_{k=0}^{n}\|x_k\|^2$$
$$\ge \frac{c_V^{-1}}{\|T\|^2\|T^{-1}\|^2}\sum_{k=0}^{n}\|Tx_k\|^2 = \frac{c_V^{-1}}{\|T\|^2\|T^{-1}\|^2}\sum_{k=0}^{n}\|y_k\|^2.$$

Hence $(W_k)_{k\in\mathbb{N}}$ forms an l^2-decomposition. □

Corollary 2.2.3 *A sequence $(W_k)_{k\in\mathbb{N}}$ of closed subspaces in a separable Hilbert space H forms an l^2-decomposition if and only if there exists an orthogonal decomposition $H = \bigoplus_{k\in\mathbb{N}} V_k$ and an isomorphism T with $W_k = T(V_k)$, $k \in \mathbb{N}$.*

Proof. Note that for any sequence $d_k \in \mathbb{N}\cup\{\infty\}$ with $\sum_{k\in\mathbb{N}} d_k = \dim H$ (in particular for $d_k = \dim W_k$) we can find an orthogonal decomposition $H = \bigoplus_{k\in\mathbb{N}} V_k$ with $d_k = \dim V_k$. Since every orthogonal decomposition is also an l^2-decomposition, the claim is an immediate consequence of the previous proposition. □

This last characterisation explains the notion "basis of subspaces equivalent to an orthogonal one" used by Gohberg and Krein [22, §VI.5].

Our next aim is to derive a condition for the existence of an l^2-decomposition in terms of norms of the associated projections.

Lemma 2.2.4 *Let V be a Banach space and $(x_n)_{n\in\mathbb{N}}$ a sequence in V. If there exists $C \ge 0$ such that for every reordering $\phi : \mathbb{N} \xrightarrow{\text{bij}} \mathbb{N}$ and every $n \in \mathbb{N}$ we have $\|\sum_{k=0}^{n} x_{\phi(k)}\| \le C$, then*

$$\sup_{n\in\mathbb{N},\varepsilon_k=\pm1}\Big\|\sum_{k=0}^{n}\varepsilon_k x_k\Big\| \le 2C.$$

Proof. Let $\varepsilon_0,\dots,\varepsilon_n \in \{-1,1\}$ and consider reorderings ϕ_1 and ϕ_2 that move all $+1$ and all -1 in the sequence $(\varepsilon_0,\dots,\varepsilon_n)$, respectively, to its beginning. Then, with n_1, n_2 appropriate, we obtain

$$\Big\|\sum_{k=0}^{n}\varepsilon_k x_k\Big\| \le \Big\|\sum_{\substack{k=0\\ \varepsilon_k=+1}}^{n} x_k\Big\| + \Big\|\sum_{\substack{k=0\\ \varepsilon_k=-1}}^{n} x_k\Big\| = \Big\|\sum_{k=0}^{n_1} x_{\phi_1(k)}\Big\| + \Big\|\sum_{k=0}^{n_2} x_{\phi_2(k)}\Big\| \le 2C.$$

□

Lemma 2.2.5 *Let H be a Hilbert space, $x_0, \dots, x_n \in H$, and*

$$E = \{(\varepsilon_0, \dots, \varepsilon_n) \mid \varepsilon_k = \pm 1\}.$$

Then

$$2^{n+1} \sum_{k=0}^{n} \|x_k\|^2 = \sum_{\varepsilon \in E} \|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|^2.$$

Proof. We use induction on n. The statement is true for the case $n = 0$ since $2\|x_0\|^2 = \|x_0\|^2 + \|-x_0\|^2$. Now suppose the statement holds for some $n \geq 0$; let

$$\widetilde{E} = \{(\varepsilon_0, \dots, \varepsilon_{n+1}) \mid \varepsilon_k = \pm 1\}$$

and write $x_\varepsilon = \varepsilon_0 x_0 + \dots + \varepsilon_n x_n$. Then

$$\begin{aligned}\sum_{\varepsilon \in \widetilde{E}} \|\varepsilon_0 x_0 + \dots + \varepsilon_{n+1} x_{n+1}\|^2 &= \sum_{\varepsilon \in E} \left(\|x_\varepsilon + x_{n+1}\|^2 + \|x_\varepsilon - x_{n+1}\|^2\right) \\ &= \sum_{\varepsilon \in E} \left(2\|x_\varepsilon\|^2 + 2\|x_{n+1}\|^2\right) = 2 \sum_{\varepsilon \in E} \|x_\varepsilon\|^2 + 2 \cdot 2^{n+1} \|x_{n+1}\|^2 \\ &= 2^{n+2} \left(\sum_{k=0}^{n} \|x_k\|^2 + \|x_{n+1}\|^2 \right).\end{aligned}$$

□

Lemma 2.2.6 *Let $P_0, \dots, P_n$ be projections in a Hilbert space H with $P_j P_k = 0$ for $j \neq k$. Then*

$$C^{-2} \sum_{k=0}^{n} \|P_k x\|^2 \leq \Big\| \sum_{k=0}^{n} P_k x \Big\|^2 \leq C^2 \sum_{k=0}^{n} \|P_k x\|^2 \quad \text{for all} \quad x \in H$$

where $C = \max\{\| \sum_{k=0}^{n} \varepsilon_k P_k \| \mid \varepsilon_k = \pm 1\}$.

Proof. We write $x_k = P_k x$ and use the last lemma considering that $\varepsilon \in E$ for which $\|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|$ becomes maximal. Then we obtain

$$\sum_{k=0}^{n} \|P_k x\|^2 \leq \|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|^2 = \Big\| \Big(\sum_{k=0}^{n} \varepsilon_k P_k \Big) \Big(\sum_{k=0}^{n} x_k \Big) \Big\|^2 \leq C^2 \Big\| \sum_{k=0}^{n} P_k x \Big\|^2.$$

On the other hand, if we choose $\varepsilon \in E$ such that $\|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|$ is minimal, we find

$$\begin{aligned}\Big\| \sum_{k=0}^{n} P_k x \Big\|^2 &= \Big\| \Big(\sum_{k=0}^{n} \varepsilon_k P_k \Big) \Big(\sum_{k=0}^{n} \varepsilon_k x_k \Big) \Big\|^2 \\ &\leq C^2 \|\varepsilon_0 x_0 + \dots + \varepsilon_n x_n\|^2 \leq C^2 \sum_{k=0}^{n} \|P_k x\|^2.\end{aligned}$$

□

The following statement yields a sufficient condition for a sequence of projections to generate an l^2-decomposition. It is a slight modification[1] of a result by Markus [36, Lemma 6.2] and will be used in the next chapter to obtain determining l^2-decompositions for non-normal operators.

Proposition 2.2.7 *Let H be a Hilbert space with scalar product $(\cdot|\cdot)$ and $(P_k)_{k\in\mathbb{N}}$ a sequence of projections in H satisfying $P_jP_k = 0$ for $j \neq k$. Suppose that $\sum_{k\in\mathbb{N}} \mathcal{R}(P_k) \subset H$ is dense and that*

$$\sum_{k=0}^{\infty} |(P_k x|y)| \leq C\|x\|\|y\| \quad \textit{for all} \quad x, y \in H \tag{2.10}$$

with some constant $C \geq 0$. Then the projections generate an l^2-decomposition

$$H = \bigoplus_{k\in\mathbb{N}}^{2} \mathcal{R}(P_k)$$

with constant $c = 4C^2$.

Proof. From

$$\Big|\Big(\sum_{k=0}^{n} P_k x\Big|y\Big)\Big| \leq \sum_{k=0}^{n} |(P_k x|y)| \leq C\|x\|\|y\|$$

we conclude that $\|\sum_{k=0}^{n} P_k\| \leq C$ for all $n \in \mathbb{N}$. This assertion remains valid after an arbitrary rearrangement of the sequence $(P_k)_{k\in\mathbb{N}}$ since the assumptions of the proposition still hold for the rearranged sequence. An application of Lemmas 2.2.4 and 2.2.6 now completes the proof. □

Remark 2.2.8 Suppose that we have a sequence $(Q_k)_{k\in\mathbb{N}}$ of orthogonal projections with $Q_jQ_k = 0$ for $j \neq k$. Then

$$\begin{aligned}\sum_k |(P_k x|y)| &\leq \sum_k \big|\big((P_k - Q_k)x\big|y\big)\big| + \sum_k |(Q_k x|y)| \\ &\leq \sum_k \big|\big((P_k - Q_k)x\big|y\big)\big| + \|x\|\|y\|.\end{aligned}$$

Therefore, in order to show $\sum_{k=0}^{\infty} |(P_k x|y)| \leq C\|x\|\|y\|$, it is also possible to show

$$\sum_{k=0}^{\infty} \big|\big((P_k - Q_k)x\big|y\big)\big| \leq \widetilde{C}\|x\|\|y\| \quad \text{for all} \quad x, y \in H \tag{2.11}$$

with some constant $\widetilde{C}$. ⌟

[1] Under the weaker assumption $\sum_{k=0}^{\infty} |(P_k x|y)| < \infty$ for all $x, y \in H$, Markus proved the existence of the decomposition $H = \bigoplus_{k\in\mathbb{N}}^{2} \mathcal{R}(P_k)$, but without obtaining a formula for the constant c.

The conditions in Proposition 2.2.7 are actually one of several equivalent criteria for a sequence of subspaces to form an l^2-decomposition. We say that the sequence $(V_k)_{k\in\mathbb{N}}$ is an *unconditional basis* for H if every $x \in H$ can be uniquely written as $x = \sum_{k=0}^{\infty} x_k$, $x_k \in V_k$, and the convergence of the series $\sum_{k=0}^{\infty} x_k$ is unconditional; compare Singer [46, page 534].

Theorem 2.2.9 *For a sequence of closed subspaces $(V_k)_{k\in\mathbb{N}}$ in a separable Hilbert space H the following conditions are equivalent:*

(i) $(V_k)_{k\in\mathbb{N}}$ *forms an l^2-decomposition for H.*

(ii) There is an isomorphism $T : H \to H$ such that the subspaces $T(V_k)$, $k \in \mathbb{N}$, form an orthogonal decomposition of H.

(iii) $(V_k)_{k\in\mathbb{N}}$ *is an unconditional basis for H.*

(iv) The sum $\sum_k V_k \subset H$ is dense and there exist projections P_k, $k \in \mathbb{N}$, such that $V_k = \mathcal{R}(P_k)$, $P_j P_k = 0$ for $j \neq k$, and there is a constant $C > 0$ with

$$\Big\| \sum_{k\in F} P_k \Big\| \leq C \quad \textit{for every finite} \quad F \subset \mathbb{N}.$$

(v) The sum $\sum_k V_k \subset H$ is dense and there exist projections P_k, $k \in \mathbb{N}$, such that $V_k = \mathcal{R}(P_k)$, $P_j P_k = 0$ for $j \neq k$, and there is a constant $C > 0$ with

$$\sum_{k=0}^{\infty} |(P_k x|y)| \leq C\|x\|\|y\| \quad \textit{for all} \quad x, y \in H.$$

Moreover, in the two last statements the density condition can be replaced by the condition that $P_k x = 0$ for all k implies $x = 0$.

Proof. We already know that

$$(i) \Leftrightarrow (ii) \quad \text{and} \quad (v) \Rightarrow (iv) \Rightarrow (i) \Rightarrow (iii),$$

compare Proposition 2.1.6, Corollary 2.2.3, and the proof of Proposition 2.2.7. We only sketch the remaining implications, see Gohberg and Krein [22, §VI.5] and Singer [46, §15] for more details:

(ii) $\Rightarrow$ *(v)*: Let $W_k = T(V_k)$ and denote by Q_k the orthogonal projections corresponding to the decomposition $H = \bigoplus_k W_k$. Then $\bigoplus_k^2 V_k$ and $\bigoplus_k^2 T^*(W_k)$ are both l^2-decompositions with corresponding projections $P_k = T^{-1}Q_k T$ and $P_k^* = T^* Q_k T^{-*}$ and constants c and $\tilde{c}$, respectively. This yields

$$\begin{aligned}\sum_k |(P_k x|y)| &= \sum_k |(P_k x|P_k^* y)| \leq \sum_k \|P_k x\|\|P_k^* y\| \\ &\leq \Big(\sum_k \|P_k x\|^2\Big)^{1/2} \Big(\sum_k \|P_k^* y\|^2\Big)^{1/2} \leq \sqrt{c\tilde{c}}\,\|x\|\|y\|.\end{aligned}$$

$(iii) \Rightarrow (i)$: Since $(V_k)_{k\in\mathbb{N}}$ is a basis, the sum $\sum_k V_k$ is dense in H and the projections P_k onto the components x_k given by the unique expansion $x = \sum_{k=0}^{\infty} x_k$ are bounded. Moreover, the projections $\sum_{k=0}^{n} P_k$ are uniformly bounded in n. Since the basis is even unconditional, this remains true after an arbitrary rearrangement of the sequence $(P_k)_{k\in\mathbb{N}}$. Using the principle of uniform boundedness in the version for continuous, convex, positively homogeneous functionals (cf. [3, §18]), one can deduce that[2]

$$\sup_{n\in\mathbb{N},\varepsilon_k=\pm 1} \Big\| \sum_{k=0}^{n} \varepsilon_k P_k \Big\| < \infty.$$

Then Lemma 2.2.6 yields the l^2-property.

Now suppose we have (iv) with the density condition replaced by the assumption that $P_k x = 0$ for all k implies $x = 0$. By Lemmas 2.2.4 and 2.2.6 we have, for every $x \in H$,

$$\begin{aligned}
&\frac{1}{4C^2}\sum_{k=0}^{n}\|P_k x\|^2 \le \Big\|\sum_{k=0}^{n} P_k x\Big\|^2 \le \Big\|\sum_{k=0}^{n} P_k\Big\|^2 \|x\|^2 \quad \text{for all} \quad n\in\mathbb{N}\\
\Longrightarrow \quad &\sum_{k=0}^{\infty}\|P_k x\|^2 \le 4C^4\|x\|^2 < \infty\\
\Longrightarrow \quad &\Big\|\sum_{k=n}^{m} P_k x\Big\|^2 \le 4C^2\sum_{k=n}^{m}\|P_k x\|^2 \to 0 \quad \text{as} \quad n,m\to\infty.
\end{aligned}$$

Consequently, $\sum_{k=0}^{\infty} P_k x$ converges for every $x \in H$. Let $y = x - \sum_{k=0}^{\infty} P_k x$. Then

$$P_j y = P_j x - \sum_{k=0}^{\infty} P_j P_k x = P_j x - P_j x = 0 \quad \text{for all} \quad j\in\mathbb{N}$$

and thus $x = \sum_{k=0}^{\infty} P_k x$. In particular, $\sum_k V_k$ is dense in H.

Finally, if $\bigoplus_k^2 V_k$ is an l^2-decomposition, we know that $P_k x = 0$ for all k implies $x = 0$. □

We end this section with statements about the connection between l^2-decompositions and Riesz bases, see also Gohberg and Krein [22, §VI.2].

Proposition 2.2.10 *For a sequence $(x_k)_{k\in\mathbb{N}}$ in a Hilbert space the following properties are equivalent:*

(i) $(x_k)_{k\in\mathbb{N}}$ is a Riesz basis.

---

[2]Note that we can not use Lemma 2.2.4 here since a priori we have different bounds for each rearrangement.

(ii) $(x_k)_{k\in\mathbb{N}}$ *is complete and there exist constants* $m, M > 0$ *such that*

$$m\sum_{k=0}^{n}|\alpha_k|^2 \le \Big\|\sum_{k=0}^{n}\alpha_k x_k\Big\|^2 \le M\sum_{k=0}^{n}|\alpha_k|^2 \tag{2.12}$$

holds for all $n\in\mathbb{N}$, $\alpha_k\in\mathbb{C}$.

(iii) $(x_k)_{k\in\mathbb{N}}$ *is an unconditional basis with* $\inf_{k\in\mathbb{N}}\|x_k\| > 0$, $\sup_{k\in\mathbb{N}}\|x_k\| < \infty$.

(iv) The subspaces $V_k = \mathbb{C}x_k$ *form an* l^2*-decomposition and* $\inf_{k\in\mathbb{N}}\|x_k\| > 0$, $\sup_{k\in\mathbb{N}}\|x_k\| < \infty$.

Proof. The equivalence $(i) \Leftrightarrow (iv)$ is immediate from Definition 2.1.1 and Corollary 2.2.3. $(ii) \Leftrightarrow (iv)$ holds by definition of an l^2-decomposition and $(iii) \Leftrightarrow (iv)$ follows from Theorem 2.2.9 and Definition 2.1.1. □

A generalisation of the concept of bases are bases with parentheses, see e.g. Markus [36, page 27] and Vizitei and Markus [50, §1].

Definition 2.2.11 A sequence $(x_k)_{k\in\mathbb{N}}$ in a Banach space V is called a *basis with parentheses* if there is a strictly increasing sequence $k_n\in\mathbb{N}$ with $k_0 = 0$ such that every $x\in V$ has a unique representation

$$x = \sum_{n=0}^{\infty}\left(\sum_{k=k_n}^{k_{n+1}-1}\alpha_k x_k\right), \qquad \alpha_k\in\mathbb{C}, \tag{2.13}$$

i.e., instead of (2.1) only the subsequence $\big(\sum_{k=0}^{k_n-1}\alpha_k x_k\big)_{n\in\mathbb{N}}$ of the sequence of all partial sums converges to x. If the convergence in (2.13) is unconditional, $(x_k)_{k\in\mathbb{N}}$ is called an *unconditional basis with parentheses*. ⌟

In a Hilbert space an unconditional basis with parentheses is also called a *Riesz basis with parentheses (or brackets)*, see Shkalikov [43].

Proposition 2.2.12 *The sequence* $(x_k)_{k\in\mathbb{N}}$ *in a Hilbert space is an unconditional basis with parentheses if and only if it is finitely linearly independent and the subspaces* $V_n = \operatorname{span}\{x_{k_n},\dots,x_{k_{n+1}-1}\}$ *form an* l^2*-decomposition.*

Proof. This is immediate from Theorem 2.2.9. □

2.3 Finitely determining l^2-decompositions

In this section we introduce the class of (generally non-normal) operators with a finitely determining l^2-decomposition. This amounts to the existence of an l^2-decomposition into finite-dimensional invariant subspaces such that the properties of the whole operator are determined by its restriction to these subspaces. For example, we obtain formulas for the domain of definition, the spectrum, and the resolvent. If the spectra of the restrictions are pairwise disjoint, the decomposition is called finitely spectral.

The notion of a finitely determining l^2-decomposition is equivalent to the existence of a Riesz basis with parentheses of Jordan chains such that each Jordan chain is contained inside some parenthesis, see Proposition 2.3.11. Riesz bases of this kind are frequently used in the literature, e.g. by Markus [36] and Tretter [47].

Other classes of non-normal operators that provide similar descriptions of properties of the operator are spectral and Riesz-spectral operators. The notion of a spectral operator was introduced by Dunford [18] (see [20] for a comprehensive presentation) and is in general not comparable with a finitely determining or spectral l^2-decomposition. However, a spectral operator with compact resolvent has a finitely spectral l^2-decomposition such that all restrictions of the operator to the subspaces of the decomposition have one eigenvalue only. Riesz-spectral operators are used for example in control theory (see [14] and [29]) and allow for a finitely spectral l^2-decomposition where all subspaces are one-dimensional.

The relations of finitely determining l^2-decompositions to the above and other classes of non-normal operators are summarised in Theorem 2.3.17.

Definition 2.3.1 Let $T(H \to H)$ be a closed operator on a separable Hilbert space H. We say that an l^2-decomposition $H = \bigoplus_{k\in\mathbb{N}}^2 V_k$ is *finitely determining* for T if

$$\dim V_k < \infty, \quad V_k \subset \mathcal{D}(T), \quad T(V_k) \subset V_k,$$

and $\sum_{k\in\mathbb{N}} V_k$ is a core for T. ⌟

A finitely determining l^2-decomposition is not uniquely determined since any finite collection of the subspaces V_k can be replaced by its sum.

Note that the restrictions $T|_{V_k} : V_k \to V_k$ are bounded since the V_k are finite-dimensional. The assumption of $\sum_k V_k$ being a core for T will then enable us to carry over results for the finite-dimensional parts $T|_{V_k}$ to the whole operator T. In Proposition 2.3.8 we show that this "core property" is automatically satisfied for operators with non-empty resolvent set. Without the core property, the theory still applies to an operator generated by the parts $T|_{V_k}$:

Lemma 2.3.2 *Let $T(H \to H)$ be an operator and $H = \bigoplus_{k\in\mathbb{N}}^2 V_k$ an l^2-decomposition with $\dim V_k < \infty$, $V_k \subset \mathcal{D}(T)$, and $T(V_k) \subset V_k$. Then the restriction $T_0 = T|_{\sum_{k\in\mathbb{N}} V_k}$ is closable and $\bigoplus_{k\in\mathbb{N}}^2 V_k$ is finitely determining for the closure $\overline{T_0}$.*

Proof. Let P_k be the projection onto V_k corresponding to the given l^2-decomposition. Suppose we have $y_n \in \mathcal{D}(T_0)$ with $y_n \to 0$ and $T_0 y_n \to z$. We may write

$y_n = \sum_{j\in\mathbb{N}} P_j y_n$, where the sum is actually finite since $\mathcal{D}(T_0) = \sum_k V_k$. The T-invariance of the V_k's yields $P_k T_0 y_n = P_k \sum_j T P_j y_n = T P_k y_n$. Therefore

$$\begin{aligned} P_k z = P_k \lim_{n\to\infty} T_0 y_n &= \lim_{n\to\infty} P_k T_0 y_n = \lim_{n\to\infty} T|_{V_k} P_k y_n \\ &= T|_{V_k} \lim_{n\to\infty} P_k y_n = T|_{V_k}(0) = 0, \end{aligned}$$

where we have used the fact that $T|_{V_k}$ is a bounded operator because V_k is finite-dimensional. Now, from $P_k z = 0$ for all k we conclude that $z = 0$, i.e., T_0 is closable. $\sum_k V_k$ is then a core for $\overline{T_0}$ and the assertion follows. □

The next proposition shows that an operator with a finitely determining l^2-decomposition is in fact determined by its finite-dimensional parts $T|_{V_k}$. For the case of an orthogonal decomposition, the spectrum of an operator defined by (2.14) and (2.15) was calculated by Davies [15, Theorem 8.1.12].

Proposition 2.3.3 *Let $T(H \to H)$ be a closed operator with finitely determining l^2-decomposition $H = \bigoplus^2_{k\in\mathbb{N}} V_k$. Then*

$$\mathcal{D}(T) = \left\{ \sum_{k\in\mathbb{N}} x_k \in \bigoplus_{k\in\mathbb{N}}^2 V_k \,\middle|\, \sum_{k\in\mathbb{N}} \|Tx_k\|^2 < \infty \right\}, \tag{2.14}$$

$$Tx = \sum_{k\in\mathbb{N}} Tx_k \quad \textit{for} \quad x = \sum_{k\in\mathbb{N}} x_k \in \mathcal{D}(T)\,. \tag{2.15}$$

T is bounded if and only if the restrictions $T|_{V_k}$ are uniformly bounded and in this case

$$\|T\| \le c \sup_{k\in\mathbb{N}} \|T|_{V_k}\|.$$

The point spectrum, residual spectrum and resolvent set are given by

$$\sigma_p(T) = \bigcup_{k\in\mathbb{N}} \sigma(T|_{V_k}), \quad \sigma_r(T) = \varnothing,$$

$$\varrho(T) = \left\{ z \in \mathbb{C} \setminus \sigma_p(T) \,\middle|\, \sup_{k\in\mathbb{N}} \|(T|_{V_k} - z)^{-1}\| < \infty \right\}. \tag{2.16}$$

Proof. We denote again by P_k the projections onto V_k corresponding to the l^2-decomposition.

(i): We derive (2.14) and (2.15). Let $y \in \mathcal{D}(T)$. Since $\sum_k V_k$ is a core for T, there is a sequence $y_n \in \sum_k V_k$ with $y_n \to y$, $Ty_n \to Ty$. Analogously to the proof of Lemma 2.3.2, we obtain $P_k T y_n = T P_k y_n$ and

$$P_k T y = P_k \lim_{n\to\infty} T y_n = \lim_{n\to\infty} P_k T y_n = T|_{V_k} \lim_{n\to\infty} P_k y_n = T P_k y.$$

Hence $\sum_k \|TP_k y\|^2 = \sum_k \|P_k Ty\|^2 \le c\|Ty\|^2 < \infty$ and

$$y = \sum_k P_k y \in \Big\{ \sum_k x_k \in \bigoplus_{k\in\mathbb{N}}^2 V_k \;\Big|\; \sum_k \|Tx_k\|^2 < \infty \Big\} \quad \text{with}$$
$$Ty = \sum_k P_k Ty = \sum_k TP_k y.$$

If on the other hand $\sum_k x_k \in \bigoplus_k^2 V_k$ with $\sum_k \|Tx_k\|^2 < \infty$, then

$$\mathcal{D}(T) \ni \sum_{k=0}^n x_k \to \sum_{k=0}^\infty x_k \quad \text{and} \quad T\sum_{k=0}^n x_k = \sum_{k=0}^n Tx_k \to \sum_{k=0}^\infty Tx_k.$$

Hence $\sum_k x_k \in \mathcal{D}(T)$ since T is closed.

(ii): Suppose that $L = \sup_k \|T|_{V_k}\| < \infty$. Then for $x = \sum_k x_k \in \mathcal{D}(T)$:

$$\|Tx\|^2 = \Big\| \sum_k T|_{V_k} x_k \Big\|^2 \le c \sum_k \|T|_{V_k} x_k\|^2$$
$$\le cL^2 \sum_k \|x_k\|^2 \le c^2 L^2 \|x\|^2;$$

thus T is bounded with norm $\le c\,L$.

(iii): Next we compute the point spectrum. We use the notation $\sigma_k = \sigma(T|_{V_k})$. Evidently $\sigma_k \subset \sigma_p(T)$ for all $k \in \mathbb{N}$. Now suppose that $\lambda \in \sigma_p(T)$. Then there exists $\sum_k x_k \in \mathcal{D}(T) \setminus \{0\}$ such that

$$0 = (T - \lambda) \sum_{k\in\mathbb{N}} x_k = \sum_{k\in\mathbb{N}} (T-\lambda)x_k\,,$$

i.e. $(T|_{V_k} - \lambda)x_k = 0$ for all k. Since $x_{k_0} \ne 0$ for some k_0, we find $\lambda \in \sigma_{k_0}$.

(iv): To see that $\sigma_r(T) = \varnothing$, note that for $z \notin \sigma_p(T)$ the injective operator $T - z$ maps each finite-dimensional T-invariant subspace V_k onto itself. This implies $\sum_k V_k \subset \mathcal{R}(T - z)$; the range is thus dense.

(v): Now we want to derive the formula for the resolvent set. For one inclusion, consider $z \in \mathbb{C} \setminus \bigcup_k \sigma_k$ such that $L = \sup_k \|(T|_{V_k} - z)^{-1}\| < \infty$. Using steps (i) and (ii), we see that

$$S : \sum_{k\in\mathbb{N}} x_k \mapsto \sum_{k\in\mathbb{N}} (T|_{V_k} - z)^{-1} x_k$$

defines a bounded operator $S : V \to V$, which has the finitely determining decomposition $\bigoplus_k^2 V_k$ and $\mathcal{R}(S) \subset \mathcal{D}(T - z)$. Obviously, we have $(T - z)Sx = x$ for all $x \in V$. Since $z \notin \sigma_p(T)$, i.e., $T - z$ is injective, we obtain $z \in \varrho(T)$ with $(T - z)^{-1} = S$. For the other inclusion, if $z \in \varrho(T)$ then clearly $z \notin \sigma_k$ for all k. Since $T|_{V_k} \subset T$, we also have $(T|_{V_k} - z)^{-1} \subset (T - z)^{-1}$ and thus

$$\|(T|_{V_k} - z)^{-1}\| \le \|(T - z)^{-1}\| \quad \text{for all} \quad k.$$

□

Corollary 2.3.4 *If T is closed with a finitely determining decomposition $\bigoplus_{k\in\mathbb{N}}^2 V_k$, then the point spectrum of T is non-empty and at most countably infinite. For $x = \sum_k x_k \in \bigoplus_k^2 V_k$ we have*

$$x \in \mathcal{L}(\lambda) \quad \Leftrightarrow \quad x_k \in \mathcal{L}(\lambda) \text{ for all } k \in \mathbb{N}. \tag{2.17}$$

Moreover, $\bigoplus_{k\in\mathbb{N}}^2 V_k$ is finitely determining for $(T-z)^{-1}$, $z \in \varrho(T)$, and

$$(T-z)^{-1}x = \sum_{k\in\mathbb{N}} (T|_{V_k} - z)^{-1} x_k \quad \text{for} \quad x = \sum_{k\in\mathbb{N}} x_k \in \bigoplus_{k\in\mathbb{N}}^2 V_k\,. \tag{2.18}$$

□

Example 2.3.5 Let $H = \bigoplus_{k\in\mathbb{N}} V_k$ be an orthogonal decomposition of a Hilbert space into finite-dimensional subspaces V_k and $T_k : V_k \to V_k$ linear. We can define an operator $T(H \to H)$ by

$$\mathcal{D}(T) = \sum_{k\in\mathbb{N}}^{\dotplus} V_k\,, \qquad T|_{V_k} = T_k.$$

Lemma 2.3.2 implies that T is closable and that $\bigoplus_k V_k$ is a finitely determining l^2-decomposition for $\overline{T}$. Proposition 2.3.3 then yields $\sigma_p(\overline{T}) = \bigcup_k \sigma(T_k)$.

In particular, for any given non-empty subset $\sigma \subset \mathbb{C}$ which is at most countable, we may choose the operators T_k such that $\sigma_p(\overline{T}) = \sigma$. ⌟

Proposition 2.3.6 *Let $H = \bigoplus_{k\in\mathbb{N}}^2 V_k$ be a finitely determining l^2-decomposition for a closed operator $T(H \to H)$.*

(i) If $\dim V_k = 1$ for almost all k, then

$$\varrho(T) = \Big\{ z \in \mathbb{C} \,\Big|\, \operatorname{dist}\Big(z, \bigcup_{k\in\mathbb{N}} \sigma(T|_{V_k})\Big) > 0 \Big\}, \quad \textit{i.e.} \quad \sigma(T) = \overline{\bigcup_{k\in\mathbb{N}} \sigma(T|_{V_k})}\,.$$

(ii) $(T-z)^{-1}$ compact $\Leftrightarrow$ $\lim_{k\to\infty} \|(T|_{V_k} - z)^{-1}\| = 0$.

Proof. (i): Let $J \subset \mathbb{N}$ be the subset of those k for which $\dim V_k = 1$ and let λ_k be the corresponding eigenvalues. Then

$$\|(T|_{V_k} - z)^{-1}\| = |\lambda_k - z|^{-1} \quad \text{for} \quad k \in J.$$

With $\sigma_k = \sigma(T|_{V_k})$ and since $\{\sigma_k \,|\, k \in \mathbb{N} \setminus J\}$ is a finite collection of finite sets, we have

$$\overline{\bigcup_{k\in\mathbb{N}} \sigma_k} = \overline{\{\lambda_k \,|\, k \in J\}} \cup \bigcup_{k\in\mathbb{N}\setminus J} \sigma_k\,.$$

For $z \in \mathbb{C} \setminus \bigcup_k \sigma_k$ we thus obtain

$$\begin{aligned} z \notin \overline{\bigcup_k \sigma_k} \quad &\Leftrightarrow \quad z \notin \overline{\{\lambda_k \,|\, k \in J\}} \\ &\Leftrightarrow \quad \inf_{k \in J} |\lambda_k - z| > 0 \quad \Leftrightarrow \quad \sup_{k \in \mathbb{N}} \|(T|_{V_k} - z)^{-1}\| < \infty\,; \end{aligned}$$

for the last equivalence, we used again that $\mathbb{N} \setminus J$ is finite. Applying the characterisation (2.16) of the resolvent set, the proof is complete.

(ii): Suppose first that $\|(T|_{V_k} - z)^{-1}\| \to 0$ as $k \to \infty$. Then the sequence of finite-rank operators $\sum_{k=0}^{n} (T|_{V_k} - z)^{-1}$, $n \in \mathbb{N}$, converges uniformly to the resolvent $(T - z)^{-1}$ since

$$\Big\| \sum_{k>n} (T|_{V_k} - z)^{-1} \Big\| \le c \sup_{k>n} \|(T|_{V_k} - z)^{-1}\| \to 0 \quad \text{as} \quad n \to \infty$$

by Proposition 2.3.3. The resolvent is thus compact. If on the other hand we have $\|(T|_{V_k} - z)^{-1}\| \not\to 0$, there is a monotonically increasing sequence of indices k_l and elements $x_l \in V_{k_l}$ with $\|x_l\| = 1$ such that $y_l = (T - z)^{-1} x_l$ satisfies $\inf_l \|y_l\| > 0$. Let P_k be the projections corresponding to the l^2-decomposition. From $y_l \in V_{k_l}$ it follows that $\lim_{l \to \infty} P_k y_l = 0$. Consequently every converging subsequence of $(y_l)_{l \in \mathbb{N}}$ must converge to zero. But this is impossible, so $(y_l)_{l \in \mathbb{N}}$ has no converging subsequence. Therefore $(T - z)^{-1}$ is not compact. □

Now we show that the "core property" from Definition 2.3.1 is automatically satisfied if T has a point of regular type.

Definition 2.3.7 For an operator T on a Banach space we say that $z \in \mathbb{C}$ is a *point of regular type* of T if there is a constant $C > 0$ such that

$$\|(T - z)x\| \ge C\|x\| \quad \text{for all} \quad x \in \mathcal{D}(T).$$

The set of all points of regular type of T will be denoted by $r(T)$. ⌟

Evidently $z \in r(T)$ if and only if $T - z$ is injective with bounded inverse $(T - z)^{-1}$. The set $r(T)$ is open and satisfies $\varrho(T) \subset r(T)$ and $\sigma_p(T) \cap r(T) = \varnothing$, see Akhiezer and Glazman [3, §78].

Proposition 2.3.8 *Let $T(H \to H)$ be a closed operator satisfying $r(T) \neq \varnothing$ and $H = \bigoplus_{k \in \mathbb{N}}^2 V_k$ an l^2-decomposition into finite-dimensional T-invariant subspaces such that $V_k \subset \mathcal{D}(T)$. Then $\bigoplus_{k \in \mathbb{N}}^2 V_k$ is finitely determining for T.*

Proof. By Lemma 2.3.2, the restriction $T_0 = T|_{\sum_k V_k}$ is closable, and $\bigoplus_k^2 V_k$ is finitely determining for $\overline{T_0}$. Let $z \in r(T)$. As $\overline{T_0} \subset T$ we have $z \notin \sigma_p(\overline{T_0})$ and

$$\|(T|_{V_k} - z)^{-1}\| \le \|(T - z)^{-1}\| \quad \text{for all} \quad k \in \mathbb{N}.$$

Hence $z \in \varrho(\overline{T_0})$ by (2.16). Now if $\overline{T_0} \subsetneqq T$ then the surjectivity of $\overline{T_0} - z$ would imply that $T - z$ could not be injective, which is a contradiction; thus $\overline{T_0} = T$. □

As a consequence of the previous proposition, $T = \overline{T|_{\sum_k V_k}}$ is the only possible extension of $T|_{\sum_k V_k}$ with $\varrho(T) \neq \varnothing$. Also note that in the proof we have shown that $r(T) = \varrho(T)$. This property actually holds for a larger class of operators:

Definition 2.3.9 We say that an operator T on a Banach space V has a *dense system of root subspaces* if

$$\sum_{\lambda \in \sigma_p(T)} \mathcal{L}(\lambda) \subset V \quad \text{is dense.}$$

⌟

Obviously, the density of the system of root subspaces is equivalent to the completeness of the family of root vectors. Also observe that an operator with a finitely determining l^2-decomposition has a dense system of root subspaces.

Lemma 2.3.10 *If $T(V \to V)$ is closed with a dense system of root subspaces, then $r(T) = \varrho(T)$.*

Proof. Let $z \in r(T)$, i.e., the operator $(T - z)^{-1} : \mathcal{R}(T - z) \to \mathcal{D}(T)$ exists and is bounded. It is also closed since T is closed. Consequently $\mathcal{R}(T - z)$ is closed. Now let $\lambda \in \sigma_p(T)$ and consider the T-invariant subspace U generated by a Jordan chain in $\mathcal{L}(\lambda)$. Then U is finite-dimensional and the injective operator $T - z$ maps U onto itself; in particular $U \subset \mathcal{R}(T - z)$. Therefore $\mathcal{R}(T - z) \subset V$ is dense, which implies $\mathcal{R}(T - z) = V$ and $z \in \varrho(T)$. □

Another class of operators related to finitely determining l^2-decompositions are operators having a Riesz basis with parentheses of root vectors.

Proposition 2.3.11 *Let $T(H \to H)$ be an operator with $\varrho(T) \neq \varnothing$. Then T has a finitely determining l^2-decomposition if and only if T has a Riesz basis with parentheses of Jordan chains such that each Jordan chain is entirely contained in some parenthesis.*

Proof. If $H = \bigoplus^2_{k \in \mathbb{N}} V_k$ is finitely determining for T, the choice of a basis of Jordan chains in every subspace V_k yields the desired Riesz basis with parentheses by Proposition 2.2.12. On the other hand, suppose that T has a Riesz basis with parentheses of Jordan chains where each Jordan chain lies inside some parenthesis. Then the subspaces generated by the parentheses are T-invariant and form an l^2-decomposition which is finitely determining for T by Proposition 2.3.8. □

Riesz bases with parentheses of Jordan chains are frequently constructed in the literature, see e.g. Markus [36, §6] or Tretter [47]; the condition that each chain lies inside some parenthesis is typically satisfied due to the methods used for constructing the basis. However, not every Riesz basis with parentheses of root vectors needs to satisfy this additional condition:

Example 2.3.12 Consider the shift operator $S : l^2 \to l^2$, $Se_0 = 0$, $Se_{k+1} = e_k$, where $(e_k)_{k\in\mathbb{N}}$ is the standard orthonormal basis of l^2. Then we have $0 \in \sigma_p(S)$ and $e_k \in \mathcal{L}(0)$ for every k. Hence $(e_k)_{k\in\mathbb{N}}$ is an orthonormal basis of root vectors, but it is not possible to place parentheses such that the corresponding subspaces become S-invariant. ⌟

A natural subclass of finitely determining l^2-decompositions are finitely spectral l^2-decompositions:

Definition 2.3.13 If $\bigoplus^2_{k\in\mathbb{N}} V_k$ is a finitely determining l^2-decomposition for a closed operator T with the additional property that the sets $\sigma(T|_{V_k})$ are pairwise disjoint, then we say that $\bigoplus^2_{k\in\mathbb{N}} V_k$ is *finitely spectral*. ⌟

As for the case of finitely determining decompositions, finitely spectral l^2-decompositions are not uniquely determined.

Lemma 2.3.14 *Let $T(H \to H)$ be a closed operator. A finitely determining decomposition $H = \bigoplus^2_{k\in\mathbb{N}} V_k$ for T is finitely spectral if and only if*

$$V_k = \sum_{\lambda\in\sigma(T|_{V_k})} \mathcal{L}(\lambda) \quad \textit{for all} \quad k \in \mathbb{N}. \tag{2.19}$$

In this case $\sigma_p(T)$ is countably infinite (provided $\dim H = \infty$) and all root subspaces $\mathcal{L}(\lambda)$ are finite-dimensional.

Proof. Let the l^2-decomposition $\bigoplus^2_k V_k$ be spectral for T. Let $\lambda \in \sigma(T|_{V_k})$ and $x \in \mathcal{L}(\lambda)$ with $x = \sum_j x_j$, $x_j \in V_j$. Then $x_j \in \mathcal{L}(\lambda)$ for all j by (2.17). Since the decomposition is spectral, we have $\lambda \notin \sigma(T|_{V_j})$ for $j \neq k$ and hence $x_j = 0$ for $j \neq k$. This implies $x = x_k$, i.e. $\mathcal{L}(\lambda) \subset V_k$. As V_k is the sum of all the root subspaces of $T|_{V_k}$, (2.19) holds. On the other hand, if (2.19) holds, then each $\mathcal{L}(\lambda)$ is completely contained in some V_k. Hence the $\sigma(T|_{V_k})$ are pairwise disjoint and the decomposition is spectral. The other assertions are immediate. □

Lemma 2.3.15 *Consider an operator $T(H \to H)$ with $\varrho(T) \neq \varnothing$.*

(i) If T has a Riesz basis of Jordan chains, then there exists a finitely determining l^2-decomposition for T. If in addition $\dim \mathcal{L}(\lambda) < \infty$ for all $\lambda \in \sigma_p(T)$, then the root subspaces $\mathcal{L}(\lambda)$ form a finitely spectral l^2-decomposition for T.

(ii) T admits a finitely spectral l^2-decomposition $H = \bigoplus^2_{k\in\mathbb{N}} V_k$ that satisfies $\dim V_k = 1$ for almost all k if and only if almost all eigenvalues of T are

simple, $\dim \mathcal{L}(\lambda) < \infty$ *for all* $\lambda \in \sigma_p(T)$*, and* T *has a Riesz basis of eigenvectors and at most finitely many Jordan chains. The subspaces* V_k *can be chosen as the root subspaces of* T*.*

Proof. (i): If T has a Riesz basis of Jordan chains, the subspaces V_k generated by each Jordan chain form an l^2-decomposition of H, see Proposition 2.2.10 and (2.9); it is finitely determining by Proposition 2.3.8. Now suppose that $\dim \mathcal{L}(\lambda) < \infty$ for all λ. Since every $T|_{V_k}$ has only one eigenvalue λ_k, (2.17) implies

$$\mathcal{L}(\lambda) = \sum_{\lambda_k = \lambda} V_k \quad \text{for all} \quad \lambda \in \sigma_p(T),$$

where the sum is finite. Using again (2.9), we see that the root subspaces form a finitely spectral l^2-decomposition.

(ii): If $\bigoplus_k^2 V_k$ is finitely spectral for T, Lemma 2.3.14 yields $\dim \mathcal{L}(\lambda) < \infty$ and that each V_k is the sum of root subspaces. Then $\dim V_k = 1$ for almost all k implies that almost all root subspaces are one-dimensional, i.e., the corresponding eigenvalues are simple. To construct the Riesz basis, we choose a normalised eigenvector in every V_k with dimension one and a basis of Jordan chains in those finitely many V_k with dimension bigger than one. Due to Lemma 2.1.10 and Proposition 2.2.10 this procedure yields a Riesz basis.

For the other implication, the system of root subspaces forms a finitely spectral l^2-decomposition by (i), and since almost all eigenvalues λ are simple, the corresponding $\mathcal{L}(\lambda)$ are one-dimensional. □

The classes of *spectral operators* (see Dunford and Schwartz [20]) and operators with finitely determining or spectral l^2-decomposition are in general not comparable. On the one hand, spectral operators (which include selfadjoint operators) may have empty point spectrum which is not possible for operators with a finitely determining l^2-decomposition. On the other hand, there are operators with a finitely spectral l^2-decomposition whose spectrum is separated into two parts but corresponding spectral subspaces do not exist (cf. Section 2.4 and Example 5.1.1); spectral operators always have corresponding spectral subspaces.

For the case of operators with compact resolvent, the situation is different:

Proposition 2.3.16 *Let* T *be an operator with compact resolvent and* P_k*,* $k \in \mathbb{N}$*, the Riesz projections associated with its eigenvalues. Then* T *is spectral if and only if*

(i) there exists $C > 0$ *such that* $\|\sum_{k\in F} P_k\| \le C$ *for every finite* $F \subset \mathbb{N}$ *and*

(ii) $P_k x = 0$ *for all* $k \in \mathbb{N}$ *implies* $x = 0$*.*

Proof. This is an immediate consequence of the definition of a spectral operator in [20, Definition XVIII.2.1]. □

With Theorem 2.2.9 we conclude that operators with compact resolvent are spectral if and only if their root subspaces form an l^2-decomposition.[3]

A closed operator T is called *Riesz-spectral* (see Curtain and Zwart [14] and Kuiper and Zwart [29]) if all its eigenvalues are simple, T has a Riesz basis of eigenvectors, and $\overline{\sigma_p(T)}$ is totally disconnected[4]. In [29, Corollary 4.6] it is shown that the Riesz-spectral operators with compact resolvent are exactly the spectral operators with compact resolvent and simple eigenvalues.

The various classes of operators considered so far can be put into a hierarchy as follows:

Theorem 2.3.17 *Let $T(H \to H)$ be an operator with $\varrho(T) \neq \varnothing$ and $\dim \mathcal{L}(\lambda) < \infty$ for all $\lambda \in \sigma_p(T)$. For the properties*

(i) T has a dense system of root subspaces,

(ii) T has a Riesz basis with parentheses of root vectors,

(iii) T has a finitely determining l^2-decomposition,
($\Leftrightarrow$ T has a Riesz basis with parentheses of Jordan chains such that each Jordan chain lies inside some parenthesis)

(iv) T has a finitely spectral l^2-decomposition,
($\Leftrightarrow$ T has an l^2-decomposition of finite sums of root subspaces)

(v) T has an l^2-decomposition of root subspaces,
(If T has a compact resolvent, this is equivalent to T being a spectral operator.)

(vi) T has a Riesz basis of Jordan chains,

(vii) T has a Riesz basis of eigenvectors and finitely many Jordan chains, and almost all eigenvalues are simple,
($\Leftrightarrow$ T has a finitely spectral l^2-decomposition with almost all subspaces one-dimensional)

(viii) T is a Riesz-spectral operator,

we have the implications

$$(viii) \Rightarrow (vii) \Rightarrow (vi) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i).$$

If we drop the assumption $\dim \mathcal{L}(\lambda) < \infty$, we still have the implications

$$(vi) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i).$$

□

[3]Spectral operators with compact resolvent are also called “discrete spectral”.

[4]A set $S \subset \mathbb{C}$ is totally disconnected if no two points from S can be joined by a path lying in S.

In this thesis, the properties (iii), (iv), (vi), and (occasionally) (i) will be used as assumptions in theorems. The perturbation results from Sections 3.4 and 4.4 yield operators of type (iv), (v), and (vii).

With the help of Example 2.3.5 it is not hard to see that the implications $(viii) \Rightarrow \cdots \Rightarrow (iii)$ in Theorem 2.3.17 are strict. An example of an operator with compact resolvent and a finitely spectral l^2-decomposition that is not a spectral operator is the Hamiltonian operator in Example 5.1.1.

We end this section with the example of an operator with a finitely spectral l^2-decomposition whose spectrum is not the closure of its point spectrum, compare (2.16) and Proposition 2.3.6(i).

Example 2.3.18 Consider an orthogonal decomposition $H = \bigoplus_{k\geq 1} V_k$ such that $\dim V_k = 2$ and an operator $T_0(H \to H)$ with $\mathcal{D}(T_0) = \sum_k V_k$ such that all V_k are invariant and the restrictions $T_0|_{V_k}$ have eigenvalues k and $k+i$. By Lemma 2.3.2 and Proposition 2.3.3, T_0 is closable and $\sigma_p(\overline{T_0}) = \bigcup_k \{k, k+i\}$. Moreover, if there are unit length eigenvectors $v_k, w_k \in V_k$ corresponding to k and $k+i$, respectively, which satisfy

$$(v_k|w_k) = 1 - k^{-q}$$

with $q > 6$, then $\sigma(\overline{T_0}) = \mathbb{C}$.

Proof. Let $z \in \mathbb{C} \setminus \bigcup_k \{k, k+i\}$. Consider some $k \geq 1$ and let

$$\lambda_1 = k - z, \quad \lambda_2 = k + i - z, \quad \omega = (v_k|w_k).$$

Then we get

$$\|v_k - w_k\|^2 = \|v_k\|^2 - 2(v_k|w_k) + \|w_k\|^2 = 2(1-\omega),$$
$$\big(T_0|_{V_k} - z\big)^{-1}(v_k - w_k) = \lambda_1^{-1} v_k - \lambda_2^{-1} w_k,$$

and, using $0 \leq \omega \leq 1$,

$$\begin{aligned}\big\|\big(T_0|_{V_k} - z\big)^{-1}\big\|^2 &\geq \frac{\|\lambda_1^{-1} v_k - \lambda_2^{-1} w_k\|^2}{\|v_k - w_k\|^2} = \frac{|\lambda_1^{-1}|^2 - 2\operatorname{Re}\big(\lambda_1^{-1}\overline{\lambda_2^{-1}}\big)\omega + |\lambda_2^{-1}|^2}{2(1-\omega)} \\ &\geq \frac{|\lambda_1^{-1}|^2 - 2|\lambda_1^{-1}| \cdot |\lambda_2^{-1}| + |\lambda_2^{-1}|^2}{2(1-\omega)} = \frac{(|\lambda_1^{-1}| - |\lambda_2^{-1}|)^2}{2(1-\omega)}.\end{aligned}$$

With $z = x + iy$, $x, y \in \mathbb{R}$, we find

$$\begin{aligned}\frac{|\lambda_1^{-1}| - |\lambda_2^{-1}|}{\sqrt{1-\omega}} &= \frac{|\lambda_2|^2 - |\lambda_1|^2}{\sqrt{1-\omega}\,|\lambda_1| \cdot |\lambda_2| \cdot (|\lambda_1| + |\lambda_2|)} \\ &= \frac{(k-x)^2 + (1-y)^2 - \big((k-x)^2 + y^2\big)}{\sqrt{k^{-q}}\,|k-z| \cdot |k+i-z|\big(|k-z| + |k+i-z|\big)} \\ &= \frac{k^{q/2}(1-2y)}{|k-z| \cdot |k+i-z|\big(|k-z| + |k+i-z|\big)}.\end{aligned}$$

Since $q/2 > 3$ and if $y \neq 1/2$, this last expression tends to $\pm\infty$ as $k \to \infty$ and we can conclude that $\sup_{k\geq 1} \|(T_0|_{V_k} - z)^{-1}\| = \infty$ in this case. Using the characterisation (2.16) of the resolvent set, we see that

$$\{z \in \mathbb{C} \mid \operatorname{Im} z \neq 1/2\} \subset \sigma(\overline{T_0}) .$$

Since the spectrum is a closed set, this implies $\sigma(\overline{T_0}) = \mathbb{C}$. □

2.4 Compatible subspaces of determining l^2-decompositions

In this section we show that for every operator with a finitely determining l^2-decomposition there exists a large class of invariant subspaces, so-called compatible subspaces. In particular we obtain compatible subspaces associated with arbitrary subsets of the point spectrum. We argue that these associated subspaces are a natural generalisation of spectral subspaces for operators with a finitely determining l^2-decomposition.

Existence results for invariant and spectral subspaces of unbounded non-normal operators are known in special cases only: For a bounded isolated component of the spectrum the corresponding Riesz projection yields a spectral subspace. Dichotomous operators as defined by Langer, Ran and van de Rotten [31], see also Langer and Tretter [33] and Definition 2.4.8, have spectral subspaces associated with the spectrum in the right and left half-plane.

Lemma 2.4.1 *Let $H = \bigoplus^2_{k\in\mathbb{N}} V_k$ be a finitely determining l^2-decomposition for a closed operator $T(H \to H)$. If $U_k \subset V_k$ are T-invariant subspaces, then the subspace*

$$\bigoplus_{k\in\mathbb{N}}^2 U_k \quad \textit{is } T\textit{-invariant and } (T-\lambda)^{-1}\textit{-invariant for all } \lambda \in \varrho(T).$$

In particular, $\bigoplus^2_{k\in J} V_k$ is T- and $(T-\lambda)^{-1}$-invariant for every $J \subset \mathbb{N}$.

Proof. This is evident from the formulas (2.15) and (2.18) for T and $(T-\lambda)^{-1}$. □

The statement of the lemma suggests the next definition.

Definition 2.4.2 We say that a T-invariant subspace $U \subset H$ is *compatible* with the finitely determining decomposition $H = \bigoplus^2_{k\in\mathbb{N}} V_k$ if

$$U = \bigoplus_{k\in\mathbb{N}}^2 U_k \quad \text{with} \quad U_k \subset V_k \quad T\text{-invariant}.$$

┘

Let $\sigma \subset \sigma_p(T)$ be an arbitrary subset of the point spectrum of an operator $T(H \to H)$. A subspace naturally associated with σ is the closure of the sum of the root subspaces corresponding to σ,

$$U = \overline{\sum_{\lambda\in\sigma} \mathcal{L}(\lambda)}.$$

If T is bounded, it is immediate that U is T-invariant; for unbounded T this need not be the case. However, U is $(T-\lambda)^{-1}$-invariant for every $\lambda \in \varrho(T)$ and $\sum_{\lambda\in\sigma} \mathcal{L}(\lambda)$ is T-invariant.

Now let us assume that $H = \bigoplus_{k\in\mathbb{N}}^2 V_k$ is a finitely determining l^2-decomposition for T. Since V_k is finite-dimensional, $\sigma(T|_{V_k})$ is a finite set consisting of the eigenvalues of $T|_{V_k}$, and we can decompose V_k into the invariant subspaces U_k and W_k corresponding to the eigenvalues in σ and $\sigma_p(T) \setminus \sigma$, respectively:

$$V_k = U_k \oplus W_k, \quad \sigma(T|_{U_k}) = \sigma(T|_{V_k}) \cap \sigma, \quad \sigma(T|_{W_k}) = \sigma(T|_{V_k}) \setminus \sigma\,. \tag{2.20}$$

We can then show that U is compatible with $\bigoplus_{k\in\mathbb{N}}^2 V_k$:

Proposition 2.4.3 *Let $T(H \to H)$ be an operator with a finitely determining l^2-decomposition $H = \bigoplus_{k\in\mathbb{N}}^2 V_k$ and $\sigma \subset \sigma_p(T)$ a subset of its point spectrum. Let U_k, W_k be the invariant subspaces of V_k corresponding to σ and $\tau = \sigma_p(T) \setminus \sigma$, as defined in (2.20). Then the subspaces*

$$U = \overline{\sum_{\lambda\in\sigma} \mathcal{L}(\lambda)} \quad \textit{and} \quad W = \overline{\sum_{\lambda\in\tau} \mathcal{L}(\lambda)}$$

are T-invariant compatible with $\bigoplus_k^2 V_k$,

$$U = \bigoplus_{k\in\mathbb{N}}^2 U_k, \qquad W = \bigoplus_{k\in\mathbb{N}}^2 W_k, \tag{2.21}$$

and we have $\sigma_p(T|_U) = \sigma$, $\sigma_p(T|_W) = \tau$. Moreover,

(i) $U \dotplus W \subset H$ is algebraic direct and dense and

(ii) $(\mathcal{D}(T) \cap U) \dotplus (\mathcal{D}(T) \cap W) \subset \mathcal{D}(T)$ is a core for T.

Proof. First we derive (2.21). Let $x \in \mathcal{L}(\lambda)$ with $\lambda \in \sigma$. Applying (2.17) to the decomposition $x = \sum_k x_k$, $x_k \in V_k$, we obtain $x_k \in U_k$ for all k. Therefore $\mathcal{L}(\lambda) \subset \bigoplus_k^2 U_k$. Together with the inclusion $U_k \subset \sum_{\lambda\in\sigma} \mathcal{L}(\lambda)$ this yields (2.21). Hence U is a compatible T-invariant subspace and $\sigma_p(T|_U) = \sigma$. The sum $U + W$ is algebraic direct and dense by (2.8), and $\sum_k V_k$ is a core for T which is contained in $(\mathcal{D}(T) \cap U) \dotplus (\mathcal{D}(T) \cap W)$. □

The above invariance result justifies the following definition:

Definition 2.4.4 Let $T(H \to H)$ be an operator with a finitely determining l^2-decomposition. For a subset $\sigma \subset \sigma_p(T)$ of the point spectrum we call

$$U = \overline{\sum_{\lambda \in \sigma} \mathcal{L}(\lambda)} \tag{2.22}$$

the *compatible subspace associated with* σ. ⌟

If the l^2-decomposition of the operator is finitely spectral, the subspace U defined by (2.22) has the following uniqueness property:

Proposition 2.4.5 *Suppose that T has a compact resolvent and a finitely spectral l^2-decomposition $\bigoplus_{k\in\mathbb{N}}^2 V_k$. Then the compatible subspace U associated with a subset $\sigma \subset \sigma_p(T)$ is the unique maximal closed T-invariant subspace with $\sigma(T|_U) = \sigma$ that is also $(T-\lambda)^{-1}$-invariant for all $\lambda \in \varrho(T)$.*

Proof. Suppose that U is closed, T- and $(T-\lambda)^{-1}$-invariant, and $\sigma(T|_U) = \sigma$. Note that the projections P_k onto V_k corresponding to the decomposition are the Riesz projections of T associated with the respective part of the spectrum. The invariance of U then implies $P_k(U) \subset U$ and hence $U = \bigoplus_k^2 (U \cap V_k)$. Moreover with U_k from (2.20) we have $U \cap V_k \subset U_k$ and the claim follows by (2.21). □

For unbounded operators, the notion of a *spectral subspace* is typically used only for certain classes of operators. Often it comes in conjunction with a corresponding class of projections whose images are the spectral subspaces. For example, if the spectrum of an operator has a bounded isolated component, then the range and kernel of the associated Riesz projection are spectral subspaces. For normal operators, spectral subspaces appear as images of the spectral projections.

The notion of an *exponentially dichotomous* operator $T(V \to V)$ was introduced by Bart, Gohberg and Kaashoek [7], see also Krein and Savčenko [28]. Such an operator admits a decomposition $V = U_+ \oplus U_-$ into T-invariant subspaces such that $-T|_{U_+}$ and $T|_{U_-}$ are generators of C_0-semigroups of negative exponential type. As a consequence, a strip around the imaginary axis belongs to $\varrho(T)$, and $\sigma(T|_{U_+})$ and $\sigma(T|_{U_-})$ lie in the right and left half-plane, respectively. Here U_+ and U_- are the spectral subspaces.

The properties shared by the above examples may be used to give a general definition of a spectral subspace:

Definition 2.4.6 Consider an operator $T(V \to V)$ on a Banach space, a partition $\mathbb{C} = \Sigma_1 \mathbin{\dot\cup} \Sigma_2$, and a topological direct sum $V = U_1 \oplus U_2$ such that

$$\mathcal{D}(T) = (\mathcal{D}(T) \cap U_1) \oplus (\mathcal{D}(T) \cap U_2) \tag{2.23}$$

and U_1, U_2 are T-invariant. If

$$\sigma_p(T|_{U_j}) \subset \Sigma_j \quad \text{and} \quad \sigma(T|_{U_j}) \subset \overline{\Sigma_j} \quad \text{for} \quad j = 1, 2,$$

then $V = U_1 \oplus U_2$ is called a *spectral decomposition* corresponding to the partition and U_j is the *spectral subspace* associated with Σ_j. ⌟

It is easy to see that (2.23) implies that the subspaces U_1, U_2 are also $(T-\lambda)^{-1}$-invariant for every $\lambda \in \varrho(T)$ and

$$\sigma(T) = \sigma(T|_{U_1}) \cup \sigma(T|_{U_2}), \quad \sigma_p(T) = \sigma_p(T|_{U_1}) \cup \sigma_p(T|_{U_2}). \tag{2.24}$$

In particular $\sigma_p(T|_{U_j}) = \sigma_p(T) \cap \Sigma_j$ for a spectral decomposition.

The next proposition shows that, for operators with a finitely determining l^2-decomposition, compatible subspaces associated with subsets of the point spectrum are a natural generalisation of spectral subspaces.

Proposition 2.4.7 *Let $T(H \to H)$ be an operator on a Hilbert space and consider a partition $\mathbb{C} = \Sigma_1 \mathbin{\dot\cup} \Sigma_2$ of the complex plane.*

(i) If $H = U_1 \oplus U_2$ is a spectral decomposition for T corresponding to Σ_1, Σ_2 and T has a dense system of root subspaces, then

$$U_j = \overline{\sum_{\lambda \in \sigma_p(T|_{U_j})} \mathcal{L}(\lambda)} \quad \textit{for} \quad j = 1, 2.$$

(ii) Let T have a compact resolvent and a finitely determining l^2-decomposition $H = \bigoplus^2_{k\in\mathbb{N}} V_k$ such that for all k

$$\textit{either} \quad \sigma(T|_{V_k}) \subset \Sigma_1 \quad \textit{or} \quad \sigma(T|_{V_k}) \subset \Sigma_2.$$

Then the compatible subspaces U and W associated with $\sigma = \sigma_p(T) \cap \Sigma_1$ and $\tau = \sigma_p(T) \cap \Sigma_2$, respectively, have the form

$$U = \bigoplus_{k\in J}^{2} V_k \quad \textit{and} \quad W = \bigoplus_{k\in\mathbb{N}\setminus J}^{2} V_k \quad \textit{with} \quad J = \big\{k \in \mathbb{N} \,\big|\, \sigma(T|_{V_k}) \subset \Sigma_1\big\}$$

and constitute a spectral decomposition for T corresponding to Σ_1, Σ_2.

Proof. (i): It is easy to show that for $\lambda \in \sigma_p(T)$ either $\mathcal{L}(\lambda) \subset U_1$ or $\mathcal{L}(\lambda) \subset U_2$. Hence $\sum_{\sigma_p(T|_{U_j})} \mathcal{L}(\lambda) \subset U_j$. That these inclusions are also dense follows from the density of the system of root subspaces.

(ii): With the notation from Proposition 2.4.3, either $U_k = V_k$ or $U_k = \{0\}$ holds. Hence U and W have the stated form and their sum is topological direct by (2.7). From (2.14) we obtain the formula for $\mathcal{D}(T)$ in (2.23). Finally we have $\sigma(T) = \sigma_p(T)$ since T has a compact resolvent, and the proof is complete. □

Langer, Ran and van de Rotten [31] generalised the concept of exponential dichotomy as follows:

Definition 2.4.8 A closed, densely defined operator T is called *dichotomous* if a strip around the imaginary axis belongs to $\varrho(T)$ and there exists a spectral decomposition corresponding to the parts of the spectrum in the left and right half-plane. ⌟

Corollary 2.4.9 *Let T be an operator with compact resolvent and a finitely determining l^2-decomposition $\bigoplus_{k\in\mathbb{N}}^2 V_k$. If a strip around the imaginary axis belongs to $\varrho(T)$ and every $\sigma(T|_{V_k})$ is contained either in the right or left half-plane, then T is dichotomous.* □

Note that for an operator with a finitely determining l^2-decomposition the compatible subspaces associated with the point spectrum in the right and left half-plane, respectively, even exist in cases where the operator is not dichotomous; see Example 5.1.1.

2.5 J-symmetric operators and neutral invariant subspaces

We apply the theory of finitely determining l^2-decompositions to symmetric operators in Krein spaces. For a J-symmetric operator with a dense system of root subspaces we obtain the symmetry of its point spectrum with respect to the real axis and a J-orthogonal decomposition in terms of root subspaces, see Theorem 2.5.12. In Theorem 2.5.16 we show that if the operator has a finitely spectral l^2-decomposition and no eigenvalues on the real axis, then the compatible subspaces associated with a partition of the point spectrum which separates conjugate pairs are hypermaximal neutral, i.e., they coincide with their J-orthogonal complements.

Orthogonality relations for the root subspaces of a J-symmetric operator are well known [5, 16]. For a J-selfadjoint operator with compact resolvent, the symmetry of the point spectrum immediately follows from the symmetry of the spectrum. Langer, Ran and van de Rotten [31] considered a dichotomous operator T such that iT is J-selfadjoint and showed that the spectral subspaces associated with the right and left half-plane are hypermaximal neutral.

For an introduction to Krein spaces and operators therein we refer to the monographs of Azizov and Iokhvidov [5], Bognar [9], and Dijksma and Langer [17]. One possible way to define a Krein space is as follows:

Definition 2.5.1 A complex vector space V together with a Hermitian sesquilinear form $\langle\cdot|\cdot\rangle$ is called a *Krein space* if there exists an involution $J : V \to V$ such that

$$(x|y) = \langle Jx|y\rangle \quad \text{for} \quad x, y \in V \tag{2.25}$$

defines a scalar product and $(V, (\cdot|\cdot))$ is a Hilbert space. ⌟

The involution J is called a *fundamental symmetry*. While it is not uniquely determined, the Hilbert space norms induced by different fundamental symmetries are equivalent. We will always consider a fixed J and denote by $\|\cdot\|$ the norm induced by the scalar product. It is easy to see that J is selfadjoint with respect to $(\cdot|\cdot)$ and

$$\langle x|y\rangle = (Jx|y) \quad \text{for all} \quad x, y \in V. \tag{2.26}$$

The inner product $\langle\cdot|\cdot\rangle$ is typically indefinite: We say that an element $x \in V$ is *positive*, *neutral*, and *negative* if $\langle x|x\rangle > 0$, $= 0$, and < 0, respectively. A subspace $U \subset V$ is called *nonnegative*, *positive*, and *uniformly positive* if $\langle x|x\rangle \geq 0$, > 0, and $\geq \alpha\|x\|^2$ for all $x \in V \setminus \{0\}$ and some constant $\alpha > 0$. The notions of a *nonpositive*, *negative*, and *uniformly negative* subspace are defined accordingly. The subspace is called *neutral* if $\langle x|x\rangle = 0$ for all $x \in U$. The closure of a neutral subspace is again neutral.

We may define orthogonality with respect to the inner product $\langle\cdot|\cdot\rangle$: Two elements $x, y \in V$ are called *orthogonal* if $\langle x|y\rangle = 0$. Two subspaces $U, W \subset V$ are *orthogonal*, denoted by $U\langle\perp\rangle W$, if $\langle x|y\rangle = 0$ for all $x \in U$, $y \in W$. The *orthogonal complement* of U is defined by

$$U^{\langle\perp\rangle} = \big\{x \in V \bigm| \langle x|y\rangle = 0 \text{ for all } y \in U\big\}. \tag{2.27}$$

A subspace U is neutral if and only if $U \subset U^{\langle\perp\rangle}$. If necessary, we will use the term *J-orthogonal* to distinguish orthogonality with respect to the Krein space inner product $\langle\cdot|\cdot\rangle$ from orthogonality with respect to the scalar product $(\cdot|\cdot)$.

Definition 2.5.2 We say that the algebraic direct sum $\sum_{\lambda\in\Lambda}^{\dotplus} U_\lambda$ is *orthogonal direct* if the subspaces U_λ are mutually orthogonal. In this case we use the notation

$$\sum_{\lambda\in\Lambda}^{\langle\dotplus\rangle} U_\lambda.$$

For an orthogonal direct sum with two components we write $U\langle\dotplus\rangle W$. ⌟

Note that the orthogonal direct sum of neutral subspaces is again neutral.

In contrast to the Hilbert space case, two orthogonal subspaces of a Krein space need not form a direct sum. As an extreme example, a neutral subspace is orthogonal to itself. And even if a sum is orthogonal direct, it is not necessarily topological direct.

A subspace $U \subset V$ is called *non-degenerate* if for every $x \in U \setminus \{0\}$ there exists $y \in U$ such that $\langle x|y\rangle \neq 0$ or, equivalently, if $U \cap U^{\langle\perp\rangle} = \{0\}$. The Krein space V itself is non-degenerate since $\langle Jx|x\rangle = \|x\|^2$ for all $x \in V$.

Lemma 2.5.3 *Consider a family of subspaces $(U_\lambda)_{\lambda\in\Lambda}$ of V forming an orthogonal direct sum*

$$\sum_{\lambda\in\Lambda}^{\langle\dot{+}\rangle} U_\lambda$$

which is dense in V. Then each U_λ is non-degenerate.

Proof. Let $x \in U_\lambda \setminus \{0\}$. Since V is non-degenerate and the direct sum is dense, we have $\langle x|y\rangle \neq 0$ for some $y = y_{\lambda_1} + \cdots + y_{\lambda_n}$, $y_{\lambda_j} \in U_{\lambda_j}$. Now $\langle x|y_{\lambda_j}\rangle = 0$ for every index $\lambda_j \neq \lambda$ by orthogonality of the sum. Therefore one of the indices $\lambda_1, \ldots, \lambda_n$ is equal to λ and $\langle x|y_\lambda\rangle \neq 0$. □

Definition 2.5.4 Two systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ of elements in a Krein space V are called *biorthogonal* if $\langle x_j|y_k\rangle = \delta_{jk}$ for all j, k. ⌟

As a consequence of the definition, if two systems $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are biorthogonal, then they are both linearly independent.

Lemma 2.5.5 *Let U, W be subspaces of a Krein space such that $U \cap W^{\langle\perp\rangle} = \{0\}$. Then for $n \leq \dim U$ there are systems $(x_1, \ldots, x_n)$ in U and $(y_1, \ldots, y_n)$ in W which are biorthogonal. In particular we have $\dim U \leq \dim W$.*

Proof. We use induction. For $n = 1$ take $x_1 \in U \setminus \{0\}$. Since $U \cap W^{\langle\perp\rangle} = \{0\}$ there exists $y_1 \in W$ with $\langle x_1|y_1\rangle = 1$. Now suppose we have $n + 1 \leq \dim U$ and biorthogonal systems $(x_1, \ldots, x_n)$ in U, $(y_1, \ldots, y_n)$ in W. We choose an element $x \in U \setminus \operatorname{span}\{x_1, \ldots, x_n\}$ and set

$$x_{n+1} = x - \sum_{j=1}^{n} \langle x|y_j\rangle x_j.$$

This yields $\langle x_{n+1}|y_k\rangle = 0$ for $k = 1, \ldots, n$. Moreover $x_{n+1} \neq 0$ by the choice of x and hence there exists a $y \in W$ with $\langle x_{n+1}|y\rangle = 1$. We set

$$y_{n+1} = y - \sum_{j=1}^{n} \langle y|x_j\rangle y_j$$

and find $\langle y_{n+1}|x_k\rangle = 0$ for $k = 1, \ldots, n$ as well as $\langle x_{n+1}|y_{n+1}\rangle = \langle x_{n+1}|y\rangle = 1$. □

Corollary 2.5.6 *Let U, W be two neutral subspaces. If their sum $U + W$ is non-degenerate, then $\dim U = \dim W$ and the sum is algebraic direct.*

Proof. Let $x \in U \setminus \{0\}$. By assumption there exist elements $x_1 \in U$, $y_1 \in W$ such that $\langle x|x_1 + y_1\rangle \neq 0$. Furthermore $\langle x|x_1 + y_1\rangle = \langle x|y_1\rangle$ by neutrality of U and hence $x \notin W^{\langle\perp\rangle}$. An application of the previous lemma yields $\dim U \leq \dim W$. Analogously we obtain $\dim W \leq \dim U$ and thus equality. Finally, an element

$x_0 \in U \cap W$ satisfies $\langle x_0 | x + y \rangle = 0$ for all $x \in U$, $y \in W$; consequently $x_0 = 0$ by the non-degeneracy of $U + W$. □

The definitions of symmetric and selfadjoint operators in Krein spaces are analogous to the Hilbert space case:

Definition 2.5.7 Let $T(V \to V)$ be a densely defined operator. Then

(i) T is *symmetric* if $\langle Tx|y \rangle = \langle x|Ty \rangle$ for all $x, y \in \mathcal{D}(T)$;

(ii) the adjoint operator $T^{\langle * \rangle}$ is defined by

$$\langle Tx|y \rangle = \langle x|T^{\langle * \rangle} y \rangle \quad \text{for all} \quad x \in \mathcal{D}(T),\, y \in \mathcal{D}(T^{\langle * \rangle}) \quad \text{where}$$
$$\mathcal{D}(T^{\langle * \rangle}) = \big\{ y \in V \,\big|\, \mathcal{D}(T) \ni x \mapsto \langle Tx|y \rangle \text{ is a bounded linear form} \big\};$$

(iii) T is *selfadjoint* if $T = T^{\langle * \rangle}$;

(iv) T is *skew-symmetric* if $\langle Tx|y \rangle = -\langle x|Ty \rangle$ for $x, y \in \mathcal{D}(T)$ and *skew-adjoint* if $T = -T^{\langle * \rangle}$.

⌟

Again we shall use the terms J-symmetric, J-selfadjoint and so forth if we need to distinguish the Krein space concepts from those in a Hilbert space.

Remark 2.5.8 It is easy to see that T is J-symmetric/-selfadjoint if and only if JT is symmetric/selfadjoint with respect to the scalar product $(\cdot|\cdot)$. As in the Hilbert space case we have that

(i) $T^{\langle * \rangle}$ is closed;

(ii) T is symmetric if and only if $T \subset T^{\langle * \rangle}$, and T is closable in this case;

(iii) T is skew-symmetric (skew-adjoint) if and only if iT is symmetric (selfadjoint);

(iv) $\ker T^{\langle * \rangle} = \mathcal{R}(T)^{\langle \perp \rangle}$;

(v) if T is symmetric and there exist $\lambda, \overline{\lambda} \in \varrho(T)$, then T is selfadjoint;

(vi) if T is bijective with bounded inverse, then the same holds for $T^{\langle * \rangle}$ and $(T^{\langle * \rangle})^{-1} = (T^{-1})^{\langle * \rangle}$.

⌟

A new phenomenon in the Krein space context is that a selfadjoint operator may have spectrum outside the real axis. The next proposition shows that the spectrum is symmetric with respect to the real axis:

Proposition 2.5.9 *Let $T(V \to V)$ be a selfadjoint operator. Then we have*

$$\lambda \in \varrho(T) \quad \Longleftrightarrow \quad \overline{\lambda} \in \varrho(T).$$

Proof. Let $\lambda \in \varrho(T)$, i.e., $T - \lambda$ is bijective with bounded inverse. Then the adjoint $(T-\lambda)^{\langle * \rangle} = T^{\langle * \rangle} - \overline{\lambda} = T - \overline{\lambda}$ is also bijective with bounded inverse. □

Another new aspect in Krein spaces is the possible existence of generalised eigenvectors for (skew-)symmetric operators, see Example 5.1.5. Yet it is possible to derive orthogonality properties similar to the situation in a Hilbert space. The corresponding result for linear relations in a Krein space was obtained by Dijksma and de Snoo [16, Proposition 3.2].

Proposition 2.5.10 *Let $T(V \to V)$ be a densely defined operator and $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \overline{\mu}$. Then*

$$\ker(T-\lambda)^k \;\langle\perp\rangle\; \ker(T^{\langle * \rangle}-\mu)^k \quad \textit{for all} \quad k \in \mathbb{N}.$$

Proof. The proof is by induction on k. The case $k = 0$ is clear. Suppose the assertion is true for some $k \in \mathbb{N}$ and let $x \in \ker(T-\lambda)^{k+1}$, $y \in \ker(T^{\langle * \rangle} - \mu)^k$. We set $x_0 = (T-\lambda)x \in \ker(T-\lambda)^k$. Then $\langle x_0 | y \rangle = 0$ which yields

$$\lambda \langle x|y \rangle = \langle Tx|y \rangle - \langle x_0|y \rangle = \langle x|T^{\langle * \rangle} y \rangle = \langle x|(T^{\langle * \rangle} - \mu)y \rangle + \overline{\mu}\langle x|y \rangle,$$

thus

$$(\lambda - \overline{\mu})\langle x|y \rangle = \langle x|(T^{\langle * \rangle} - \mu)y \rangle.$$

Since also $(T^{\langle * \rangle}-\mu)y, \dots, (T^{\langle * \rangle}-\mu)^{k-1}y \in \ker(T^{\langle * \rangle}-\mu)^k$, we can use the last formula repeatedly and find

$$\begin{aligned}(\lambda - \overline{\mu})^k \langle x|y \rangle &= (\lambda - \overline{\mu})^{k-1} \langle x|(T^{\langle * \rangle} - \mu)y \rangle \\ &= (\lambda - \overline{\mu})^{k-2} \langle x|(T^{\langle * \rangle} - \mu)^2 y \rangle = \ldots = \langle x|(T^{\langle * \rangle} - \mu)^k y \rangle = 0\,;\end{aligned}$$

therefore $\langle x|y \rangle = 0$. Now consider x as above and $y \in \ker(T^{\langle * \rangle} - \mu)^{k+1}$. With $y_0 = (T^{\langle * \rangle} - \mu)y \in \ker(T^{\langle * \rangle} - \mu)^k$ we have $\langle x|y_0 \rangle = 0$,

$$\overline{\mu}\langle x|y \rangle = \langle x|T^{\langle * \rangle} y \rangle - \langle x|y_0 \rangle = \langle Tx|y \rangle = \langle (T-\lambda)x|y \rangle + \lambda \langle x|y \rangle,$$

and therefore

$$(\overline{\mu} - \lambda)\langle x|y \rangle = \langle (T-\lambda)x|y \rangle.$$

As above, iterated use of this formula yields

$$(\overline{\mu} - \lambda)^{k+1}\langle x|y \rangle = \langle (T-\lambda)^{k+1}x|y \rangle = 0.$$

Consequently $\langle x|y \rangle = 0$ and the proof is complete. □

Corollary 2.5.11 *Let $T(V \to V)$ be symmetric and $\lambda, \mu \in \mathbb{C}$ with $\lambda \neq \overline{\mu}$. Then the root subspaces $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$ of T are orthogonal. In particular every $\mathcal{L}(\lambda)$ with $\lambda \notin \mathbb{R}$ is a neutral subspace.*

Proof. Since $T \subset T^{\langle * \rangle}$, we have $\ker(T-\mu)^k \subset \ker(T^{\langle * \rangle}-\mu)^k$. The claim thus follows from the previous proposition. In particular $\mathcal{L}(\lambda)\langle\perp\rangle\mathcal{L}(\lambda)$ for $\lambda \notin \mathbb{R}$, i.e., $\mathcal{L}(\lambda)$ is neutral. □

For symmetric operators with a dense system of root subspaces, we can now show that their point spectrum is symmetric with respect to the real axis and compute an orthogonal decomposition in terms of root subspaces.

Theorem 2.5.12 *Suppose that a symmetric operator T on a Krein space V has a dense system of root subspaces. Then the point spectrum $\sigma_p(T)$ is symmetric with respect to the real axis and we have an orthogonal decomposition*

$$\sum_{\lambda \in \sigma_p(T)} \mathcal{L}(\lambda) \;=\; \sum_{t \in \sigma_p(T) \cap \mathbb{R}}^{\langle \dotplus \rangle} \mathcal{L}(t) \;\langle \dotplus \rangle \sum_{\substack{\lambda \in \sigma_p(T) \\ \operatorname{Im}\lambda > 0}}^{\langle \dotplus \rangle} \big(\mathcal{L}(\lambda) + \mathcal{L}(\overline{\lambda})\big), \tag{2.28}$$

in which each summand $\mathcal{L}(t)$ and $\mathcal{L}(\lambda) + \mathcal{L}(\overline{\lambda})$ is non-degenerate. Moreover, the root subspaces $\mathcal{L}(\lambda)$ and $\mathcal{L}(\overline{\lambda})$ with $\operatorname{Im}\lambda > 0$ are neutral and satisfy $\dim \mathcal{L}(\lambda) = \dim \mathcal{L}(\overline{\lambda})$.

Proof. We start by defining

$$\sigma_0 = \big\{ \lambda \in \mathbb{C} \;\big|\; \operatorname{Im}\lambda > 0 \text{ and } \big(\lambda \in \sigma_p(T) \text{ or } \overline{\lambda} \in \sigma_p(T)\big) \big\};$$

so $\lambda \in \sigma_0$ need not necessarily be an eigenvalue of T, but if not then $\overline{\lambda}$ is. We may thus write the sum of all root subspaces as

$$\sum_{\lambda \in \sigma_p(T)} \mathcal{L}(\lambda) = \sum_{t \in \sigma_p(T) \cap \mathbb{R}}^{\dotplus} \mathcal{L}(t) \;\dotplus\; \sum_{\lambda \in \sigma_0}^{\dotplus} \big(\mathcal{L}(\lambda) + \mathcal{L}(\overline{\lambda})\big).$$

By the preceding corollary, two root subspaces $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$ can be non-orthogonal only in case of $\mu = \overline{\lambda}$. Therefore, we get the orthogonal direct sum

$$\sum_{\lambda \in \sigma_p(T)} \mathcal{L}(\lambda) = \sum_{t \in \sigma_p(T) \cap \mathbb{R}}^{\langle \dotplus \rangle} \mathcal{L}(t) \;\langle \dotplus \rangle \sum_{\lambda \in \sigma_0}^{\langle \dotplus \rangle} \big(\mathcal{L}(\lambda) + \mathcal{L}(\overline{\lambda})\big).$$

Since this sum is dense, Lemma 2.5.3 shows that its summands are non-degenerate. Applying Corollary 2.5.6 to the neutral subspaces $\mathcal{L}(\lambda)$ and $\mathcal{L}(\overline{\lambda})$ for $\lambda \in \sigma_0$, we can now conclude that their dimensions coincide. Consequently the point spectrum of T is symmetric with respect to the real axis and $\sigma_0 = \{\lambda \in \sigma_p(T) \mid \operatorname{Im}\lambda > 0\}$. □

We will now study neutral invariant subspaces of symmetric operators. Recall that a subspace U is neutral if and only if $U \subset U^{\langle\perp\rangle}$; we are in fact interested in the stronger condition $U = U^{\langle\perp\rangle}$. In Chapter 4, invariant subspaces of this type will be used to construct selfadjoint solutions of Riccati equations.

Definition 2.5.13 A subspace U of a Krein space satisfying $U = U^{\langle\perp\rangle}$ is called *hypermaximal neutral*. ⌟

The notion is justified by the following observations, see also Azizov and Iokhvidov [5, §I.4] and Dijksma and de Snoo [16].

Remark 2.5.14 Let $U = U^{\langle\perp\rangle}$. Then U is neutral, in particular nonnegative and nonpositive. Consider a nonnegative subspace W such that $U \subset W$. For $u \in U$, $w \in W$, the relation

$$0 \leq \langle \lambda u + w | \lambda u + w \rangle = 2\,\mathrm{Re}(\lambda \langle u|w \rangle) + \langle w|w \rangle \quad \text{for all} \quad \lambda \in \mathbb{C}$$

shows that $\langle u|w\rangle = 0$. Consequently $W \subset U^{\langle\perp\rangle} = U$, i.e., U is maximal nonnegative. Analogously we see that U is maximal nonpositive.

Now suppose that U is neutral and also maximal nonnegative or maximal nonpositive. If W is neutral and $U \subset W$, then, as W is in particular nonnegative (nonpositive), we find $U = W$. Hence U is maximal neutral.

In fact the following equivalences were shown by Azizov and Iokhvidov [5, §I.4]: U is maximal neutral if and only if U is neutral and, additionally, maximal nonnegative or maximal nonpositive; moreover $U = U^{\langle\perp\rangle}$ if and only if U is maximal nonnegative and maximal nonpositive. ⌟

In order to obtain invariant subspaces, we use finitely determining l^2-decompositions and consider the compatible subspaces U associated with subsets $\sigma \subset \sigma_p(T)$ of the point spectrum. Then the requirement $U = U^{\langle\perp\rangle}$ has certain consequences for σ and the point spectrum of T:

Proposition 2.5.15 *Consider a symmetric operator $T(V \to V)$ with a dense system of root subspaces, a subset $\sigma \subset \sigma_p(T)$ of the point spectrum, and the subspace*

$$U = \overline{\sum_{\lambda \in \sigma} \mathcal{L}(\lambda)}. \tag{2.29}$$

Then U is neutral if and only if σ does not contain any conjugate pair of eigenvalues.

Moreover if $U = U^{\langle\perp\rangle}$, then we have $\sigma_p(T) \cap \mathbb{R} = \varnothing$ and σ induces a partition $\sigma_p(T) = \sigma \uplus \tau$ which separates conjugate points*, i.e.,*

$$\lambda \in \sigma \quad \Leftrightarrow \quad \overline{\lambda} \in \tau.$$

Proof. The first assertion is an immediate consequence of Theorem 2.5.12. Let $U = U^{\langle\perp\rangle}$ and assume that we have $t \in \sigma_p(T)$ for some $t \in \mathbb{R}$, i.e. $\mathcal{L}(t) \neq \{0\}$. From Theorem 2.5.12 we know that $\mathcal{L}(t)$ is non-degenerate and since U is neutral this implies $\mathcal{L}(t) \not\subset U$ and $t \notin \sigma$. Moreover, $\mathcal{L}(t)$ is orthogonal to any other root subspace of T, in particular to all $\mathcal{L}(\lambda)$ with $\lambda \in \sigma$. Therefore we get $\mathcal{L}(t) \subset U^{\langle\perp\rangle} = U$, a contradiction.

Suppose now that there is a conjugate pair $\lambda_1 \neq \overline{\lambda_1}$ of eigenvalues such that neither $\lambda_1 \in \sigma$ nor $\overline{\lambda_1} \in \sigma$. Consider U_1 given by (2.29) with σ replaced by $\sigma \cup \{\lambda_1\}$. Then $U \subsetneqq U_1$ which implies $U_1^{\langle\perp\rangle} \subsetneqq U^{\langle\perp\rangle}$. Furthermore U_1 is neutral, $U_1 \subset U_1^{\langle\perp\rangle}$, and we obtain the contradiction $U \subsetneqq U^{\langle\perp\rangle}$. □

The necessary condition for $U = U^{\langle\perp\rangle}$ from the previous proposition is also sufficient if T has a finitely spectral l^2-decomposition:

Theorem 2.5.16 *Consider a symmetric operator T on a Krein space V with a finitely spectral l^2-decomposition $V = \bigoplus_{k\in\mathbb{N}}^2 V_k$ and $\sigma_p(T) \cap \mathbb{R} = \varnothing$. If the partition $\sigma_p(T) = \sigma \cup \tau$ separates conjugate points, then the associated subspaces*

$$U = \overline{\sum_{\lambda\in\sigma} \mathcal{L}(\lambda)}, \qquad W = \overline{\sum_{\lambda\in\tau} \mathcal{L}(\lambda)}$$

satisfy

$$U = U^{\langle\perp\rangle}, \quad W = W^{\langle\perp\rangle}.$$

Note that due to Proposition 2.4.3, U and W are of the form

$$U = \bigoplus_{k\in\mathbb{N}}^2 U_k, \quad W = \bigoplus_{k\in\mathbb{N}}^2 W_k$$

where U_k and W_k are the spectral subspaces of V_k corresponding to σ and τ, respectively.

Proof of the theorem. As σ contains no conjugate pairs, Proposition 2.5.15 shows that U is neutral, $U \subset U^{\langle\perp\rangle}$. To prove the other inclusion, let

$$x \in U^{\langle\perp\rangle} \quad \text{with} \quad x = \sum_{k\in\mathbb{N}} x_k, \quad x_k = u_k + w_k \in V_k \quad \text{and} \quad u_k \in U_k,\, w_k \in W_k.$$

We aim to show that all w_k are zero. Consider one particular $k \in \mathbb{N}$. Since every V_j is the sum of root subspaces of T, there is a finite subset $\tau_0 \subset \tau$ such that

$$W_k = \sum_{\lambda\in\tau_0} \mathcal{L}(\lambda) \quad \text{and} \quad W_j \subset \sum_{\lambda\in\tau\setminus\tau_0} \mathcal{L}(\lambda) \quad \text{for all} \quad j \neq k.$$

Hence by Theorem 2.5.12, every

$$y \in \sum_{\lambda\in\tau_0} \mathcal{L}(\overline{\lambda}) \subset U$$

is orthogonal to W_j for $j \neq k$ and to all U_j. Therefore

$$0 = \langle x|y\rangle = \sum_{j\in\mathbb{N}} \langle x_j|y\rangle = \sum_{j\in\mathbb{N}} \langle w_j|y\rangle = \langle w_k|y\rangle.$$

Since the subspace $W_k \dotplus \sum_{\lambda\in\tau_0} \mathcal{L}(\overline{\lambda})$ is non-degenerate, we conclude that $w_k = 0$. Consequently $x = \sum_{k\in\mathbb{N}} u_k \in U$, i.e. $U = U^{\langle\perp\rangle}$. The assertion for W follows by symmetry. □

2.6 J-accretive operators and positive invariant subspaces

In this section we study operators with a finitely determining l^2-decomposition which are accretive in a Krein space. We obtain a separation of the spectrum at the imaginary axis and the positivity and negativity of the compatible subspaces associated with the point spectrum in the right and left half-plane, respectively. Analogous results for dichotomous operators have been shown by Langer, Ran and van de Rotten [31] and Langer and Tretter [33].

Definition 2.6.1 An operator $T(V \to V)$ on a Krein space is called

(i) *accretive* if $\operatorname{Re}\langle Tx|x\rangle \geq 0$ for all $x \in \mathcal{D}(T)$,

(ii) *strictly accretive* if $\operatorname{Re}\langle Tx|x\rangle > 0$ for all $x \in \mathcal{D}(T) \setminus \{0\}$,

(iii) *uniformly accretive* if there exists $\gamma > 0$ such that $\operatorname{Re}\langle Tx|x\rangle \geq \gamma\|x\|^2$ for all $x \in \mathcal{D}(T)$.

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Proposition 2.6.2 *Let $T(V \to V)$ be an operator on a Krein space.*

(i) If T is strictly accretive, then $\sigma_p(T) \cap i\mathbb{R} = \varnothing$.

(ii) If T is uniformly accretive with constant γ, then a strip around the imaginary axis belongs to the set of points of regular type for T,

$$\big\{\lambda \in \mathbb{C} \,\big|\, |\operatorname{Re}\lambda| < \gamma\big\} \subset r(T).$$

If in addition T is closed with a dense system of root subspaces, then

$$\big\{\lambda \in \mathbb{C} \,\big|\, |\operatorname{Re}\lambda| < \gamma\big\} \subset \varrho(T).$$

Proof. (i): Consider an eigenvalue $\lambda \in \sigma_p(T)$ and a corresponding eigenvector $x \neq 0$. Then

$$0 < \operatorname{Re}\langle Tx|x\rangle = \operatorname{Re}\langle \lambda x|x\rangle = \operatorname{Re}\lambda \cdot \langle x|x\rangle,$$

in particular $\operatorname{Re}\lambda \neq 0$.

(ii): Let $\lambda \in \mathbb{C} \setminus r(T)$. Then there exists a sequence $x_n \in \mathcal{D}(T)$ with $\|x_n\| = 1$ and $(T-\lambda)x_n \to 0$ as $n \to \infty$. For $\alpha_n = \operatorname{Re}\langle (T-\lambda)x_n|x_n\rangle$ this implies $\alpha_n \to 0$. Using the fundamental symmetry J, in particular $\|J\| = 1$, we obtain

$$\begin{aligned}\gamma = \gamma\|x_n\|^2 &\leq \operatorname{Re}\langle Tx_n|x_n\rangle = \alpha_n + \operatorname{Re}\lambda \cdot \langle x_n|x_n\rangle \\ &\leq |\alpha_n| + |\operatorname{Re}\lambda|\,|(Jx_n|x_n)| \leq |\alpha_n| + |\operatorname{Re}\lambda|\|x_n\|^2 \to |\operatorname{Re}\lambda|\end{aligned}$$

as $n \to \infty$, i.e. $\gamma \leq |\operatorname{Re}\lambda|$. The additional assertion immediately follows from Lemma 2.3.10. □

For operators with a finitely determining l^2-decomposition and no spectrum on the imaginary axis there are the compatible subspaces U_+ and U_- associated with the part of the spectrum in the right and left half-plane, respectively. The algebraic projections $P_\pm$ corresponding to the direct sum $U_+ \dot{+} U_-$ can be represented by a resolvent integral along the imaginary axis. Integrals of this kind have also been studied by Langer, Ran and van de Rotten [31] and Langer and Tretter [33].

Lemma 2.6.3 *Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \neq 0$. Then we have*

$$\int_{i\mathbb{R}}' \frac{dz}{\lambda - z} = \begin{cases} \pi i & \text{if } \operatorname{Re}\lambda > 0, \\ -\pi i & \text{if } \operatorname{Re}\lambda < 0, \end{cases} \qquad \text{and} \qquad \int_{i\mathbb{R}}' \frac{dz}{(\lambda - z)^k} = 0 \quad \text{for} \quad k \geq 2\,,$$

where the prime denotes the Cauchy principal value at infinity, that is $\int_{i\mathbb{R}}' f\,dz = \lim_{r\to\infty} \int_{-ir}^{ir} f\,dz$.

Proof. For $k \geq 2$ we compute

$$\int_{-ir}^{ir} \frac{dz}{(\lambda - z)^k} = \frac{1}{(k-1)(\lambda - z)^{k-1}}\bigg|_{-ir}^{ir} \to 0 \qquad \text{as} \quad r \to \infty,$$

which proves the second assertion. To show the first one, we consider the two branches of the complex logarithm defined by

$$\begin{aligned} \log_+(z) &= \log|z| + i\arg_+(z) \quad \text{with} \quad \arg_+(z) \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\qquad \text{for } \operatorname{Re} z > 0, \\ \log_-(z) &= \log|z| + i\arg_-(z) \quad \text{with} \quad \arg_-(z) \in \left]\frac{\pi}{2}, \frac{3\pi}{2}\right[\qquad \text{for } \operatorname{Re} z < 0. \end{aligned}$$

For $\operatorname{Re}\lambda > 0$ and < 0, respectively, this yields

$$\int_{-ir}^{ir} \frac{dz}{\lambda - z} = -\log_\pm(\lambda - z)\Big|_{-ir}^{ir} = \log\frac{|\lambda + ir|}{|\lambda - ir|} + i\big(\arg_\pm(\lambda + ir) - \arg_\pm(\lambda - ir)\big).$$

The first summand vanishes as r goes to infinity whereas for the arguments we obtain

$$\arg_\pm(\lambda + ir) \to \frac{\pi}{2} \quad \text{and} \quad \arg_\pm(\lambda - ir) \to \begin{cases} -\pi/2 \\ 3\pi/2 \end{cases} \qquad \text{as} \quad r \to \infty.$$

Consequently, the integral converges to $i\pi$ and $-i\pi$, respectively. □

For an operator T we denote by $\sigma_p^+(T)$ and $\sigma_p^-(T)$ the set of all eigenvalues in the right and left half-plane, respectively.

Proposition 2.6.4 *Let T be an operator on a Banach space with $\sigma_p(T) \cap i\mathbb{R} = \varnothing$. Consider the algebraic direct decomposition of the sum of all root subspaces*

$$\sum_{\lambda \in \sigma_p(T)} \mathcal{L}(\lambda) = \sum_{\lambda \in \sigma_p^+(T)} \mathcal{L}(\lambda) \dotplus \sum_{\lambda \in \sigma_p^-(T)} \mathcal{L}(\lambda)$$

and the associated algebraic projections P_+ and P_- onto the first and second component, respectively. Then we have

$$\frac{1}{i\pi} \int_{i\mathbb{R}}' (T-z)^{-1} x \, dz = P_+ x - P_- x \qquad \text{for all} \quad x \in \sum_{\lambda \in \sigma_p(T)} \mathcal{L}(\lambda) . \tag{2.30}$$

Note that we do not need the stronger assumption $i\mathbb{R} \subset \varrho(T)$: In the integrand, the inverse $(T-z)^{-1}$ acts, for each x, on a finite sum of finite-dimensional subspaces generated by Jordan chains. Therefore $(T-z)^{-1}x$ is a continuous function in z.

Proof of the proposition. By linearity and since every $x \in \sum_{\sigma_p(T)} \mathcal{L}(\lambda)$ is a finite sum $x = x_1 + \cdots + x_n$ of elements $x_k \in \mathcal{L}(\lambda_k)$, each contained in some Jordan chain, it suffices to consider $x \in \mathcal{L}(\lambda)$ and the Jordan chain generated by x. This Jordan chain is the basis of an invariant subspace and in this basis T is represented by a Jordan matrix of the form

$$A = \begin{pmatrix} \lambda & 1 & \\ & \ddots & \ddots \\ & & \lambda \end{pmatrix} .$$

So, we only have to show that

$$\int_{i\mathbb{R}}' (A-z)^{-1} dz = \pm i\pi I$$

for $\operatorname{Re}\lambda > 0$ and $\operatorname{Re}\lambda < 0$, respectively. As the inverse of $A - z$ is given by

$$(A-z)^{-1} = \begin{pmatrix} (\lambda-z)^{-1} & -(\lambda-z)^{-2} & (\lambda-z)^{-3} & \dots \\ & (\lambda-z)^{-1} & -(\lambda-z)^{-2} & \dots \\ & & (\lambda-z)^{-1} & \\ & & & \ddots \end{pmatrix} ,$$

an application of the previous lemma completes the proof. □

Using a Riesz basis of Jordan chains, we derive an estimate for the integral over the squared norm of the resolvent along the imaginary axis:

Proposition 2.6.5 *Let $T(H \to H)$ be an operator on a Hilbert space with a Riesz basis of Jordan chains S. Suppose that $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ and that the eigenvalues of T are contained in a strip around the imaginary axis, i.e.*

$$a = \sup\bigl\{ |\operatorname{Re}\lambda| \bigm| \lambda \in \sigma_p(T) \bigr\} < \infty.$$

Then

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \|(T-it)^{-1}x\|^2\,dt \ \geq\ \frac{m}{2M\sqrt{1+a^2}}\|x\|^2 \qquad \textit{for} \quad x \in \operatorname{span} S, \tag{2.31}$$

where m and M are the constants from (2.12) associated with the Riesz basis.

Proof. Let $x \in \operatorname{span} S$. Then there is a finite system $B = (x_1, \dots, x_n) \subset S$ consisting of Jordan chains such that $x = \alpha_1 x_1 + \ldots + \alpha_n x_n$. $\operatorname{span} B$ is an invariant subspace of T with basis B. The matrix representing T with respect to B is block diagonal with blocks of the form

$$A = \begin{pmatrix} \lambda & 1 & \\ & \ddots & \ddots \\ & & \lambda \end{pmatrix},$$

one for each Jordan chain in B. Accordingly, $(T-it)^{-1}$ is represented by a block diagonal matrix C with blocks of the form $(A-it)^{-1}$. Then

$$(T-it)^{-1}x = \sum_{k=1}^{n} \alpha_k (T-it)^{-1}x_k = \sum_{j,k=1}^{n} \alpha_k C_{jk} x_j.$$

Putting $\xi = (\alpha_1, \dots, \alpha_n)$ and using the Euclidean norm on $\mathbb{C}^n$ we find

$$\|(T-it)^{-1}x\|^2 \geq m \sum_{j=1}^{n} \Big| \sum_{k=1}^{n} \alpha_k C_{jk} \Big|^2 = m\|C\xi\|^2.$$

Now $\|C\xi\|^2$ is the sum of terms of the form $\|(A-it)^{-1}\nu\|^2$, one for each Jordan chain in B with ν the part of ξ corresponding to that Jordan chain. So in order to estimate $\int \|(T-it)^{-1}x\|^2\,dt$, it suffices to estimate $\int \|(A-it)^{-1}\nu\|^2\,dt$. From

$$\|A-it\| \leq |\lambda - it| + \left\| \begin{pmatrix} 0 & 1 & \\ & \ddots & \ddots \\ & & 0 \end{pmatrix} \right\| \leq |\lambda - it| + 1$$

it follows that

$$\|(A-it)^{-1}\nu\|^2 \geq \frac{1}{(|\lambda - it| + 1)^2}\|\nu\|^2.$$

With $u = \operatorname{Re}\lambda$, $v = \operatorname{Im}\lambda$, the calculation

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dt}{(|\lambda - it|+1)^2} &\geq \int_{-\infty}^{\infty} \frac{dt}{2(|\lambda - it|^2+1)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dt}{1+u^2+(t-v)^2} \\ &= \frac{1}{2\sqrt{1+u^2}} \arctan\left(\frac{t-v}{\sqrt{1+u^2}}\right)\Big|_{t=-\infty}^{\infty} = \frac{\pi}{2\sqrt{1+u^2}} \geq \frac{\pi}{2\sqrt{1+a^2}}\end{aligned}$$

yields

$$\int_{-\infty}^{\infty} \|(A-it)^{-1}\nu\|^2\,dt \geq \frac{\pi}{2\sqrt{1+a^2}}\|\nu\|^2.$$

Putting it all together, we arrive at

$$\int_{-\infty}^{\infty} \|(T-it)^{-1}x\|^2\,dt \geq m\frac{\pi}{2\sqrt{1+a^2}}\|\xi\|^2 \geq \frac{m\pi}{2M\sqrt{1+a^2}}\|x\|^2.$$

□

Part (i) of the following proposition was obtained by Azizov and Iokhvidov [5, Corollary 2.2.22].

Proposition 2.6.6 *Let $T(V \to V)$ be an accretive operator on a Krein space with $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ and*

$$U_+ = \overline{\sum_{\lambda\in\sigma_p^+(T)} \mathcal{L}(\lambda)}, \qquad U_- = \overline{\sum_{\lambda\in\sigma_p^-(T)} \mathcal{L}(\lambda)} \tag{2.32}$$

the closed subspaces generated by the root subspaces corresponding to the right and left half-planes, respectively. Then

(i) U_+ is nonnegative, U_- is nonpositive.

(ii) If T is closed, uniformly accretive with constant γ, has a Riesz basis of Jordan chains, and $\sigma_p(T)$ is contained in a strip around the imaginary axis,

$$a = \sup\bigl\{|\operatorname{Re}\lambda| \bigm| \lambda \in \sigma_p(T)\bigr\} < \infty,$$

then U_+ and U_- are uniformly positive and negative, respectively, with constant

$$\alpha = \frac{m\gamma}{2M\sqrt{1+a^2}}.$$

Here m, M are the constants from (2.12) associated with the Riesz basis.

Proof. (i): Let

$$W_+ = \sum_{\lambda\in\sigma_p^+(T)} \mathcal{L}(\lambda) \quad \text{and} \quad W_- = \sum_{\lambda\in\sigma_p^-(T)} \mathcal{L}(\lambda).$$

Then $U_+ = \overline{W_+}$, $U_- = \overline{W_-}$ and we get an algebraic decomposition $W_+ \dotplus W_-$ of the sum of all root subspaces. Let P_+ and P_- be the corresponding algebraic projections onto W_+ and W_-, respectively. Using Proposition 2.6.4, we have

$$\frac{1}{\pi}\int_{\mathbb{R}}' (T-it)^{-1}x\,dt = P_+x - P_-x \quad \text{for} \quad x \in W_+ \dotplus W_-\,.$$

For $x \in W_+$ this yields

$$\langle x|x\rangle = \langle P_+x - P_-x|x\rangle = \frac{1}{\pi}\int_{\mathbb{R}}' \langle (T-it)^{-1}x|x\rangle\,dt\,.$$

We rewrite the integrand as

$$\langle (T-it)^{-1}x|x\rangle = \langle (T-it)^{-1}x|T(T-it)^{-1}x\rangle + it\langle (T-it)^{-1}x|(T-it)^{-1}x\rangle,$$

where the last summand is purely imaginary. Since $\langle x|x\rangle \in \mathbb{R}$ and using the accretivity of T, we obtain

$$\langle x|x\rangle = \frac{1}{\pi}\int_{\mathbb{R}}' \mathrm{Re}\langle (T-it)^{-1}x|x\rangle\,dt = \frac{1}{\pi}\int_{\mathbb{R}}' \underbrace{\mathrm{Re}\langle T(T-it)^{-1}x|(T-it)^{-1}x\rangle}_{\geq 0}\,dt \geq 0.$$

Thus W_+ and hence also U_+ are nonnegative. For $x \in W_-$ the similar calculation

$$-\langle x|x\rangle = \langle P_+x - P_-x|x\rangle = \frac{1}{\pi}\int_{\mathbb{R}}' \mathrm{Re}\langle T(T-it)^{-1}x|(T-it)^{-1}x\rangle\,dt \geq 0$$

implies that W_- and hence also U_- are nonpositive.

(ii): We use the same notations as in (i) and Proposition 2.6.5 to estimate the resolvent integral. Denote by $W_\pm^0$ the span of the Jordan chains from the Riesz basis corresponding to $\sigma_p^\pm(T)$. Then $W_\pm^0 \subset W_\pm$ and for $x \in W_+^0$ we find

$$\begin{aligned}\langle x|x\rangle = \langle P_+x - P_-x|x\rangle &= \frac{1}{\pi}\int_{\mathbb{R}}' \mathrm{Re}\langle T(T-it)^{-1}x|(T-it)^{-1}x\rangle\,dt\\ &\geq \frac{\gamma}{\pi}\int_{\mathbb{R}} \|(T-it)^{-1}x\|^2\,dt \geq \frac{m\gamma}{2M\sqrt{1+a^2}}\|x\|^2.\end{aligned}$$

By Proposition 2.6.2 we know that $\varrho(T) \neq \varnothing$. The subspaces generated by the Jordan chains of the Riesz basis thus form a finitely determining l^2-decomposition, see the proof of Lemma 2.3.15(i). Then (2.17) implies $\mathcal{L}(\lambda) \subset \overline{W_+^0}$ for $\lambda \in \sigma_p^+(T)$ and hence $\overline{W_+^0} = \overline{W_+}$. Consequently $U_+ = \overline{W_+}$ is uniformly positive with the specified constant. For $x \in W_-^0$, the relation $-\langle x|x\rangle = \langle P_+x - P_-x|x\rangle$ again leads to the corresponding result. □

Chapter 3

Perturbation theory for spectral l^2-decompositions

The purpose of this chapter is to prove the existence of finitely spectral l^2-decompositions for non-normal operators with compact resolvent. Compared to normal operators, a number of new problems arise: First, apart from eigenvectors, the existence of generalised eigenvectors is possible too. Second, in contrast to a normal operator with compact resolvent, which always has an orthonormal basis of eigenvectors, the system of root vectors of a non-normal operator with compact resolvent need not be complete. And third, even if the system is complete, this does not imply that it has additional basis properties.

To solve these problems we use an approach due to Markus and Matsaev [37], [36, §§5,6], and consider an operator $T = G + S$ where G is normal with compact resolvent and S is p-subordinate to G. Under appropriate conditions on the spectrum of G we prove that T has a compact resolvent and admits a finitely spectral l^2-decomposition. Strengthening the assumptions we even obtain an l^2-decomposition of root subspaces, i.e., T is a spectral operator. These results extend theorems due to Kato [24], Dunford and Schwartz [20], and Clark [11].

In the first section we prove an auxiliary result on the completeness of the system of root vectors. In Section 3.2 the notion of a p-subordinate perturbation is defined and differential operators are considered as examples. Section 3.3 contains several estimates for Riesz projections corresponding to T. The main perturbation theorems are proved in Section 3.4 and applied to diagonally dominant block operator matrices. In the last section we show the existence of a finitely spectral l^2-decomposition for an ordinary differential operator with possibly unbounded coefficient functions.

3.1 Completeness of the system of root subspaces

We derive a completeness result for the system of root subspaces of an operator with compact resolvent. In the proof we use ideas from a similar theorem for a relatively compact perturbation of a normal operator due to Keldysh [25], cf. [36, §4]. Our result is of auxiliary nature and will be used in the proof of the main perturbation theorems in Section 3.4. Hence we do not consider a perturbation here and instead assume that the resolvent is appropriately bounded.

Recall that the adjoint of an operator with compact resolvent on a Hilbert space also has a compact resolvent.

Lemma 3.1.1 *Let $T(H \to H)$ be a densely defined operator with compact resolvent on a Hilbert space H and*

$$M = \sum_{\lambda \in \sigma(T)} \mathcal{L}(\lambda)$$

the sum of all root subspaces of T. If P is the Riesz projection of T^* corresponding to an eigenvalue $\lambda \in \sigma(T^*)$, then $M^\perp \subset \ker P$. Moreover, $M^\perp$ is T^*-invariant and $(T^* - z)^{-1}$-invariant for every $z \in \varrho(T^*)$; in particular $\varrho(T^*) \subset \varrho(T^*|_{M^\perp})$.

Proof. We have $\lambda \in \sigma(T^*)$ if and only if $\overline{\lambda} \in \sigma(T)$. Observe that if P is the Riesz projection of T^* corresponding to λ, then P^* is the Riesz projection of T corresponding to $\overline{\lambda}$. Since $\mathcal{R}(P^*) \subset M$ we find $M^\perp \subset \mathcal{R}(P^*)^\perp = \ker P$. Now let $v \in M$ and $z \in \varrho(T^*)$. Then $Tv, (T - \bar{z})^{-1}v \in M$ and we find

$$\begin{aligned} u \in M^\perp \cap \mathcal{D}(T^*) \quad &\Rightarrow \quad (T^*u|v) = (u|Tv) = 0, \\ u \in M^\perp \quad &\Rightarrow \quad \big((T^* - z)^{-1}u\big|v\big) = \big(u\big|(T - \bar{z})^{-1}v\big) = 0. \end{aligned}$$

Therefore $M^\perp$ is T^*- and $(T^* - z)^{-1}$-invariant, and this in turn implies the inclusion $\varrho(T^*) \subset \varrho(T^*|_{M^\perp})$. □

Corollary 3.1.2 *Let T and M be as above. Then $\varrho(T^*|_{M^\perp}) = \mathbb{C}$.*

Proof. Since T has a compact resolvent, the same holds for T^* and $T^*|_{M^\perp}$. Consequently if $\lambda \in \sigma(T^*|_{M^\perp})$, then λ is an eigenvalue of $T^*|_{M^\perp}$, i.e., $T^*u = \lambda u$ for some $u \in M^\perp \setminus \{0\}$. In particular λ is an eigenvalue of T^* and we have $u \in \mathcal{R}(P)$ where P is the Riesz projection of T^* corresponding to λ. Now the previous lemma implies $u \in M^\perp \subset \ker P$ and hence $u = 0$, which is a contradiction. Therefore $\sigma(T^*|_{M^\perp}) = \varnothing$. □

Proposition 3.1.3 *Let H be a Hilbert space and $T(H \to H)$ a densely defined operator with compact resolvent. Suppose that the eigenvalues of T all lie in a finite number of pairwise disjoint sectors*

$$\Omega_j = \big\{z \in \mathbb{C} \,\big|\, |\arg z - \theta_j| < \psi_j\big\} \quad \text{with} \quad 0 < \psi_j \leq \frac{\pi}{4}, \quad j = 1, \dots, n.$$

If there is a constant $M_0 \geq 0$ *such that*

$$\|(T - z)^{-1}\| \leq M_0 \quad \textit{for} \quad z \notin \Omega_1 \cup \ldots \cup \Omega_n$$

and for each sector Ω_j *there is a sequence* $(x_k)_{k \in \mathbb{N}}$ *with* $x_k \to \infty$ *and*

$$\|(T - z)^{-1}\| \leq M_0 \quad \textit{for} \quad z \in \Omega_j,\ \mathrm{Re}(e^{-i\theta_j} z) = x_k,\ k \in \mathbb{N},$$

then T *has a dense system of root subspaces.*

Proof. Let M be as before. For $u, v \in M^\perp$ we consider the holomorphic function defined by

$$f(z) = \big((T^*|_{M^\perp} - z)^{-1} u | v\big).$$

From the previous corollary we know that its domain of definition is $\mathbb{C}$. Since

$$\|(T^*|_{M^\perp} - z)^{-1}\| \leq \|(T^* - z)^{-1}\| = \|(T - \bar{z})^{-1}\| \quad \text{for} \quad z \in \varrho(T^*),$$

we see that $|f(z)| \leq M_0 \|u\| \|v\|$ holds for $\bar{z} \in \Omega_j$ with $\mathrm{Re}(e^{-i\theta_j} z) = x_k$ as well as for $\bar{z} \notin \Omega_1 \cup \ldots \cup \Omega_n$. Using the maximum principle, we find that $|f(z)| \leq M_0 \|u\| \|v\|$ for every $z \in \mathbb{C}$; by Liouville's theorem f is constant. Since u and v have been arbitrary, the mapping $z \mapsto (T^*|_{M^\perp} - z)^{-1}$ is also constant. For $u \in M^\perp$ we obtain

$$\begin{aligned}(T^*|_{M^\perp})^{-1} u = (T^*|_{M^\perp} - I)^{-1} u \quad &\Rightarrow \quad (T^*|_{M^\perp} - I)(T^*|_{M^\perp})^{-1} u = u \\ \Rightarrow \quad (T^*|_{M^\perp})^{-1} u = 0 \quad &\Rightarrow \quad u = 0\,.\end{aligned}$$

Hence $M^\perp = \{0\}$, i.e., $M \subset H$ is dense. □

3.2 p-subordinate perturbations

The concept of p-subordination is taken from the book of Markus [36, §5], see also Krein [27, §I.7.1]. In a certain sense it is an interpolation between the notions of boundedness and relative boundedness. As examples of p-subordination we consider differential operators with boundary conditions and bounded as well as unbounded coefficient functions.

Definition 3.2.1 Let $G(V \to V)$ and $S(V \to V)$ be operators in a Banach space and $p \in [0, 1]$. The operator S is said to be *p-subordinate to G* if $\mathcal{D}(G) \subset \mathcal{D}(S)$ and there exists $b \geq 0$ such that

$$\|Su\| \leq b \|u\|^{1-p} \|Gu\|^p \quad \text{for all} \quad u \in \mathcal{D}(G). \tag{3.1}$$

The minimal constant $b \geq 0$ such that (3.1) holds is called the *p-subordination bound* of S to G. ⌟

For the case $p = 0$, subordination simply reduces to the boundedness of S. For $p > 0$, the following proposition gives a connection to relative boundedness, cf. Krein [27, page 146]. The operator $S(V \to V)$ is called *relatively bounded with respect to* $G(V \to V)$, or simply G-*bounded*, if $\mathcal{D}(G) \subset \mathcal{D}(S)$ and there exist $a, b \geq 0$ such that

$$\|Su\| \leq a\|u\| + b\|Gu\| \quad \text{for all} \quad u \in \mathcal{D}(G). \tag{3.2}$$

The infimum of all such b is called the G-*bound* of S.

Proposition 3.2.2 *Let G, S be operators in a Banach space with $\mathcal{D}(G) \subset \mathcal{D}(S)$ and $0 < p \leq 1$. Then S is p-subordinate to G if and only if there is a constant $C > 0$ such that*

$$\|Su\| \leq C(\varepsilon^{-p}\|u\| + \varepsilon^{1-p}\|Gu\|) \quad \textit{for all} \quad u \in \mathcal{D}(G),\, \varepsilon > 0. \tag{3.3}$$

Proof. First note that

$$\lambda^p + \lambda^{p-1} \geq 1 \quad \text{for} \quad \lambda > 0. \tag{3.4}$$

Indeed, we have $\lambda^p \geq 1$ for $\lambda \geq 1$ and $\lambda^{p-1} \geq 1$ for $0 < \lambda \leq 1$.

As the case $u = 0$ is trivial, we may assume $u \neq 0$. Suppose first that S is p-subordinate to G. If $\|Gu\| = 0$ then $\|Su\| = 0$ and (3.3) holds. If $\|Gu\| \neq 0$, we use (3.4) with $\lambda = \|u\|(\varepsilon\|Gu\|)^{-1}$ and obtain

$$\|Su\| \leq b\|u\|^{1-p}\|Gu\|^p \left(\left(\frac{\|u\|}{\varepsilon\|Gu\|} \right)^p + \left(\frac{\|u\|}{\varepsilon\|Gu\|} \right)^{p-1} \right) = b(\varepsilon^{-p}\|u\| + \varepsilon^{1-p}\|Gu\|).$$

Vice versa, suppose that (3.3) holds. If $\|Gu\| = 0$ then

$$\|Su\| \leq C\varepsilon^{-p}\|u\| \to 0 \quad \text{as} \quad \varepsilon \to \infty,$$

that is $\|Su\| = 0$. If $\|Gu\| \neq 0$, we use (3.3) with $\varepsilon = \|u\|/\|Gu\|$ to get

$$\|Su\| \leq C \left(\left(\frac{\|u\|}{\|Gu\|} \right)^{-p} \|u\| + \left(\frac{\|u\|}{\|Gu\|} \right)^{1-p} \|Gu\| \right) = 2C\|u\|^{1-p}\|Gu\|^p.$$

□

Corollary 3.2.3 *If the operator S is p-subordinate to G with bound b, then S is G-bounded with G-bound 0 for $0 \leq p < 1$ and G-bound $\leq b$ for $p = 1$.* □

While boundedness implies relative boundedness, there is in general no relation between p-subordination for different p. For example, if $\ker G \neq \{0\}$, then the condition $\ker G \subset \ker S$ is necessary for a bounded (i.e. 0-subordinate) operator S to be p-subordinate to G with $p > 0$. The situation is different for $0 \in \varrho(G)$:

Lemma 3.2.4 *If S is p-subordinate to G and $0 \in \varrho(G)$, then S is q-subordinate to G for all $q \in [p, 1]$.*

Proof. For $u \in \mathcal{D}(G)$ we have

$$\|Su\| \le b\|u\|^{1-p}\|Gu\|^p = b\|u\|^{1-q}\|G^{-1}Gu\|^{q-p}\|Gu\|^p \le b\|G^{-1}\|^{q-p}\|u\|^{1-q}\|Gu\|^q.$$

□

If G has a compact resolvent, connections of p-subordination to the boundedness of SG^{-p} and to relative compactness can be obtained:

Remark 3.2.5 Let H be a Hilbert space, $G(H \to H)$ normal with compact resolvent, and $0 \in \varrho(G)$. We may then define fractional powers of G: Let $(e_k)_{k\in\mathbb{N}}$ be an orthonormal basis of eigenvectors and λ_k the corresponding eigenvalues. For $p \in \mathbb{R}$ we set

$$\mathcal{D}(G^p) = \left\{ u \in H \;\middle|\; \sum_{k\in\mathbb{N}} |\lambda_k|^{2p} |(u|e_k)|^2 < \infty \right\},$$
$$G^p u = \sum_{k\in\mathbb{N}} \lambda_k^p (u|e_k) e_k \quad \text{for} \quad u \in \mathcal{D}(G^p)$$

where $\lambda^p = |\lambda|^p e^{ip\arg\lambda}$ with $\arg\lambda \in \,]-\pi,\pi]$.

Now the following can be shown, see Markus [36, §5] and Krein [27, §I.7.1]: If the operator $S(H \to H)$ is such that $SG^{-p} \in L(H)$ with $0 \le p \le 1$, then S is p-subordinate to G; the converse implication is wrong in general. However, if S is p-subordinate to G with $0 \le p < 1$, then $SG^{-q} \in L(H)$ for all $q > p$; in particular, S is *relatively compact* to G, i.e., SG^{-1} is compact. ⌟

As an example of p-subordination we investigate differential operators. We need some facts about Sobolev spaces; see Adams [2] for a detailed treatment. Let $\Omega \subset \mathbb{R}^m$ be open. For $n \in \mathbb{N}$ we consider the *Sobolev space*

$$W^{n,2}(\Omega) = \left\{ u \in L^2(\Omega) \;\middle|\; \partial_\alpha u \in L^2(\Omega) \text{ exists for } |\alpha| \le n \right\}$$

where $\partial_\alpha u$ is the weak derivative corresponding to the multi-index α. The space $W^{n,2}(\Omega)$ is a Hilbert space with respect to the norm

$$\|u\|_{W^{n,2}(\Omega)} = \left(\sum_{|\alpha|\le n} \|\partial_\alpha u\|^2_{L^2(\Omega)} \right)^{1/2}$$

and $C^\infty(\Omega)$ is a dense subspace. $W_0^{n,2}(\Omega)$ is by definition the closure of $C_0^\infty(\Omega)$ (the space of smooth functions compactly supported in Ω) in $W^{n,2}(\Omega)$.

In the one-dimensional case, $\Omega =]a_1,a_2[$ a bounded open interval, we have the characterisation [23, Theorem VII.1.1]

$$\begin{aligned} & u \in W^{n,2}(\,]a_1,a_2[\,) \\ \Leftrightarrow \quad & u \in C^{n-1}([a_1,a_2]),\ u^{(n-1)} \text{ is absolutely continuous},\ u^{(n)} \in L^2(\,]a_1,a_2[\,). \end{aligned}$$

In particular, the point evaluations $u(x), \dots, u^{(n-1)}(x)$ are well defined for every $x \in [a_1, a_2]$, and we will therefore use the notation $W^{n,2}([a_1, a_2])$ for the Sobolev space over an interval. $C^n([a_1, a_2]) \subset W^{n,2}([a_1, a_2])$ is a dense subspace.

For differential operators with certain kinds of boundary conditions, e.g. Dirichlet or periodic boundary conditions, we obtain a subordination property in a straightforward way using partial integration:

Example 3.2.6 On $L^2([a_1, a_2])$ consider the following second order differential operator with Dirichlet boundary condition:

$$Gu = u'', \quad \mathcal{D}(G) = \big\{u \in W^{2,2}([a_1, a_2]) \,\big|\, u(a_1) = u(a_2) = 0\big\}.$$

Then the first order operator

$$Su = u' \quad \text{with} \quad \mathcal{D}(S) = C^1([a_1, a_2])$$

is 1/2-subordinate to G: Integrating by parts and using the boundary condition and the Cauchy-Schwarz inequality, we obtain for $u \in \mathcal{D}(G)$

$$\begin{aligned}\int_{a_1}^{a_2} |u'(x)|^2 \, dx &= \int_{a_1}^{a_2} u'(x)\overline{u'(x)} \, dx \\ &= -\int_{a_1}^{a_2} u(x)\overline{u''(x)} \, dx \le \|u\|_{L^2([a_1,a_2])} \|u''\|_{L^2([a_1,a_2])} \, .\end{aligned}$$

Hence

$$\|Su\|_{L^2([a_1,a_2])} \le \|u\|^{1/2}_{L^2([a_1,a_2])} \|Gu\|^{1/2}_{L^2([a_1,a_2])} \quad \text{for} \quad u \in \mathcal{D}(G). \tag{3.5}$$

Obviously, this result continues to hold for every choice of boundary conditions such that the boundary terms in the integration by parts vanish. ⌟

Example 3.2.7 Consider the Laplacian on a domain $\Omega \subset \mathbb{R}^m$ with Dirichlet boundary conditions,

$$G(L^2(\Omega) \to L^2(\Omega)), \quad Gu = \Delta u, \quad \mathcal{D}(G) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega).$$

Then the operator ∂_k of taking the kth partial derivative with domain $W^{1,2}(\Omega)$ is 1/2-subordinate to G: Analogously to the previous example we find for $u \in \mathcal{D}(G)$

$$\begin{aligned}\int_\Omega |\partial_k u(x)|^2 \, dx &\le \sum_{j=1}^m \int_\Omega \partial_j u(x)\overline{\partial_j u(x)} \, dx = -\sum_{j=1}^m \int_\Omega u(x)\overline{\partial_j^2 u(x)} \, dx \\ &= -\int_\Omega u(x)\overline{\Delta u(x)} \, dx \le \|u\|_{L^2(\Omega)} \|Gu\|_{L^2(\Omega)}.\end{aligned}$$

⌟

In the case of periodic boundary conditions we can derive a subordination property for higher order derivatives.

Proposition 3.2.8 *For $n \in \mathbb{N}$ consider the operator D_n on $L^2([a_1, a_2])$ given by*

$$D_n u = u^{(n)},$$
$$\mathcal{D}(D_n) = \big\{u \in W^{n,2}([a_1, a_2]) \,\big|\, u^{(k)}(a_1) = u^{(k)}(a_2) \text{ for } k = 0, \ldots, n-1\big\}.$$

Then for $0 \le k \le n$ and $n \ge 1$, D_k is k/n-subordinate to D_n,

$$\|D_k u\| \le \|u\|^{1-k/n} \|D_n u\|^{k/n} \quad \text{for} \quad u \in \mathcal{D}(D_n). \tag{3.6}$$

Proof. As the cases $k = 0$ and $k = n$ are trivial, we consider $0 < k < n$ and use induction on n. The calculation in Example 3.2.6 shows that the assertion is true for $n = 2$. Now suppose that (3.6) holds for some $n \ge 2$ and let $u \in \mathcal{D}(D_{n+1})$. Using (3.6) twice, one time with $n = 2$, we find

$$\begin{aligned} & \|D_1 u^{(n-1)}\|^2 \le \|u^{(n-1)}\| \|D_2 u^{(n-1)}\| \le \|u\|^{\frac{1}{n}} \|D_n u\|^{\frac{n-1}{n}} \|D_{n+1} u\| \\ \Rightarrow \quad & \|D_n u\|^{2-\frac{n-1}{n}} = \|D_n u\|^{\frac{n+1}{n}} \le \|u\|^{\frac{1}{n}} \|D_{n+1} u\| \\ \Rightarrow \quad & \|D_n u\| \le \|u\|^{\frac{1}{n+1}} \|D_{n+1} u\|^{\frac{n}{n+1}}. \end{aligned}$$

Using (3.6) again, we obtain for $k \le n-1$

$$\begin{aligned} \|D_k u\| &\le \|u\|^{1-\frac{k}{n}} \|D_n u\|^{\frac{k}{n}} \le \|u\|^{1-\frac{k}{n}} \Big(\|u\|^{\frac{1}{n+1}} \|D_{n+1} u\|^{\frac{n}{n+1}}\Big)^{\frac{k}{n}} \\ &= \|u\|^{1-\frac{k}{n}+\frac{k}{n(n+1)}} \|D_{n+1} u\|^{\frac{k}{n+1}} = \|u\|^{1-\frac{k}{n+1}} \|D_{n+1} u\|^{\frac{k}{n+1}}. \end{aligned}$$

□

The next example shows that differential operators without boundary conditions do not satisfy a subordination property in general:

Example 3.2.9 Let G and S be operators on $L^2([0, 1])$ defined by

$$\begin{aligned} Gu &= u'', \quad \mathcal{D}(G) = W^{2,2}([0, 1]), \\ Su &= u', \quad \mathcal{D}(S) = W^{1,2}([0, 1]). \end{aligned}$$

For $\lambda \in \mathbb{C}$ consider the function $u_\lambda \in \mathcal{D}(G)$ given by

$$u_\lambda(x) = \frac{\lambda}{2} x^2 + x.$$

We have $u_\lambda'(x) = \lambda x + 1$, $u_\lambda''(x) = \lambda$ and hence $S u_\lambda \ne 0$, $(G - \lambda) u_\lambda = 0$. Therefore S is not p-subordinate to $G - \lambda$ for $0 < p \le 1$. As S is also not bounded, S is not p-subordinate to $G - \lambda$ for any $p \in [0, 1]$ and $\lambda \in \mathbb{C}$. ⌟

Now we derive a subordination property for ordinary differential operators with general boundary conditions. The proof is based on the following interpolation inequality for Sobolev spaces. While such an inequality holds on arbitrary domains $\Omega \subset \mathbb{R}^m$ with sufficiently smooth boundary, see Adams [2, Theorem 4.14], we will only need the simpler version over a compact interval. For a proof we also refer to [2, Theorem 4.14].

Proposition 3.2.10 *Let $a_1 < a_2$ and $n \geq 1$. Then there exists $K \geq 0$ such that for $0 < \varepsilon \leq 1$ and $0 \leq k < n$ we have*

$$\|u^{(k)}\|_{L^2([a_1,a_2])} \leq K\varepsilon^{-k/(n-k)}\|u\|_{L^2([a_1,a_2])} + K\varepsilon\|u^{(n)}\|_{L^2([a_1,a_2])} \tag{3.7}$$

for all $u \in W^{n,2}([a_1, a_2])$. □

Remark 3.2.11 Replacing ε with $\varepsilon^{(n-k)/n}$ in inequality (3.7), we obtain

$$\|u^{(k)}\|_{L^2} \leq K\big(\varepsilon^{-k/n}\|u\|_{L^2} + \varepsilon^{1-k/n}\|u^{(n)}\|_{L^2}\big)$$

for $u \in W^{n,2}([a_1, a_2])$ and $0 < \varepsilon \leq 1$. While this inequality is of the form (3.3), we can not use it directly to proof k/n-subordination since it does not hold for all $\varepsilon > 0$. On the other hand, no boundary conditions are involved in Proposition 3.2.10. ⌟

Corollary 3.2.12 *Given $a_1 < a_2$, $n \geq 1$, there are constants $K \geq 0$, $L \geq 0$ such that*

$$\|u\|_{W^{n,2}([a_1,a_2])} \leq K\big(\|u\|_{L^2([a_1,a_2])} + \|u^{(n)}\|_{L^2([a_1,a_2])}\big) \quad \textit{and}$$
$$\|u^{(k)}\|_{\infty} \leq L\big(\|u\|_{L^2([a_1,a_2])} + \|u^{(n)}\|_{L^2([a_1,a_2])}\big)$$

for all $u \in W^{n,2}([a_1, a_2])$, $0 \leq k < n$.

Proof. The first estimate is obtained from (3.7) with $\varepsilon = 1$ and

$$\|u\|_{W^{n,2}} \leq \|u\|_{L^2} + \cdots + \|u^{(n)}\|_{L^2}.$$

The second one then follows by the Sobolev imbedding theorem [2, Theorem 5.4]

$$W^{n,2}([a_1, a_2]) \hookrightarrow C^{n-1}([a_1, a_2]) \quad \text{continuous.}$$

□

The following inequality also holds on arbitrary domains $\Omega \subset \mathbb{R}^m$ with sufficiently smooth boundary, see Adams [2, Theorem 4.17].

Corollary 3.2.13 *For $a_1 < a_2$, $n \geq 1$, and $0 \leq k \leq n$ there is a constant $C \geq 0$ such that*

$$\|u\|_{W^{k,2}([a_1,a_2])} \leq C\|u\|_{L^2([a_1,a_2])}^{1-k/n}\|u\|_{W^{n,2}([a_1,a_2])}^{k/n}$$

for all $u \in W^{n,2}([a_1, a_2])$.

Proof. The inequality is trivial for $k = n$ and $k = 0$, so let $0 < k < n$. By the previous corollary there exists $K_0 \geq 0$ such that

$$\|u\|_{W^{k,2}} \leq K_0\big(\|u\|_{L^2} + \|u^{(k)}\|_{L^2}\big).$$

For $\varepsilon \in]0,1]$ we have

$$\|u\|_{L^2} \leq \varepsilon^{-k/(n-k)}\|u\|_{L^2};$$

together with (3.7) this yields

$$\begin{aligned}\|u\|_{W^{k,2}} &\leq K_0(K+1)\big(\varepsilon^{-k/(n-k)}\|u\|_{L^2} + \varepsilon\|u^{(n)}\|_{L^2}\big)\\ &\leq K_0(K+1)\big(\varepsilon^{-k/(n-k)}\|u\|_{L^2} + \varepsilon\|u\|_{W^{n,2}}\big).\end{aligned}$$

Since $\|u\|_{L^2} \leq \|u\|_{W^{n,2}}$ we may choose $\varepsilon = (\|u\|_{L^2}/\|u\|_{W^{n,2}})^{(n-k)/n}$ and obtain the assertion. □

For a differential operator of order n on the interval $[a_1, a_2]$, boundary conditions can be specified as follows: For $V : \mathbb{C}^{2n} \to \mathbb{C}$ linear and $u \in W^{n,2}([a_1, a_2])$ we define

$$V(u) = V\big(u(a_1), u'(a_1), \ldots, u^{(n-1)}(a_1), u(a_2), u'(a_2), \ldots, u^{(n-1)}(a_2)\big).$$

Then $V(u) = 0$ is a *linear, homogeneous boundary condition*. A treatment of boundary conditions for ordinary differential operators and their relation to eigenvalues and eigenfunctions may be found, for example, in the book of Naimark [40].

The next proposition yields an a priori estimate for solutions u of $u^{(n)} - \lambda u = f$ subject to boundary conditions, see also Goldberg [23, Theorem VI.6.2].

Proposition 3.2.14 *Let $V_1, \ldots, V_n : \mathbb{C}^{2n} \to \mathbb{C}$ be linear and linearly independent and consider the nth order differential operator G on $L^2([a_1, a_2])$ defined by*

$$Gu = u^{(n)}, \quad \mathcal{D}(G) = \big\{u \in W^{n,2}([a_1, a_2]) \,\big|\, V_1(u) = \cdots = V_n(u) = 0\big\}. \tag{3.8}$$

Then for every $\lambda \in \mathbb{C} \setminus \sigma_p(G)$ there is a constant $C \geq 0$ such that

$$\|u\|_{W^{n,2}([a_1,a_2])} \leq C\|(G - \lambda)u\|_{L^2([a_1,a_2])} \quad \textit{for} \quad u \in \mathcal{D}(G).$$

Proof. Since $C^n([a_1, a_2])$ is dense in $W^{n,2}([a_1, a_2])$, we may assume $u \in C^n([a_1, a_2])$ with $V_1(u) = \cdots = V_n(u) = 0$. We set $f = u^{(n)} - \lambda u$ and consider the solution $u_0 \in C^n([a_1, a_2])$ of the Cauchy problem

$$u_0^{(n)} - \lambda u_0 = f, \quad u_0(a_1) = u_0'(a_1) = \cdots = u_0^{(n-1)}(a_1) = 0.$$

Setting $z = (u_0, \ldots, u_0^{(n-1)})$, we may rewrite this as the first order system $z' = Az + g$, $z(a_1) = 0$ with $g = (0, \ldots, 0, f)$ and $A \in \mathbb{C}^{n\times n}$. Denoting by $|\cdot|_\infty$ the maximum norm on $\mathbb{C}^n$, we find

$$z(x) = \int_{a_1}^{x} \big(Az(t) + g(t)\big)\,dt \quad \Rightarrow \quad |z(x)|_\infty \leq \int_{a_1}^{a_2} |g(t)|_\infty\,dt + \int_{a_1}^{x} \|A\|\,|z(t)|_\infty\,dt.$$

By the Gronwall inequality it follows that

$$|z(x)|_\infty \le \int_{a_1}^{a_2} |g(t)|_\infty \, dt \cdot e^{\|A\|(x-a_1)}$$

and thus

$$|u_0(x)|^2 \le |z(x)|_\infty^2 \le e^{2\|A\|(a_2-a_1)}(a_2-a_1) \int_{a_1}^{a_2} |g(t)|_\infty^2 \, dt = C_0 \int_{a_1}^{a_2} |f(t)|^2 \, dt$$

with $C_0 \ge 0$. Therefore

$$\begin{aligned} \|u_0\|_{L^2} &\le \sqrt{C_0(a_2-a_1)}\, \|f\|_{L^2}, \\ \|u_0^{(n)}\|_{L^2} &= \|\lambda u_0 + f\|_{L^2} \le |\lambda| \|u_0\|_{L^2} + \|f\|_{L^2} \le \big(|\lambda|\sqrt{C_0(a_2-a_1)} + 1\big)\|f\|_{L^2}. \end{aligned}$$

Now let $u_1, \dots, u_n$ be a fundamental system of solutions of the homogeneous equation $u^{(n)} - \lambda u = 0$. Set $M = (V_j(u_k))_{j,k=1,\dots,n}$ and $\beta = (V_1(u_0), \dots, V_n(u_0))$. The matrix M is invertible since $\lambda \notin \sigma_p(G)$. Then u is of the form

$$u = \alpha_1 u_1 + \dots + \alpha_n u_n + u_0,$$

and writing $\alpha = (\alpha_1, \dots, \alpha_n)$ we have

$$V_1(u) = \dots = V_n(u) = 0 \quad \Longleftrightarrow \quad M\alpha = -\beta.$$

We obtain

$$\begin{aligned} \|u\|_{L^2} &\le \big(\|u_1\|_{L^2} + \dots + \|u_n\|_{L^2}\big)|\alpha|_\infty + \|u_0\|_{L^2}, \\ |\alpha|_\infty &\le \|M^{-1}\| |\beta|_\infty \le C_1 \|M^{-1}\| \max\big\{\|u_0\|_\infty, \dots, \|u_0^{(n-1)}\|_\infty\big\} \end{aligned}$$

with $C_1 = \max\{\|V_1\|, \dots, \|V_n\|\}$. Due to the above calculations and Corollary 3.2.12, there is a constant $C_2 \ge 0$ such that $\|u_0^{(k)}\|_\infty \le C_2 \|f\|_{L^2}$ for $k = 0, \dots, n-1$. Altogether this yields

$$\|u\|_{L^2} \le \Big(\big(\|u_1\|_{L^2} + \dots + \|u_n\|_{L^2}\big) C_1 \|M^{-1}\| C_2 + \sqrt{C_0(a_2-a_1)}\Big)\|f\|_{L^2} = C_3 \|f\|_{L^2}$$

with $C_3 > 0$. Since moreover

$$\|u^{(n)}\|_{L^2} \le |\lambda| \|u\|_{L^2} + \|f\|_{L^2} \le (|\lambda| C_3 + 1)\|f\|_{L^2},$$

the proof is complete in view of Corollary 3.2.12. □

We can now prove a subordination property for ordinary differential operators with general boundary conditions and bounded coefficients.

Proposition 3.2.15 *Let G be an nth order differential operator on $L^2([a_1, a_2])$ as in (3.8) and $\lambda \in \mathbb{C} \setminus \sigma_p(G)$. Then for $0 \le k \le n$ and $g_0, \dots, g_k \in L^\infty([a_1, a_2])$, the differential operator*

$$Su = \sum_{j=0}^{k} g_j u^{(j)}, \quad \mathcal{D}(S) = W^{k,2}([a_1, a_2])$$

of order k is k/n-subordinate to $G - \lambda$.

Proof. Using Corollary 3.2.13, we have

$$\|Su\|_{L^2} \le \sum_{j=0}^{k} \|g_j\|_\infty \|u^{(j)}\|_{L^2} \le \sum_{j=0}^{k} \|g_j\|_\infty \cdot \|u\|_{W^{k,2}} \le b_0 \|u\|_{L^2}^{1-k/n} \|u\|_{W^{n,2}}^{k/n}$$

with some constant b_0. The claim is thus an immediate consequence of Proposition 3.2.14. □

When the coefficients of S are L^2-functions, we can still prove a subordination property, though with larger constant p.

Proposition 3.2.16 *For $0 \le k \le n-1$ and $g_0, \dots, g_k \in L^2([a_1, a_2])$ consider the differential operator S on $L^2([a_1, a_2])$ given by*

$$Su = \sum_{j=0}^{k} g_j u^{(j)}, \quad \mathcal{D}(S) = C^k([a_1, a_2]).$$

If G is a differential operator as in (3.8) and $\lambda \in \mathbb{C} \setminus \sigma_p(G)$, then S is $(k+1)/n$-subordinate to $G - \lambda$.

Proof. Let $u \in W^{n,2}([a_1, a_2])$. Using Corollaries 3.2.12 and 3.2.13 we find

$$\begin{aligned} \|Su\|_{L^2} &\le \sum_{j=0}^{k} \|g_j\|_{L^2} \|u^{(j)}\|_\infty \le L \sum_{j=0}^{k} \|g_j\|_{L^2} \big(\|u\|_{L^2} + \|u^{(j+1)}\|_{L^2} \big) \\ &\le b_0 \|u\|_{W^{k+1,2}} \le b_1 \|u\|_{L^2}^{1-(k+1)/n} \|u\|_{W^{n,2}}^{(k+1)/n} \end{aligned}$$

with some constants $b_0, b_1 \ge 0$. The assertion is again a consequence of Proposition 3.2.14. □

3.3 Estimates for Riesz projections

In this section we consider the operator $T = G + S$ where G is normal and S is p-subordinate to G. We derive estimates for the resolvent and for Riesz projections of T. They will be used to prove the perturbation theorems for finitely spectral l^2-decompositions in the following section. Lemma 3.3.2 and Propositions 3.3.8 and 3.3.12 may be of interest on their own. The key ideas are taken from the book of Markus [36, §§5,6].

Lemma 3.3.1 *Let G be a normal operator on a Hilbert space, S p-subordinate to G with bound b, and $T = G + S$. If $0 < \varepsilon < 1$ and $z \in \varrho(G)$ such that*

$$b\Big(1 + \frac{|z|}{\operatorname{dist}(z, \sigma(G))}\Big)^p \frac{1}{\operatorname{dist}(z, \sigma(G))^{1-p}} \leq \varepsilon,$$

then $z \in \varrho(T)$ and

$$\|S(G-z)^{-1}\| \leq \varepsilon, \quad \|(T-z)^{-1}\| \leq \frac{(1-\varepsilon)^{-1}}{\operatorname{dist}(z, \sigma(G))}, \quad \|S(T-z)^{-1}\| \leq \frac{\varepsilon}{1-\varepsilon}.$$

Proof. Using the spectral theorem for normal operators [19, Theorem XII.2.3, Exercises XII.9.9 and XII.9.12], see also [24, §V.3.8], we have

$$\|(G-z)^{-1}\| = \sup_{\lambda \in \sigma(G)} \frac{1}{|\lambda - z|} = \frac{1}{\operatorname{dist}(z, \sigma(G))}$$

and

$$\|G(G-z)^{-1}\| = \|I + z(G-z)^{-1}\| \leq 1 + \frac{|z|}{\operatorname{dist}(z, \sigma(G))}.$$

With the definition of p-subordination this yields

$$\begin{aligned}\|S(G-z)^{-1}u\| &\leq b\|G(G-z)^{-1}u\|^p \|(G-z)^{-1}u\|^{1-p} \\ &\leq b\Big(1 + \frac{|z|}{\operatorname{dist}(z, \sigma(G))}\Big)^p \frac{1}{\operatorname{dist}(z, \sigma(G))^{1-p}} \|u\| \leq \varepsilon \|u\|\end{aligned}$$

for every $u \in H$, hence $\|S(G-z)^{-1}\| \leq \varepsilon < 1$. Since

$$T - z = \big(I + S(G-z)^{-1}\big)(G-z),$$

a Neumann series argument shows that $z \in \varrho(T)$ with

$$\begin{aligned}\|(T-z)^{-1}\| &\leq \|(G-z)^{-1}\| \|(I + S(G-z)^{-1})^{-1}\| \\ &\leq \|(G-z)^{-1}\| \frac{1}{1 - \|S(G-z)^{-1}\|} \leq \frac{(1-\varepsilon)^{-1}}{\operatorname{dist}(z, \sigma(G))}.\end{aligned}$$

Finally, the identity $S(T-z)^{-1} = S(G-z)^{-1}(I + S(G-z)^{-1})^{-1}$ implies that $\|S(T-z)^{-1}\| \leq \varepsilon(1-\varepsilon)^{-1}$. □

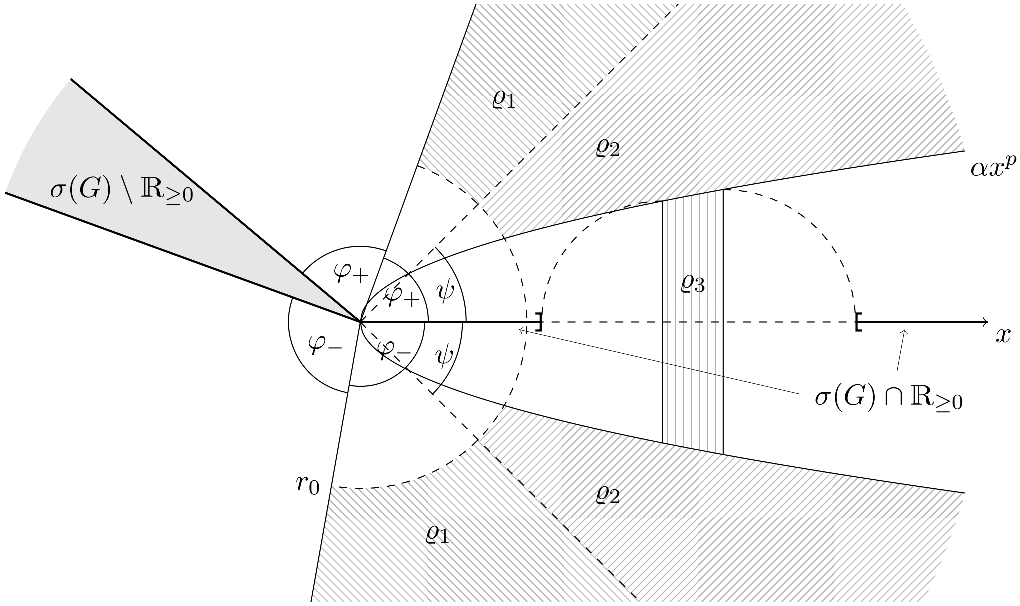

Figure 3.1: The situation of Lemma 3.3.2

In the remaining part of this section we use the notations

$$\Omega(\varphi_-, \varphi_+) = \{re^{i\varphi} \,|\, r \geq 0\,,\, \varphi_- < \varphi < \varphi_+\} \quad \text{and} \quad \Omega(\varphi) = \Omega(-\varphi, \varphi)$$

for the sectors lying between the rays with arguments φ_-, φ_+ and $-\varphi$, φ, respectively. Furthermore, we always assume that

$$\sigma(G) \cap \Omega(2\varphi_-, 2\varphi_+) \subset \mathbb{R}_{\geq 0} \quad \text{with} \quad -\pi \leq \varphi_- < 0 < \varphi_+ \leq \pi.$$

The next lemma states that in this situation the sets ϱ_1, ϱ_2, ϱ_3 belong to the resolvent set of the perturbed operator $T = G+S$, compare Figure 3.1. The set $\varrho_1 \cup \varrho_2$ comprises all points z with $|z|$ large enough, inside the closed sector $\overline{\Omega(\varphi_-, \varphi_+)}$, but outside a parabola around the real axis. The strip ϱ_3 corresponds to large gaps of $\sigma(G)$ on the positive real axis. Sufficient conditions for the existence of such gaps may be found in Proposition 3.3.12, Theorem 3.4.7 and Lemma 3.4.10; examples are the ordinary differential operators in Section 3.5.

Lemma 3.3.2 *Let G be a normal operator such that $\sigma(G) \cap \Omega(2\varphi_-, 2\varphi_+) \subset \mathbb{R}_{\geq 0}$ with $-\pi \leq \varphi_- < 0 < \varphi_+ \leq \pi$. Let S be p-subordinate to G with bound b, $0 \leq p < 1$, and $T = G + S$.*

Then for $\alpha > b$, $b/\alpha < \varepsilon < 1$, and $0 < \psi < \min\{-\varphi_-, \varphi_+, \pi/2\}$ there exists

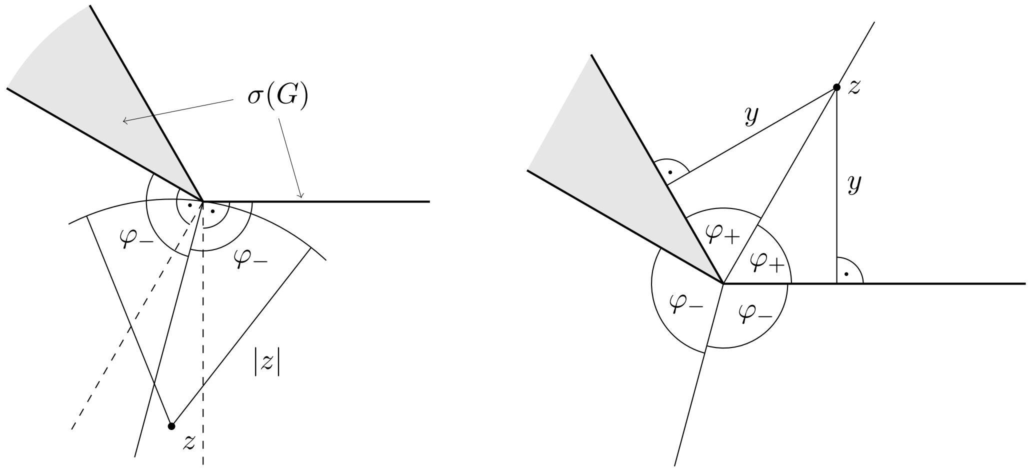

Figure 3.2: Estimates for $\operatorname{dist}(z,\sigma(G))$ in Lemma 3.3.2

$r_0 > 0$ *such that the sets*

$$\begin{aligned}\varrho_1 &= \bigl\{z \in \overline{\Omega(\varphi_-,\varphi_+)} \bigm| |z| \ge r_0,\ z \notin \Omega(\psi)\bigr\},\\ \varrho_2 &= \bigl\{z = x+iy \in \overline{\Omega(\psi)} \bigm| |z| \ge r_0,\ |y| \ge \alpha x^p\bigr\},\\ \varrho_3 &= \bigl\{z = x+iy \in \overline{\Omega(\psi)} \bigm| |z| \ge r_0,\ |y| \le \alpha x^p \le \operatorname{dist}(z,\sigma(G))\bigr\}\end{aligned}$$

satisfy $\varrho_1 \cup \varrho_2 \cup \varrho_3 \subset \varrho(T)$, *and for* $z \in \varrho_1 \cup \varrho_2 \cup \varrho_3$ *we have*

$$\|S(G-z)^{-1}\| \le \varepsilon, \quad \|(T-z)^{-1}\| \le \frac{(1-\varepsilon)^{-1}}{\operatorname{dist}(z,\sigma(G))}, \quad \|S(T-z)^{-1}\| \le \frac{\varepsilon}{1-\varepsilon}.$$

Furthermore there is a constant $M > 0$ *such that*

$$\|(T-z)^{-1}\| \le M \quad \textit{for all} \quad z \in \varrho_1 \cup \varrho_2 \cup \varrho_3.$$

Proof. We want to apply the last lemma and write $d = \operatorname{dist}(z,\sigma(G))$. So we have to show that

$$C = b\Bigl(1 + \frac{|z|}{d}\Bigr)^p \frac{1}{d^{1-p}} \le \varepsilon.$$

First we analyse the geometry of the situation, see Figure 3.2. For $z = x+iy$ we have the implications

$$\varphi_- \le \arg z \le -\frac{\pi}{2} \ \text{ or } \ \frac{\pi}{2} \le \arg z \le \varphi_+ \quad \Longrightarrow \quad d \ge |z|, \tag{3.9}$$

$$\max\Bigl\{\varphi_-, -\frac{\pi}{2}\Bigr\} \le \arg z \le \min\Bigl\{\varphi_+, \frac{\pi}{2}\Bigr\} \quad \Longrightarrow \quad d \ge |y|, \tag{3.10}$$

as well as

$$\psi \leq |\arg z| \leq \frac{\pi}{2} \quad \Longrightarrow \quad |y| \geq |z| \sin \psi, \tag{3.11}$$

$$|\arg z| \leq \psi \quad \Longrightarrow \quad x \geq |z| \cos \psi. \tag{3.12}$$

Now let $z \in \varrho_1$. If $\varphi_- \leq \arg z \leq -\pi/2$ or $\pi/2 \leq \arg z \leq \varphi_+$, then (3.9) yields $C \leq 2^p b|z|^{p-1} \leq \varepsilon$, provided r_0 is large enough. If $\psi \leq |\arg z| \leq \pi/2$, then (3.10) and (3.11) imply $d \geq |z| \sin \psi$ and hence

$$C \leq b\Big(1 + \frac{1}{\sin \psi}\Big)^p \frac{1}{(|z| \sin \psi)^{1-p}} \leq \varepsilon$$

for r_0 sufficiently large.

For $z \in \varrho_2$, implications (3.10) and (3.12) apply and with $|y| \geq \alpha x^p$ we find $d \geq \alpha x^p$. For $p > 0$ we use the Minkowski inequality to get the estimate

$$\Big(1 + \frac{|z|}{d}\Big)^p \leq \Big(1 + \frac{x + |y|}{d}\Big)^p \leq 1 + \frac{x^p + |y|^p}{d^p} \leq 1 + \frac{\alpha^{-1} d + d^p}{d^p} = 2 + \frac{1}{\alpha} d^{1-p},$$

i.e. $C \leq 2bd^{p-1} + b/\alpha$. Since $b/\alpha < \varepsilon$ and $d \geq \alpha(|z| \cos \psi)^p$, we obtain $C \leq \varepsilon$ for r_0 sufficiently large. On the other hand, if $p = 0$ then $d \geq \alpha$ and $C = b/d \leq b/\alpha < \varepsilon$.

In the case $z \in \varrho_3$, (3.10) and (3.12) apply, and we have $d \geq \alpha x^p$ by definition of the set ϱ_3. In the same manner as for $z \in \varrho_2$, we conclude that $C \leq \varepsilon$ if r_0 is large enough.

Finally, to prove that $\|(T - z)^{-1}\|$ is uniformly bounded, we need to show that d^{-1} is bounded independently of z. For $z \in \varrho_1$ we have

$$\text{either} \quad d \geq |z| \geq r_0 > 0 \quad \text{or} \quad d \geq |z| \sin \psi \geq r_0 \sin \psi > 0.$$

For $z \in \varrho_2 \cup \varrho_3$ we obtain

$$d \geq \alpha(|z| \cos \psi)^p \geq \alpha(r_0 \cos \psi)^p > 0.$$

□

We will now focus on the case where G is normal with compact resolvent. The next two lemmas yield estimates for some resolvent integrals along contours associated with the parabola from Figure 3.1.

Lemma 3.3.3 (Markus [36, Lemma 6.6]) *Let G be normal with compact resolvent and $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$. Then for $0 \leq p < 1$, $\alpha > 0$ there exists $r_0 > 0$ such that the contours*

$$\Gamma_\pm = \{x + iy \in \mathbb{C} \,|\, x \geq r_0,\, y = \pm \alpha x^p\} \tag{3.13}$$

satisfy $\Gamma_\pm \subset \varrho(G) \cap \overline{\Omega(\varphi)}$ and we have

$$\int_{\Gamma_\pm} |z|^p \|(G - z)^{-1} u\|^2 \,|dz| \leq C_1 \|u\|^2, \qquad \int_{\Gamma_\pm} |z|^{p-2} \|G(G - z)^{-1} u\|^2 \,|dz| \leq C_2 \|u\|^2$$

for all $u \in H$ with some constants $C_1, C_2 \geq 0$.

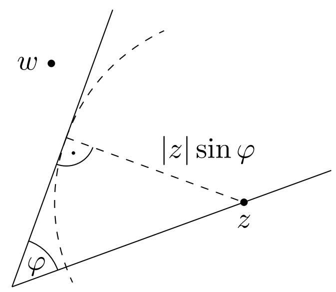

Figure 3.3: Two points separated by a sector

Proof. Since G is normal with compact resolvent, there is an orthonormal basis $(u_j)_{j\in\mathbb{N}}$ of eigenvectors with corresponding eigenvalues λ_j. For $u \in H$ and $z \in \varrho(G)$ we thus get

$$\|(G-z)^{-1}u\|^2 = \sum_j \frac{1}{|\lambda_j - z|^2} |(u|u_j)|^2.$$

We have $\Gamma_\pm \subset \overline{\Omega(\varphi)}$ if we choose r_0 large enough. Hence $\Gamma_\pm \subset \varrho(G)$,

$$\begin{aligned}\int_{\Gamma_\pm} |z|^p \|(G-z)^{-1}u\|^2 \,|dz| &= \sum_j \int_{\Gamma_\pm} \frac{|z|^p}{|\lambda_j - z|^2} \,|dz|\, |(u|u_j)|^2 \\ &\le \sup_j \int_{\Gamma_\pm} \frac{|z|^p}{|\lambda_j - z|^2} \,|dz| \cdot \|u\|^2\end{aligned}$$

and similarly

$$\int_{\Gamma_\pm} |z|^{p-2} \|G(G-z)^{-1}u\|^2 \,|dz| \le \sup_j \int_{\Gamma_\pm} \frac{|z|^{p-2}|\lambda_j|^2}{|\lambda_j - z|^2} \,|dz| \cdot \|u\|^2.$$

We need estimates for the differential forms dz, dx, dy: For $z = x + iy \in \Gamma_\pm$ and r_0 large enough we find

$$\begin{gathered}|dz|^2 = dx^2 + dy^2 = \big(1 + (\alpha p x^{p-1})^2\big)\, dx^2 \le 2\, dx^2 \quad \text{and} \\ x^2 \le |z|^2 = x^2 + \alpha^2 x^{2p} = \big(1 + \alpha^2 x^{2(p-1)}\big) x^2 \le 2x^2.\end{gathered}$$

Figure 3.3 shows that if two points w, z in the complex plane are separated by a sector of angle $\ge \varphi$, then $|w - z| \ge |z| \sin\varphi$ and also $|w - z| \ge |w| \sin\varphi$ by symmetry. So if $\lambda_j \notin \mathbb{R}_{>0}$, then $\lambda_j \notin \Omega(2\varphi)$ and we obtain $|\lambda_j - z| \ge \max\{|z|, |\lambda_j|\} \sin\varphi$ for $z \in \Gamma_\pm$. Hence

$$\int_{\Gamma_\pm} \frac{|z|^p}{|\lambda_j - z|^2} \,|dz| \le \frac{1}{\sin^2\varphi} \int_{\Gamma_\pm} |z|^{p-2} \,|dz| \le \frac{\sqrt{2}}{\sin^2\varphi} \int_{r_0}^\infty \frac{dx}{x^{2-p}} < \infty$$

as well as

$$\int_{\Gamma_\pm} \frac{|z|^{p-2}|\lambda_j|^2}{|\lambda_j - z|^2}\,|dz| \le \frac{1}{\sin^2\varphi}\int_{\Gamma_\pm} |z|^{p-2}\,|dz| < \infty.$$

If on the other hand $\lambda_j \in \mathbb{R}_{>0}$ and $z \in \Gamma_\pm$, then

$$|\lambda_j - z|^2 = (\lambda_j - x)^2 + (\alpha x^p)^2 \ge \min\{1,\alpha^2\}((\lambda_j - x)^2 + x^{2p}),$$

and it suffices to prove the two assertions

$$\sup_{t>0}\int_{r_0}^\infty \frac{x^p}{(x-t)^2 + x^{2p}}\,dx < \infty, \quad \sup_{t>0}\int_{r_0}^\infty \frac{t^2 x^{p-2}}{(x-t)^2 + x^{2p}}\,dx < \infty.$$

For $0 < t \le r_0/2$ we have

$$\int_{r_0}^\infty \frac{x^p\,dx}{(x-t)^2 + x^{2p}} \le \int_{r_0}^\infty \frac{x^p\,dx}{(x-\frac{1}{2}r_0)^2} \le \int_{r_0}^\infty \frac{x^p\,dx}{(x-\frac{1}{2}x)^2} = \int_{r_0}^\infty \frac{4\,dx}{x^{2-p}} < \infty,$$

$$\int_{r_0}^\infty \frac{t^2 x^{p-2}\,dx}{(x-t)^2 + x^{2p}} \le \int_{r_0}^\infty \frac{x^p\,dx}{(x-t)^2 + x^{2p}} \le \int_{r_0}^\infty \frac{4\,dx}{x^{2-p}} < \infty.$$

Using $1 \le t/x$ for $x \in [r_0/2, t]$ and $t/x \le 1$ for $x \in [t, \infty[$, we obtain for $t \ge r_0/2$

$$\int_{r_0}^\infty \frac{x^p\,dx}{(x-t)^2 + x^{2p}} \le \int_{r_0/2}^t \frac{t^2 x^{p-2}\,dx}{(x-t)^2 + x^{2p}} + \int_t^\infty \frac{x^p\,dx}{(x-t)^2 + x^{2p}},$$

$$\int_{r_0}^\infty \frac{t^2 x^{p-2}\,dx}{(x-t)^2 + x^{2p}} \le \int_{r_0/2}^t \frac{t^2 x^{p-2}\,dx}{(x-t)^2 + x^{2p}} + \int_t^\infty \frac{x^p\,dx}{(x-t)^2 + x^{2p}}.$$

For t_0 with $r_0/2 \le t \le t_0$ we get

$$\int_t^\infty \frac{x^p\,dx}{(x-t)^2 + x^{2p}} \le \int_{r_0/2}^{2t_0} \frac{dx}{x^p} + \int_{2t_0}^\infty \frac{x^p\,dx}{(x-t)^2} \le \int_{r_0/2}^{2t_0} \frac{dx}{x^p} + \int_{2t_0}^\infty \frac{4\,dx}{x^{2-p}} < \infty,$$

$$\int_{r_0/2}^t \frac{t^2 x^{p-2}\,dx}{(x-t)^2 + x^{2p}} \le \int_{r_0/2}^{t_0} \frac{t_0^2\,dx}{x^{2+p}} < \infty.$$

Thus it remains to be shown that

$$\limsup_{t\to\infty}\int_t^\infty \frac{x^p\,dx}{(x-t)^2 + x^{2p}} < \infty \quad \text{and} \quad \limsup_{t\to\infty}\int_{r_0/2}^t \frac{t^2 x^{p-2}\,dx}{(x-t)^2 + x^{2p}} < \infty.$$

Assuming $t + t^p \le 2t$, we have

$$\int_t^{t+t^p} \frac{x^p\,dx}{(x-t)^2 + x^{2p}} \le \int_t^{t+t^p} \frac{1}{x^p}\,dx \le \int_t^{t+t^p} \frac{1}{t^p}\,dx = 1,$$

$$\int_{t+t^p}^{2t} \frac{x^p\,dx}{(x-t)^2 + x^{2p}} \le (2t)^p \int_{t+t^p}^{2t} \frac{dx}{(x-t)^2} \le (2t)^p \int_{t+t^p}^\infty \frac{dx}{(x-t)^2} = 2^p,$$

$$\int_{2t}^\infty \frac{x^p\,dx}{(x-t)^2 + x^{2p}} \le \int_{2t}^\infty \frac{x^p\,dx}{(x-\frac{1}{2}x)^2} = 4\int_{2t}^\infty \frac{dx}{x^{2-p}} = \frac{4}{(1-p)(2t)^{1-p}},$$

which yields

$$\limsup_{t\to\infty} \int_t^\infty \frac{x^p\,dx}{(x-t)^2+x^{2p}} \le 1+2^p.$$

For $r_0/2 \le t/2 \le t-t^p$ we have the estimates

$$\int_{r_0/2}^{t/2} \frac{x^{p-2}\,dx}{(t-x)^2+x^{2p}} \le \left(\frac{2}{t}\right)^2 \int_{r_0/2}^{t/2} x^{p-2}\,dx \le \left(\frac{2}{t}\right)^2 \int_{r_0/2}^{\infty} \frac{dx}{x^{2-p}} = \frac{2^{3-p}}{t^2(1-p)r_0^{1-p}},$$

$$\int_{t/2}^{t-t^p} \frac{x^{p-2}\,dx}{(t-x)^2+x^{2p}} \le \left(\frac{t}{2}\right)^{p-2} \int_{t/2}^{t-t^p} \frac{dx}{(t-x)^2} = \left(\frac{t}{2}\right)^{p-2}\left(\frac{1}{t^p}-\frac{2}{t}\right) \le \frac{2^{2-p}}{t^2}\text{, and}$$

$$\int_{t-t^p}^{t} \frac{x^{p-2}\,dx}{(t-x)^2+x^{2p}} \le \int_{t-t^p}^{t} \frac{dx}{x^{2+p}} \le \int_{t-t^p}^{t} \frac{dx}{(t-t^p)^{2+p}} = \frac{t^p}{(t-t^p)^{2+p}}\,.$$

Therefore

$$\begin{aligned}\limsup_{t\to\infty} \int_{r_0/2}^{t} \frac{t^2x^{p-2}\,dx}{(x-t)^2+x^{2p}} &\le \frac{2^{3-p}}{(1-p)r_0^{1-p}} + 2^{2-p} + \limsup_{t\to\infty}\frac{t^{2+p}}{(t-t^p)^{2+p}}\\ &= \frac{2^{3-p}}{(1-p)r_0^{1-p}} + 2^{2-p} + 1\end{aligned}$$

and the proof is complete. □

Lemma 3.3.4 (Markus [36, Lemma 6.7]) *Let G be normal with compact resolvent and $\sigma(G)\cap\Omega(2\varphi)\subset\mathbb{R}_{\ge 0}$ with $0<\varphi\le\pi/2$. Let $(x_k)_{k\ge1}$ be a sequence of positive numbers, $0\le p<1$, and $\alpha, c_1, c_2>0$ such that $\alpha x_1^{p-1}\le\tan\varphi$ and*

$$x_n^{1-p}-x_k^{1-p}\ge c_1(n-k) \quad \textit{for} \quad n>k, \quad \operatorname{dist}(x_k,\sigma(G))\ge c_2x_k^p \quad \textit{for} \quad k\ge1.$$

Then the lines

$$\gamma_k = \left\{x_k+iy\in\mathbb{C}\,\middle|\,|y|\le\alpha x_k^p\right\} \tag{3.14}$$

satisfy $\gamma_k\subset\varrho(G)\cap\overline{\Omega(\varphi)}$ and we have

$$\sum_{k=1}^{\infty} x_k^p\int_{\gamma_k}\|(G-z)^{-1}u\|^2\,|dz|\le C_1\|u\|^2, \qquad \sum_{k=1}^{\infty} x_k^{p-2}\int_{\gamma_k}\|G(G-z)^{-1}u\|^2\,|dz|\le C_2\|u\|^2$$

for all $u\in H$ with some constants $C_1, C_2\ge 0$.

Proof. The assumptions on $(x_k)_k$ yield that the sequence is monotonically increasing and that $\gamma_k\subset\overline{\Omega(\varphi)}$; hence $\gamma_k\subset\varrho(G)$ for all k. Then, analogously to the previous proof,

$$\begin{aligned}\sum_{k=1}^{\infty} x_k^p\int_{\gamma_k}\|(G-z)^{-1}u\|^2\,|dz| &= \sum_{k=1}^{\infty} x_k^p\sum_j\int_{\gamma_k}\frac{|dz|}{|\lambda_j-z|^2}|(u|u_j)|^2\\ &= \sum_j\sum_{k=1}^{\infty} x_k^p\int_{\gamma_k}\frac{|dz|}{|\lambda_j-z|^2}|(u|u_j)|^2 \le \sup_j\sum_{k=1}^{\infty} x_k^p\int_{\gamma_k}\frac{|dz|}{|\lambda_j-z|^2}\cdot\|u\|^2\end{aligned}$$

holds; similarly

$$\sum_{k=1}^{\infty} x_k^{p-2} \int_{\gamma_k} \|G(G-z)^{-1}u\|^2 \,|dz| \le \sup_j \sum_{k=1}^{\infty} x_k^{p-2} \int_{\gamma_k} \frac{|\lambda_j|^2}{|\lambda_j - z|^2} \,|dz| \cdot \|u\|^2.$$

From the assumption on $(x_k)_k$ we conclude that $x_n^{1-p} \ge c_3 n$ for all $n \ge 1$ with $c_3 = \min\{c_1/2, x_1^{1-p}\}$. For $\lambda_j \notin \Omega(2\varphi)$ we obtain the estimates

$$\sum_{k=1}^{\infty} x_k^p \int_{\gamma_k} \frac{|dz|}{|\lambda_j - z|^2} \le \frac{1}{\sin^2\varphi} \sum_{k=1}^{\infty} x_k^p \frac{2\alpha x_k^p}{x_k^2} \le \frac{2\alpha}{c_3^2 \sin^2\varphi} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty,$$

$$\sum_{k=1}^{\infty} x_k^{p-2} \int_{\gamma_k} \frac{|\lambda_j|^2}{|\lambda_j - z|^2} \,|dz| \le \frac{1}{\sin^2\varphi} \sum_{k=1}^{\infty} x_k^{p-2} \cdot 2\alpha x_k^p \le \frac{2\alpha}{c_3^2 \sin^2\varphi} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Otherwise $\lambda_j \in \mathbb{R}_{>0}$ and we have

$$\sum_{k=1}^{\infty} x_k^p \int_{\gamma_k} \frac{|dz|}{|\lambda_j - z|^2} \le 2\alpha \sum_{k=1}^{\infty} \frac{x_k^{2p}}{(\lambda_j - x_k)^2} \qquad \text{and}$$

$$\sum_{k=1}^{\infty} x_k^{p-2} \int_{\gamma_k} \frac{|\lambda_j|^2}{|\lambda_j - z|^2} \,|dz| \le 2\alpha \sum_{k=1}^{\infty} x_k^{2p-2} \frac{\lambda_j^2}{(\lambda_j - x_k)^2}.$$

Now there exists $n \in \mathbb{N}$ with $x_n < \lambda_j < x_{n+1}$ (where we have put $x_0 = 0$). Then

$$|x_k - \lambda_j| \ge x_n - x_k \quad \text{for} \quad k < n, \qquad |x_k - \lambda_j| \ge x_k - x_{n+1} \quad \text{for} \quad k > n+1,$$
$$|x_n - \lambda_j| \ge c_2 x_n^p, \qquad \text{and} \qquad |x_{n+1} - \lambda_j| \ge c_2 x_{n+1}^p.$$

In addition, for $l > k$,

$$x_l - x_k \ge x_l^p (x_l^{1-p} - x_k^{1-p}) \ge c_1 x_l^p (l - k).$$

Using this, we obtain the estimates

$$\sum_{k=1}^{\infty} \frac{x_k^{2p}}{(\lambda_j - x_k)^2} \le \frac{2}{c_2^2} + \sum_{k<n} \frac{x_k^{2p}}{(x_n - x_k)^2} + \sum_{k>n+1} \frac{x_k^{2p}}{(x_k - x_{n+1})^2}$$
$$\le \frac{2}{c_2^2} + \sum_{k<n} \frac{x_k^{2p}}{c_1^2 x_n^{2p} (n-k)^2} + \sum_{k>n+1} \frac{1}{c_1^2 (k-n-1)^2} \le \frac{2}{c_2^2} + \frac{2}{c_1^2} \sum_{k=1}^{\infty} \frac{1}{k^2}$$

as well as

$$\sum_{k=1}^{\infty} \frac{\lambda_j^2 x_k^{2p-2}}{(\lambda_j - x_k)^2} \le \sum_{k\le n} \frac{\lambda_j^2 x_k^{2p-2}}{(\lambda_j - x_k)^2} + \sum_{k>n} \frac{x_k^{2p}}{(\lambda_j - x_k)^2}$$

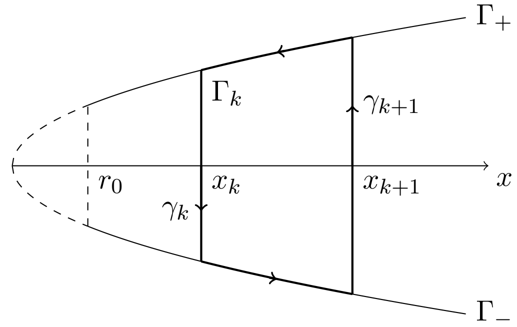

Figure 3.4: The boundary contour from Lemma 3.3.5

and

$$\sum_{k\le n}\frac{\lambda_j^2 x_k^{2p-2}}{(\lambda_j-x_k)^2}\le\sum_{x_k\le\lambda_j/2}\frac{\lambda_j^2 x_k^{2p-2}}{(\lambda_j/2)^2}+\sum_{\substack{k\le n\\ x_k>\lambda_j/2}}\frac{4x_k^{2p}}{(\lambda_j-x_k)^2}\le\frac{4}{c_3^2}\sum_{k=1}^{\infty}\frac{1}{k^2}+4\sum_{k=1}^{\infty}\frac{x_k^{2p}}{(\lambda_j-x_k)^2},$$

which complete the proof. □

With the previous resolvent estimates at hand, we derive an estimate for a sequence of Riesz projections associated with the parabola $\Gamma_\pm$, see Figure 3.4.

Lemma 3.3.5 *Let G be normal with compact resolvent, $\sigma(G)\cap\Omega(2\varphi)\subset\mathbb{R}_{\ge 0}$ with $0<\varphi\le\pi/2$, S p-subordinate to G with bound b, $0\le p<1$, and $T=G+S$.*

Let $\alpha>b$, let $(x_k)_{k\ge1}$, γ_k be as in Lemma 3.3.4, and suppose that there is a constant $M\ge 0$ such that

$$\gamma_k\subset\varrho(T)\quad\text{and}\quad\|S(T-z)^{-1}\|\le M\quad\text{for all}\quad z\in\gamma_k,\ k\ge 1.$$

Then there exist $r_0>0$, $k_0\ge 1$ such that $x_{k_0}\ge r_0$ and the following holds: If $\Gamma_\pm$ is as in (3.13) and Γ_k with $k\ge k_0$ is the positively oriented boundary contour of the region enclosed by $\gamma_k,\Gamma_-,\gamma_{k+1},\Gamma_+$, then $\Gamma_k\subset\varrho(T)$. If P_k is the Riesz projection of T associated with Γ_k, then

$$\sum_{k=k_0}^{\infty}|(P_k u|v)|\le C\|u\|\|v\|\quad\text{for all}\quad u,v\in H$$

with some constant $C\ge 0$.

Proof. We want to apply Lemmas 3.3.2, 3.3.3 and 3.3.4, and choose $\varepsilon\in\,]b/\alpha,1[$ and r_0 accordingly. The assumptions on $(x_k)_k$ imply that the sequence tends monotonically to infinity and we choose k_0 such that $x_{k_0} \geq r_0$. By Lemma 3.3.2, $\|S(T-z)^{-1}\|$ is uniformly bounded on $\Gamma_\pm$. We thus have

$$\Gamma_k \subset \varrho(G) \cap \varrho(T) \quad \text{and} \quad \|S(T-z)^{-1}\| \leq M_0 \quad \text{for all} \quad z \in \Gamma_k, k \geq k_0,$$

with some $M_0 \geq 0$. Consider now the Riesz projections Q_k of G associated with Γ_k, which are orthogonal since G is normal. In view of Remark 2.2.8 it suffices to prove

$$\sum_{k=k_0}^{\infty} \left|\left((P_k - Q_k)u\middle|v\right)\right| \leq C\|u\|\|v\|.$$

Now

$$P_k - Q_k = \frac{i}{2\pi}\int_{\Gamma_k} \left((T-z)^{-1} - (G-z)^{-1}\right) dz = \frac{-i}{2\pi}\int_{\Gamma_k} (T-z)^{-1}S(G-z)^{-1} dz$$

and hence

$$\left|\left((P_k - Q_k)u\middle|v\right)\right| \leq \frac{1}{2\pi}\int_{\Gamma_k} \|S(G-z)^{-1}u\|\|(T-z)^{-*}v\|\,|dz|.$$

Then, with the help of

$$\begin{aligned} & & G - z &= \left(I - S(T-z)^{-1}\right)(T-z) \\ \Longrightarrow & & (T-z)^{-1} &= (G-z)^{-1}\left(I - S(T-z)^{-1}\right) \\ \Longrightarrow & & (T-z)^{-*} &= \left(I - S(T-z)^{-1}\right)^{*}(G-z)^{-*} \\ \Longrightarrow & & \|(T-z)^{-*}v\| &\leq \Big(1 + \underbrace{\|S(T-z)^{-1}\|}_{\leq M_0}\Big)\|(G-z)^{-*}v\| \end{aligned}$$

and $\|(G-z)^{-*}v\| = \|(G-z)^{-1}v\|$ (since G is normal), we find

$$\begin{aligned} \sum_{k=k_0}^{\infty} \left|\left((P_k - Q_k)u\middle|v\right)\right| &\leq \frac{1+M_0}{2\pi}\sum_{k=k_0}^{\infty}\int_{\Gamma_k} \|S(G-z)^{-1}u\|\|(G-z)^{-1}v\|\,|dz| \\ &\leq \frac{1+M_0}{2\pi}\left(\int_{\Gamma_+} + \int_{\Gamma_-} + 2\sum_{k=k_0}^{\infty}\int_{\gamma_k}\right)\|S(G-z)^{-1}u\|\|(G-z)^{-1}v\|\,|dz|. \end{aligned}$$

Using p-subordination, Lemma 3.3.3, and (for $p \neq 0$) Hölder's inequality, we estimate

$$\begin{aligned} &\int_{\Gamma_\pm} \|S(G-z)^{-1}u\|\|(G-z)^{-1}v\|\,|dz| \\ &\qquad \leq \left(\int_{\Gamma_\pm} |z|^{-p}\|S(G-z)^{-1}u\|^2\,|dz|\right)^{1/2} \underbrace{\left(\int_{\Gamma_\pm} |z|^{p}\|(G-z)^{-1}v\|^2\,|dz|\right)}_{\leq C_1\|v\|^2}{}^{1/2}, \end{aligned}$$

$$\begin{aligned}
&\int_{\Gamma_\pm} |z|^{-p}\|S(G-z)^{-1}u\|^2\,|dz| \\
&\quad\le b^2\int_{\Gamma_\pm} |z|^{p(p-2)}\|G(G-z)^{-1}u\|^{2p}|z|^{p(1-p)}\|(G-z)^{-1}u\|^{2(1-p)}\,|dz| \\
&\quad\le b^2\left(\int_{\Gamma_\pm} |z|^{p-2}\|G(G-z)^{-1}u\|^2\,|dz|\right)^p\left(\int_{\Gamma_\pm} |z|^{p}\|(G-z)^{-1}u\|^2\,|dz|\right)^{1-p} \\
&\quad\le b^2C_2^pC_1^{1-p}\|u\|^2,
\end{aligned}$$

which yields

$$\int_{\Gamma_\pm} \|S(G-z)^{-1}u\|\|(G-z)^{-1}v\|\,|dz| \le b\sqrt{C_1^{2-p}C_2^p}\,\|u\|\|v\|.$$

In the same way, with Lemma 3.3.4, we see that

$$\begin{aligned}
&\sum_k\int_{\gamma_k} \|S(G-z)^{-1}u\|\|(G-z)^{-1}v\|\,|dz| \\
&\quad\le \sum_k\left(\int_{\gamma_k} x_k^{-p}\|S(G-z)^{-1}u\|^2\,|dz|\right)^{1/2}\left(\int_{\gamma_k} x_k^{p}\|(G-z)^{-1}v\|^2\,|dz|\right)^{1/2} \\
&\quad\le \left(\sum_k\int_{\gamma_k} x_k^{-p}\|S(G-z)^{-1}u\|^2\,|dz|\right)^{1/2}\underbrace{\left(\sum_k\int_{\gamma_k} x_k^{p}\|(G-z)^{-1}v\|^2\,|dz|\right)^{1/2}}_{\le C_1'\|v\|^2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_k\int_{\gamma_k} x_k^{-p}\|S(G-z)^{-1}u\|^2\,|dz| \\
&\quad\le b^2\sum_k\int_{\gamma_k} x_k^{p(p-2)}\|G(G-z)^{-1}u\|^{2p}x_k^{p(1-p)}\|(G-z)^{-1}u\|^{2(1-p)}\,|dz| \\
&\quad\le b^2\left(\sum_k\int_{\gamma_k} x_k^{p-2}\|G(G-z)^{-1}u\|^2\,|dz|\right)^p\left(\sum_k\int_{\gamma_k} x_k^{p}\|(G-z)^{-1}u\|^2\,|dz|\right)^{1-p} \\
&\quad\le b^2C_2'^pC_1'^{1-p}\|u\|^2.
\end{aligned}$$

□

We briefly review some facts about the determinant of operators [36, §2.5], see also [21, Chapter VII], [22, §IV.1] and [24, §III.4.3]. Let A be an operator of finite rank n in a Hilbert space, i.e. $\dim\mathcal{R}(A)=n$. The *determinant* of $I+A$ is defined by

$$\det(I+A)=\det\big((I+A)|_U\big) \tag{3.15}$$

where U is a finite dimensional, A-invariant subspace with $U^\perp \subset \ker A$. Such a subspace U always exists, and the value of the determinant does not depend on the choice of U.

Lemma 3.3.6 *Let* $A \in L(H)$ *with* $\dim \mathcal{R}(A) = n$. *Then*

(i) $|\det(I+A)| \le (1+\|A\|)^n$;

(ii) $I+A$ *is invertible if and only if* $\det(I+A) \neq 0$, *and in this case*

$$\|(I+A)^{-1}\| \le \frac{(1+\|A\|)^n}{|\det(I+A)|};$$

(iii) if the operator-valued function $B : \Omega \to L(H)$ *is analytic on a domain* $\Omega \subset \mathbb{C}$, *then* $z \mapsto \det(I+AB(z))$ *is analytic on* Ω *too.*

Sketch of the proof. The first two statements essentially follow from the relations

$$|\det(I+C)| = \prod_{j=1}^{m} s_j(I+C) \qquad \text{and} \qquad s_j(I+C) \le 1 + s_j(C),$$

where C is an $m \times m$-matrix and $s_j(C)$ denotes the singular values of the matrix. The third assertion is proved by approximating $B(z)$ in a neighbourhood of a point z_0 by a polynomial

$$B_0 + B_1(z-z_0) + \cdots + B_k(z-z_0)^k$$

and noting that the mapping $B \mapsto \det(I+AB)$ is uniformly continuous on sets of the form $\{B \in L(H) \,|\, \|B\| \le c\}$. □

In the proof of the next proposition, we need an auxiliary result from complex analysis, cf. [36, Lemma 1.6]:

Lemma 3.3.7 *Let* $U \subset \mathbb{C}$ *be a bounded, simply connected domain,* $F \subset U$ *compact,* z_0 *an interior point of* F, *and* $\eta > 0$. *Then there exists a constant* $C > 0$ *such that the following holds: If* $a, b \in \mathbb{C}$ *and* $f : aU + b \to \mathbb{C}$ *with* $f(az_0 + b) \neq 0$ *is holomorphic and bounded, then there is a set* $E \subset \mathbb{C}$ *being the union of finitely many discs with radii summing up to at most* $|a|\eta$ *such that*

$$|f(z)| \ge \frac{|f(az_0+b)|^{1+C}}{\|f\|_{aU+b,\infty}^{C}} \quad \textit{for all} \quad z \in (aF+b) \setminus E.$$

Proof. A proof for the special case of U and F being discs, $z_0 = 0$, $a = 1$, $b = 0$, and $f(0) = 1$ can be found in Levin [35, Theorem I.11]. The general form stated here is obtained from this particular case by means of a conformal mapping. □

The following proposition permits us to estimate the resolvent of the perturbed operator even close to its eigenvalues by artificially creating a gap in the spectrum of G. We denote by $N_+(r_1, r_2, G)$ the sum of the multiplicities of all the eigenvalues of G in the open interval $]r_1, r_2[$,

$$N_+(r_1, r_2, G) = \sum_{\lambda \in \sigma_p(G) \cap]r_1, r_2[} \dim \mathcal{L}(\lambda). \tag{3.16}$$

Proposition 3.3.8 (Markus [36, Lemma 5.6]) *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$, S p-subordinate to G with bound b, $0 \leq p < 1$, and $T = G + S$.*

Let $l > b$, $0 \leq l_0 < l - b$ and $\eta > 0$. Then there are constants $C_0, C_1, r_0 > 0$ such that for every $r \geq r_0$ there is a set $E_r \subset \mathbb{C}$ with the following properties:

(i) E_r is the union of finitely many discs with radii summing up to at most ηr^p.

(ii) For every $z \in \overline{\Omega(\varphi)} \setminus E_r$ with $|\operatorname{Re} z - r| \leq l_0 r^p$ we have

$$z \in \varrho(T) \quad \text{and} \quad \|(T - z)^{-1}\| \leq \frac{C_0 C_1^m}{r^p}, \quad \|S(T - z)^{-1}\| \leq C_0 C_1^m$$

where $m = N_+(r - lr^p, r + lr^p, G)$.

Proof. We choose $l_1 \in]l_0, l - b[$ and α, $\tilde{b}$ such that

$$b < \tilde{b} < \alpha < l - l_1.$$

Let $r \geq r_0$. We may assume that $r - lr^p > 0$ by choosing r_0 large enough. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of G in $\Delta_r =]r - lr^p, r + lr^p[$, $P_1, \dots, P_n$ the orthogonal projections onto the corresponding eigenspaces, and

$$K_r = \sum_{j=1}^{n} (\lambda_j - \widetilde{\lambda}_j) P_j \quad \text{with} \quad \widetilde{\lambda}_j = \begin{cases} r - lr^p & \text{if } \lambda_j < r, \\ r + lr^p & \text{if } \lambda_j \geq r. \end{cases}$$

Then $G_r = G - K_r$ is a normal operator with $\sigma(G_r) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ and $\Delta_r \subset \varrho(G_r)$. K_r has rank m and satisfies $\|K_r\| \leq lr^p$. Setting $P_0 = I - P_1 - \ldots - P_n$ and noting that $\lambda_j / \widetilde{\lambda}_j \leq r/(r - lr^p)$ for all j, we have

$$\begin{aligned} \|Gu\|^2 = \|GP_0 u\|^2 + \sum_{j=1}^{n} \lambda_j^2 \|P_j u\|^2 &= \|GP_0 u\|^2 + \sum_{j=1}^{n} \widetilde{\lambda}_j^2 \cdot \frac{\lambda_j^2}{\widetilde{\lambda}_j^2} \|P_j u\|^2 \\ \leq \|GP_0 u\|^2 + \sum_{j=1}^{n} \widetilde{\lambda}_j^2 \Big(\frac{r}{r - lr^p} \Big)^2 \|P_j u\|^2 &\leq \Big(\frac{r}{r - lr^p} \Big)^2 \|G_r u\|^2. \end{aligned}$$

Since $1 - lr^{p-1} \to 1$ as $r \to \infty$ and $b < \tilde{b}$, we conclude

$$\|Su\| \le b\|Gu\|^p\|u\|^{1-p} \le b\Big(\frac{1}{1 - lr^{p-1}}\Big)^p \|G_r u\|^p\|u\|^{1-p} \le \tilde{b}\|G_r u\|^p\|u\|^{1-p},$$

provided r_0 is sufficiently large. Thus S is p-subordinate to G_r with bound less or equal than $\tilde{b}$.

Next, we want to prove that

$$|x - r| \le l_1 r^p \quad \Rightarrow \quad]x - \alpha x^p, x + \alpha x^p[\subset \varrho(G_r) \tag{3.17}$$

for r_0 sufficiently large. Let $|x - r| \le l_1 r^p$. Since the function $x \mapsto x - \alpha x^p$ is monotonically increasing for large x, we have

$$x - \alpha x^p \ge r - l_1 r^p - \alpha\big(r - l_1 r^p\big)^p \ge r - l_1 r^p - \alpha r^p > r - lr^p,$$

r_0 large enough. Furthermore

$$x + \alpha x^p \le r + l_1 r^p + \alpha\big(r + l_1 r^p\big)^p \le r + lr^p,$$

where the last inequality holds if and only if

$$\alpha\big(1 + l_1 r^{p-1}\big)^p \le l - l_1,$$

and this is in turn satisfied for r_0 sufficiently large. We have thus shown

$$]x - \alpha x^p, x + \alpha x^p[\subset \Delta_r \subset \varrho(G_r).$$

In order to prove the proposition, we want to apply Lemma 3.3.7. We introduce the two sets

$$\begin{aligned} U_r &= \big\{x + iy \,\big|\, |x - r| < l_1 r^p,\ |y| < 4br^p\big\},\\ F_r &= \big\{x + iy \,\big|\, |x - r| \le l_0 r^p,\ |y| \le 3br^p\big\}. \end{aligned}$$

For r_0 sufficiently large we have $U_r \subset \Omega(\varphi)$. Using (3.17), we can apply Lemma 3.3.2 to $G_r + S$ with some $\varepsilon \in]\tilde{b}/\alpha, 1[$; we obtain $U_r \subset \varrho(G_r + S)$ and, for $z \in U_r$,

$$\operatorname{dist}(z, \sigma(G_r)) \ge lr^p - l_1 r^p > \alpha r^p$$

and

$$\|(G_r + S - z)^{-1}\| \le \frac{(1-\varepsilon)^{-1}}{\alpha r^p}, \quad \|S(G_r + S - z)^{-1}\| \le \frac{\varepsilon}{1 - \varepsilon}.$$

We set $d(z) = \det(I + K_r(G_r + S - z)^{-1})$. Then, with Lemma 3.3.6,

$$\begin{aligned} |d(z)| &\le \big(1 + \|K_r\|\|(G_r + S - z)^{-1}\|\big)^m\\ &\le \bigg(1 + lr^p\frac{(1-\varepsilon)^{-1}}{\alpha r^p}\bigg)^m = \bigg(1 + \frac{l(1-\varepsilon)^{-1}}{\alpha}\bigg)^m \end{aligned}$$

on U_r. For $z \in \varrho(T) \cap U_r$ the identity $T - z = (I + K_r(G_r + S - z)^{-1})(T - K_r - z)$ yields

$$I = \left(I + K_r(G_r + S - z)^{-1}\right)\left(I - K_r(T - z)^{-1}\right).$$

Applying Lemma 3.3.2 (now with $\alpha = 2b$ and $\varepsilon = 2/3$) to the operator T and $z_r = r + i \cdot 2br^p \in F_r$, we obtain

$$z_r \in \varrho(T) \quad \text{and} \quad \|(T - z_r)^{-1}\| \leq \frac{3}{2br^p}$$

and thus

$$\left|\frac{1}{d(z_r)}\right| = \left|\det\left(I - K_r(T - z_r)^{-1}\right)\right| \leq \left(1 + \frac{3l}{2b}\right)^m.$$

Since U_r, F_r, z_r are the images of U_1, F_1, z_1 under the affine linear transformation $z \mapsto r^p(z-1) + r$ and the mapping $z \mapsto d(z)$ is analytic, Lemma 3.3.7 is applicable: There is a constant $C > 0$ depending only on b, l_0, l_1 and η such that for every $r \geq r_0$ there exists a union E_r of discs with radii summing up to at most ηr^p and

$$|d(z)| \geq \left(1 + \frac{3l}{2b}\right)^{-m(1+C)} \left(1 + \frac{l(1-\varepsilon)^{-1}}{\alpha}\right)^{-mC} \quad \text{for all} \quad z \in F_r \setminus E_r.$$

Hence $I + K_r(G_r + S - z)^{-1}$ is invertible by Lemma 3.3.6. From

$$T - z = \left(I + K_r(G_r + S - z)^{-1}\right)(G_r + S - z)$$

we see that $z \in F_r \setminus E_r$ implies $z \in \varrho(T)$ and

$$\begin{aligned}
\|(T - z)^{-1}\| &\leq \|(G_r + S - z)^{-1}\| \cdot \left\|\left(I + K_r(G_r + S - z)^{-1}\right)^{-1}\right\| \\
&\leq \frac{(1-\varepsilon)^{-1}}{\alpha r^p}\left(1 + \frac{3l}{2b}\right)^{(1+C)m}\left(1 + \frac{l(1-\varepsilon)^{-1}}{\alpha}\right)^{(1+C)m} \leq \frac{C_0 C_1^m}{r^p}
\end{aligned}$$

with appropriate constants C_0, C_1 depending on b, l_0, l_1, η only. Accordingly we have

$$\begin{aligned}
\|S(T - z)^{-1}\| &\leq \|S(G_r + S - z)^{-1}\| \cdot \left\|\left(I + K_r(G_r + S - z)^{-1}\right)^{-1}\right\| \\
&\leq \frac{\varepsilon}{1-\varepsilon}\left(1 + \frac{3l}{2b}\right)^{(1+C)m}\left(1 + \frac{l(1-\varepsilon)^{-1}}{\alpha}\right)^{(1+C)m} \leq C_0 C_1^m.
\end{aligned}$$

Finally, we consider $z = x + iy \in \overline{\Omega(\varphi)}$ with $|x - r| \leq l_0 r^p$ and $|y| \geq 3br^p$. Using $1 + l_0 r^{p-1} \to 1$ as $r \to \infty$, we have

$$2bx^p \leq 2b\,(r + l_0 r^p)^p \leq 3br^p \leq |y|$$

for r_0 sufficiently large. Applying Lemma 3.3.2 (again with $\alpha = 2b$ and $\varepsilon = 2/3$), we obtain $z \in \varrho(T)$ and

$$\|(T - z)^{-1}\| \leq \frac{3}{|y|} \leq \frac{1}{br^p} \leq \frac{C_0 C_1^m}{r^p}, \quad \|S(T - z)^{-1}\| \leq 2 \leq C_0 C_1^m$$

for $C_0 \geq \max\{2, b^{-1}\}$ and $C_1 \geq 1$. □

Corollary 3.3.9 *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$, S p-subordinate to G with bound b, $0 \leq p < 1$, and $T = G + S$.*

Let $l > b$. Then there are constants $C_0, C_1, r_0 > 0$ such that for every $r \geq r_0$ there exists $x \in \mathbb{R}$ with the following properties:

(i) $|x - r| \leq (l-b)r^p/2$;

(ii) $z \in \overline{\Omega(\varphi)}$ *with* $\operatorname{Re} z = x$ *implies*

$$z \in \varrho(T), \quad \|(T-z)^{-1}\| \leq \frac{C_0 C_1^m}{r^p}, \quad \|S(T-z)^{-1}\| \leq C_0 C_1^m,$$

and

$$\operatorname{dist}(z, \sigma(G)) \geq \frac{l-b}{4m} r^p$$

where $m = N_+(r - lr^p, r + lr^p, G)$.

Proof. We apply the previous proposition with $l_0 = (l-b)/2$ and $\eta = l_0/2$. The sum of the diameters of the discs in E_r is at most $2\eta r^p = l_0 r^p$, and the interval

$$\widetilde{\Delta}_r = [r - l_0 r^p, r + l_0 r^p]$$

is of length $2l_0 r^p$ and contains at most m eigenvalues of G. If we remove from $\widetilde{\Delta}_r$ the projection of E_r onto the real axis and an open interval

$$\left]\lambda - \frac{l_0}{2m} r^p, \lambda + \frac{l_0}{2m} r^p\right[$$

for each $\lambda \in \sigma(G) \cap \widetilde{\Delta}_r$, then a non-empty set remains. Consequently, we can find $x \in \widetilde{\Delta}_r$ such that the line $\operatorname{Re} z = x$ does not intersect E_r and we have

$$\operatorname{dist}(x, \sigma(G)) \geq \frac{l_0}{2m} r^p.$$

□

Corollary 3.3.10 *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$, S p-subordinate to G with bound b, $0 \leq p < 1$, and $T = G + S$.*

Then for $l_0, q > 0$ there are constants $C_0, C_1, r_0 > 0$ such that for every $r \geq r_0$ the following holds: For every $z = x + iy$ with $|x - r| \leq l_0 r^p$, $|y| \leq 2bx^p$ there exists $q_1 \in]0, q[$ such that

$$|w - z| = q_1 r^p \quad \Longrightarrow \quad w \in \varrho(T), \quad \|(T-w)^{-1}\| \leq \frac{C_0 C_1^m}{r^p},$$

where $m = N_+(r - lr^p, r + lr^p, G)$ *with* $l = b + 2(l_0 + q)$.

Proof. We use Proposition 3.3.8 with $l = b + 2(l_0 + q)$, $l_0 + q$ replacing l_0, and $\eta = q/3$. For $|w - z| \le qr^p$ we have $|\arg w| \le \varphi$ (for r_0 large enough) and

$$|\operatorname{Re} w - r| \le l_0 r^p + qr^p = \frac{l-b}{2} r^p.$$

Now the sum of the diameters of the discs in E_r is at most $2\eta r^p < qr^p$. Hence there exists $q_1 \in]0, q[$ such that $w \notin E_r$ for $|w - z| = q_1 r^p$ and the claim is proved. □

Under certain assumptions on the distribution of the eigenvalues of G on the positive real axis, we now derive estimates for the Riesz projections associated with a sequence of regions that cover the interior of the parabola from Figure 3.4.

Proposition 3.3.11 *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\ge 0}$ with $0 < \varphi \le \pi/2$, S p-subordinate to G with bound b, $0 \le p < 1$, and $T = G + S$.*

Assume that there is a sequence $(r_k)_{k\ge1}$ of positive numbers tending monotonically to infinity and some $l > b$, $m \in \mathbb{N}$ such that

$$N_+(r_k - lr_k^p, r_k + lr_k^p, G) \le m \quad \textit{for all} \quad k \ge 1. \tag{3.18}$$

Then there are constants $C, r_0 > 0$, $\alpha > b$, and a sequence $(x_k)_{k\ge1}$ in $\mathbb{R}_{\ge0}$ tending monotonically to infinity such that the following holds:

(i) $z \in \overline{\Omega(\varphi)}$ with $\operatorname{Re} z = x_k$ implies $z \in \varrho(T)$, $\|(T - z)^{-1}\| \le C$.

(ii) The contours $\Gamma_\pm, \gamma_k$ from (3.13) and (3.14) satisfy $\Gamma_\pm, \gamma_k \subset \varrho(T)$.

(iii) If P_k are the Riesz projections of T associated with the regions enclosed by $\gamma_k, \Gamma_-, \gamma_{k+1}, \Gamma_+$, then

$$\sum_{k=1}^{\infty} |(P_k u|v)| \le C\|u\|\|v\| \quad \textit{for all} \quad u, v \in H.$$

Proof. Applying Corollary 3.3.9, we see that for every $k \ge k_0$, k_0 appropriate, there exists x_k with the following properties: We have

$$|x_k - r_k| \le \frac{l-b}{2} r_k^p, \quad \operatorname{dist}(x_k, \sigma(G)) \ge \frac{l-b}{4m} r_k^p,$$

and $z \in \overline{\Omega(\varphi)}$ with $\operatorname{Re} z = x_k$ implies

$$z \in \varrho(T), \quad \|(T-z)^{-1}\| \le \frac{C_0 C_1^m}{r_k^p}, \quad \|S(T-z)^{-1}\| \le C_0 C_1^m.$$

Then $x_k/r_k \to 1$ as $k \to \infty$ and we obtain

$$\operatorname{dist}(x_k, \sigma(G)) \ge c_2 x_k^p \quad \text{for} \quad k \ge k_0$$

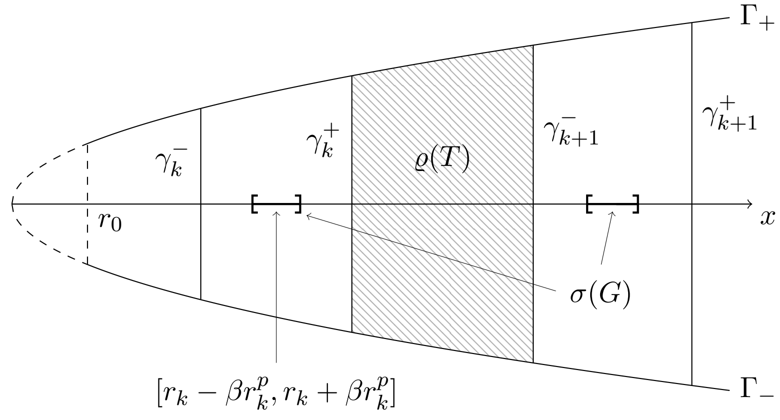

Figure 3.5: A large gap in $\sigma(G)$ yields a gap in $\sigma(T)$.

with $c_2 > 0$ and k_0 appropriately chosen. Since $x_k \to \infty$, for every k_1 there exists $k_2 > k_1$ such that

$$x_{k_2}^{1-p} - x_{k_1}^{1-p} \geq 1.$$

Passing to an appropriate subsequence, we can thus assume that

$$x_{k+1}^{1-p} - x_k^{1-p} \geq 1 \quad \text{for all} \quad k \in \mathbb{N},$$

which yields

$$x_n^{1-p} - x_k^{1-p} \geq n - k \quad \text{for} \quad n > k.$$

Now an application of Lemma 3.3.5 with $\alpha = 2b$ and the sequence $(x_k)_{k \geq k_0}$, k_0 large enough, completes the proof. □

If the spectrum of G has sufficiently large gaps on $\mathbb{R}_{\geq 0}$, then the spectrum of T has corresponding gaps (cf. Figure 3.5) and the associated Riesz projections P_k and Q_k of T and G, respectively, satisfy $\|P_k - Q_k\| < 1$; their ranges thus have the same dimension by Lemma 3.3.14.

Proposition 3.3.12 *Let G be normal with compact resolvent, $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$ with $0 < \varphi \leq \pi/2$, S p-subordinate to G with bound b, $0 \leq p < 1$, and $T = G + S$.*

Assume that there is a sequence $(r_k)_{k \geq 1}$ of nonnegative numbers tending monotonically to infinity and a constant $\beta \geq 0$ such that

$$\sigma(G) \cap \mathbb{R}_{\geq 0} \subset \bigcup_{k \geq 1} [r_k - \beta r_k^p, r_k + \beta r_k^p] \tag{3.19}$$

and

$$r_k + (\beta + \delta b) r_k^p \leq r_{k+1} - (\beta + \delta b) r_{k+1}^p \tag{3.20}$$

for almost all k with

$$\delta > \frac{4+\pi}{2\pi} + \sqrt{\frac{2\beta}{\pi b} + \Big(\frac{4+\pi}{2\pi}\Big)^2}. \tag{3.21}$$

Then for $\alpha > b$ and $\beta + \alpha < l \leq \beta + \delta b$ there are constants $C, r_0 > 0$, $k_0 \geq 1$ such that the following holds:

(i) The contours $\Gamma_\pm$ from (3.13) and

$$\gamma_k^\pm = \big\{x + iy \,\big|\, x = r_k \pm l r_k^p,\ |y| \leq \alpha x^p\big\} \quad \textit{with} \quad k \geq k_0$$

as well as the regions enclosed by $\gamma_k^+, \gamma_{k+1}^-, \Gamma_+, \Gamma_-$ belong to $\varrho(T)$, compare Figure 3.5.

(ii) $z \in \overline{\Omega(\varphi)}$ with $\operatorname{Re} z = r_k + l r_k^p$, $k \geq k_0$, implies $\|(T-z)^{-1}\| \leq C$.

(iii) If P_k and Q_k are the Riesz projections of T and G, respectively, associated with the region enclosed by $\gamma_k^-, \gamma_k^+, \Gamma_+, \Gamma_-$, then

$$\sum_{k=k_0}^{\infty} |(P_k u|v)| \leq C\|u\|\|v\| \quad \textit{for all} \quad u, v \in H$$

and

$$\|P_k - Q_k\| < 1 \quad \textit{for} \quad k \geq k_0.$$

Proof. We set $s_k^\pm = r_k \pm l r_k^p$. Then assumption (3.20) implies

$$r_k \leq s_k^+ \leq s_{k+1}^- \leq r_{k+1}.$$

Consider $s \in [s_k^+, s_{k+1}^-]$ with $k \geq k_0$. Then

$$s + \alpha s^p \leq s_{k+1}^- + \alpha r_{k+1}^p = r_{k+1} - (l-\alpha) r_{k+1}^p \leq r_{k+1} - \beta r_{k+1}^p.$$

Furthermore we have

$$s - \alpha s^p \geq s_k^+ - \alpha (s_k^+)^p$$

for k_0 large enough, since the left-hand side is monotonically increasing in s for s sufficiently large. In addition, the equivalent inequalities

$$s_k^+ - \alpha(s_k^+)^p \geq r_k + \beta r_k^p \;\Leftrightarrow\; l r_k^p - \alpha(r_k + l r_k^p)^p \geq \beta r_k^p \;\Leftrightarrow\; l - \beta \geq \alpha(1 + l r_k^{p-1})^p$$

hold for k_0 sufficiently large since $1 + l r_k^{p-1} \to 1$. Using the assumption on the spectrum of G, we have thus proved that, for $k \geq k_0$,

$$s \in [s_k^+, s_{k+1}^-] \quad \Rightarrow \quad]s - \alpha s^p, s + \alpha s^p[\subset \varrho(G).$$

With r_0 and k_0 appropriately chosen, Lemma 3.3.2 implies that the region enclosed by γ_k^+, γ_{k+1}^-, Γ^+, and Γ^- as well as the contours itself belong to $\varrho(T)$ for $k \geq k_0$. Moreover, $\|(T-z)^{-1}\|$ and $\|S(T-z)^{-1}\|$ are uniformly bounded for $z \in \overline{\Omega(\varphi)}$ with $\operatorname{Re} z = r_k + l r_k^p$, $k \geq k_0$. We also have $\operatorname{dist}(s_k^+, G) \geq \alpha (s_k^+)^p$ and

$$s_{k+1}^+ - s_k^+ = r_{k+1} - r_k + l(r_{k+1}^p - r_k^p) \geq (\beta + \delta b)(r_{k+1}^p + r_k^p) + l(r_{k+1}^p - r_k^p) \geq 2 l r_{k+1}^p.$$

The mean value theorem then yields

$$(s_{k+1}^+)^{1-p} - (s_k^+)^{1-p} \geq (1-p)(s_{k+1}^+)^{-p}(s_{k+1}^+ - s_k^+) \geq \frac{2l(1-p) r_{k+1}^p}{\left(r_{k+1} + l r_{k+1}^p\right)^p},$$

i.e., $(s_{k+1}^+)^{1-p} - (s_k^+)^{1-p} \geq l(1-p)$ for $k \geq k_0$, k_0 sufficiently large. We can thus apply Lemma 3.3.5 with $x_k = s_k^+$ to get the estimate for the sum of the Riesz projections.

To prove the final claim, we consider $c > 1$, choose $\varepsilon \in]0,1[$ such that

$$\frac{4+\pi}{2\pi} + \sqrt{\frac{2\beta}{\pi b} + \left(\frac{4+\pi}{2\pi}\right)^2} < \frac{1}{\varepsilon} < \delta, \tag{3.22}$$

and set

$$\alpha = c\frac{b}{\varepsilon} \quad \text{and} \quad l = \beta + c\alpha.$$

Then $\beta + \alpha < l \leq \beta + \delta b$ for c sufficiently near to 1. Let Γ_k be the positively oriented boundary contour of the region enclosed by γ_k^-, γ_k^+, Γ_+ and Γ_-. By the above calculations and Lemma 3.3.2 we have

$$\Gamma_k \subset \varrho(G) \cap \varrho(T), \quad \|S(G-z)^{-1}\| \leq \varepsilon, \quad \|(T-z)^{-1}\| \leq \frac{(1-\varepsilon)^{-1}}{\operatorname{dist}(z, \sigma(G))}$$

for $z \in \Gamma_k$, $k \geq k_0$, and thus

$$\begin{aligned} \|P_k - Q_k\| &= \frac{1}{2\pi}\left\| \int_{\Gamma_k} \left((T-z)^{-1} - (G-z)^{-1}\right) dz \right\| \\ &\leq \frac{1}{2\pi} \int_{\Gamma_k} \|(T-z)^{-1}\| \|S(G-z)^{-1}\| \, |dz| \leq \frac{\varepsilon}{2\pi(1-\varepsilon)} \int_{\Gamma_k} \frac{|dz|}{\operatorname{dist}(z, \sigma(G))}. \end{aligned}$$

For the integral over $\gamma_k^\pm$ we find

$$\int_{\gamma_k^\pm} \frac{|dz|}{\operatorname{dist}(z, \sigma(G))} \leq \frac{2\alpha (s_k^\pm)^p}{\alpha (s_k^\pm)^p} = 2.$$

For r_0 sufficiently large, the differential form dz can be estimated on $\Gamma_\pm$ by

$$|dz| \leq \sqrt{1 + (\alpha p x^{p-1})^2} \, dx \leq c \, dx \, ;$$

hence

$$\int_{\Gamma_\pm\cap\Gamma_k} \frac{|dz|}{\operatorname{dist}(z,\sigma(G))} \leq \int_{s_k^-}^{s_k^+} \frac{c\,dx}{\alpha x^p} \leq \frac{c(s_k^+ - s_k^-)}{\alpha s_-^p}$$
$$= \frac{2clr_k^p}{\alpha\big(r_k - lr_k^p\big)^p} = \frac{2cl}{\alpha\big(1 - lr_k^{p-1}\big)^p} \leq 2c^2\frac{l}{\alpha}$$

for k_0 sufficiently large. Putting it all together, we obtain

$$\|P_k - Q_k\| \leq \frac{\varepsilon}{2\pi(1-\varepsilon)}\left(4 + 4c^2\frac{l}{\alpha}\right) = \frac{2\varepsilon}{\pi(1-\varepsilon)}\left(1 + c^2\frac{\beta + c\alpha}{\alpha}\right)$$
$$= \frac{2\varepsilon}{\pi(1-\varepsilon)}\left(1 + c\frac{\beta + c^2 b\varepsilon^{-1}}{b\varepsilon^{-1}}\right) = \frac{2\varepsilon}{\pi(1-\varepsilon)}\Big(1 + c^3 + c\varepsilon\frac{\beta}{b}\Big).$$

Now (3.22) yields

$$\varepsilon < \left(-\frac{4+\pi}{2\pi} + \sqrt{\frac{2\beta}{\pi b} + \Big(\frac{4+\pi}{2\pi}\Big)^2}\right)\frac{\pi b}{2\beta} = -\frac{4+\pi}{4\beta b^{-1}} + \sqrt{\frac{\pi}{2\beta b^{-1}} + \Big(\frac{4+\pi}{4\beta b^{-1}}\Big)^2}$$
$$\Rightarrow \quad \Big(\varepsilon + \frac{4+\pi}{4\beta b^{-1}}\Big)^2 < \frac{\pi}{2\beta b^{-1}} + \Big(\frac{4+\pi}{4\beta b^{-1}}\Big)^2$$
$$\Rightarrow \quad \varepsilon^2 + \frac{4+\pi}{2\beta b^{-1}}\varepsilon < \frac{\pi}{2\beta b^{-1}} \quad \Rightarrow \quad \frac{2\beta}{b}\varepsilon^2 + (4+\pi)\varepsilon < \pi$$
$$\Rightarrow \quad \frac{2\beta}{b}\varepsilon^2 + 4\varepsilon < \pi(1-\varepsilon) \quad \Rightarrow \quad \frac{2\varepsilon}{\pi(1-\varepsilon)}\Big(2 + \varepsilon\frac{\beta}{b}\Big) < 1.$$

With c sufficiently near to 1 this implies

$$\frac{2\varepsilon}{\pi(1-\varepsilon)}\Big(1 + c^3 + c\varepsilon\frac{\beta}{b}\Big) < 1$$

and thus $\|P_k - Q_k\| < 1$. □

Remark 3.3.13 The constant in (3.21) is not optimal. Better estimates for the resolvent integrals along $\gamma_k^\pm$ and $\Gamma_\pm$ should yield a smaller constant.

If instead of (3.20) we only assume that

$$r_k + lr_k^p \leq r_{k+1} - lr_{k+1}^p \quad \text{for some} \quad l > \beta + \alpha,$$

then all assertions with the exception of $\|P_k - Q_k\| < 1$ still hold. ⌟

The next lemma is well known, see for example [21, Lemma II.4.3], [3, §34] and [15, Lemma 1.5.5].

Lemma 3.3.14 *Suppose that P and Q are two projections in a Banach space V with $\|P - Q\| < 1$. Then $V = \ker Q \oplus \mathcal{R}(P)$, and Q induces an isomorphism*

$$Q|_{\mathcal{R}(P)} : \mathcal{R}(P) \xrightarrow{\cong} \mathcal{R}(Q).$$

Proof. Let $u \in \ker Q \cap \mathcal{R}(P)$. Then $\|u\| = \|(P - Q)u\|$; as $\|P - Q\| < 1$, this is only possible for $u = 0$. Hence $\ker Q \cap \mathcal{R}(P) = \{0\}$ and $Q|_{\mathcal{R}(P)}$ is injective. A Neumann series argument shows that $I - Q + P$ is an isomorphism in V. Consequently, for every $v \in V$ there exists $u \in V$ such that

$$v = (I - Q + P)u = (I - Q)u + Pu.$$

This implies $V = \mathcal{R}(I - Q) + \mathcal{R}(P) = \ker Q + \mathcal{R}(P)$. Moreover, if $v \in \mathcal{R}(Q)$ then $v = Qv = QPu$. Hence $Q|_{\mathcal{R}(P)}$ maps onto $\mathcal{R}(Q)$ and the proof is complete. □

3.4 Perturbations of spectral l^2-decompositions

In this section we proof two general perturbation theorems for the non-normal operator $T = G + S$ where G is normal with compact resolvent and S is p-subordinate to G with $p < 1$. In Theorem 3.4.4 (and Proposition 3.4.1), which is a reformulation of a result of Markus and Matsaev [37], [36, Theorem 6.12], we assume that the eigenvalues of G lie on a finite number of rays from the origin and that the density of the eigenvalues has an appropriate asymptotic behaviour depending on p. Then T has a compact resolvent, almost all of its eigenvalues lie inside parabolas surrounding the rays, and T admits a finitely spectral l^2-decomposition.

In Theorem 3.4.7 we strengthen the assumptions on G by requiring that there are sequences of sufficiently large gaps in the spectrum on the rays. This allows us to control the multiplicities of the eigenvalues of T and, under an additional assumption, to obtain an l^2-decomposition of root subspaces; T is thus a spectral operator (cf. Theorem 2.3.17). This additional assumption is satisfied for example if almost all eigenvalues of G are simple, which reestablishes results due to Kato [24, Theorem V.4.15a], Dunford and Schwartz [20, Theorem XIX.2.7], and Clark [11]. Moreover, the additional assumption also holds in cases where the eigenvalues of G have multiplicity greater than one, provided we have a priori knowledge about the separation of the eigenvalues of T; see Theorem 4.4.5 for an application.

Both theorems also hold under weaker assumptions: It suffices for G to be an operator with compact resolvent and a Riesz basis of Jordan chains whose eigenvalues lie inside certain parabolas around rays from the origin, see Remark 3.4.14. With Proposition 3.4.5 we apply the theory to diagonally dominant block operator matrices.

We start by investigating how the shape of the spectrum changes under a p-subordinate perturbation. Note that we do not need the compactness of the resolvent of G here.

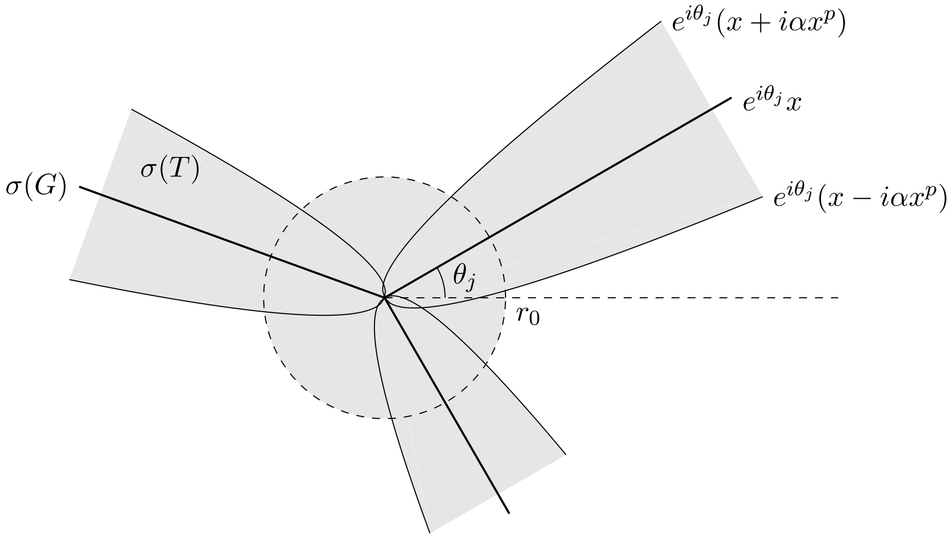

Figure 3.6: The spectrum after a p-subordinate perturbation

Proposition 3.4.1 *Let G be a normal operator on a Hilbert space whose spectrum lies on finitely many rays $e^{i\theta_j}\mathbb{R}_{\geq 0}$ with $0 \leq \theta_j < 2\pi$, $j = 1, \dots, n$. Let $T = G + S$ where S is p-subordinate to G with bound b and $0 \leq p < 1$. Then for every $\alpha > b$ there exists $r_0 > 0$ such that*

$$\sigma(T) \subset B_{r_0}(0) \cup \bigcup_{j=1}^{n} \bigl\{e^{i\theta_j}(x + iy) \bigm| x \geq 0,\, |y| \leq \alpha x^p\bigr\}, \tag{3.23}$$

cf. Figure 3.6. If G has compact resolvent, then so has T.

Proof. Without loss of generality, we assume $\theta_1 < \theta_2 < \ldots < \theta_n$ and set $\theta_0 = \theta_n - 2\pi$, $\theta_{n+1} = \theta_0 + 2\pi$. Then we may, after a rotation by θ_j, apply Lemma 3.3.2 to each sector $\Omega(\theta_{j-1}, \theta_{j+1})$. More precisely, we apply the lemma to the operators $e^{-i\theta_j}G$, $e^{-i\theta_j}S$, $e^{-i\theta_j}T$ with $\varphi_+ = (\theta_{j+1} - \theta_j)/2$, $\varphi_- = (\theta_{j-1} - \theta_j)/2$, and some suitable ε. For $z \in \sigma(T)$ this yields the implication

$$\begin{aligned} &\frac{\theta_{j-1} + \theta_j}{2} \leq \arg z \leq \frac{\theta_j + \theta_{j+1}}{2}, \quad |z| \geq r_0 \\ \Longrightarrow\quad &z \in \bigl\{e^{i\theta_j}(x + iy) \bigm| x \geq 0,\, |y| \leq \alpha x^p\bigr\} \end{aligned}$$

with some $r_0 \geq 0$ for each $j = 1, \dots, n$. If G has compact resolvent, the identity

$$(T - z)^{-1} = (G - z)^{-1}\bigl(I + S(G - z)^{-1}\bigr)^{-1} \quad \text{for} \quad z \in \varrho(G) \cap \varrho(T)$$

implies that T has compact resolvent too. $\square$

The statement about the asymptotic shape of the spectrum of T can be refined as follows:

Remark 3.4.2 To obtain a condition for $z \in \varrho(T)$, we consider without loss of generality the case $\sigma(G) \cap \Omega(2\varphi) \subset \mathbb{R}_{\geq 0}$, $0 < \varphi \leq \pi/2$, and $z = x + iy \in \overline{\Omega(\varphi)}$. Then $\operatorname{dist}(z, \sigma(G)) \geq |y|$ and, in view of Lemma 3.3.1, $b(1+|z|/|y|)^p |y|^{p-1} < 1$ is sufficient to get $z \in \varrho(T)$. For $p > 0$ this leads to the condition

$$x < \left(\frac{|y|}{b}\right)^{1/p} \sqrt{1 - 2b^{1/p}|y|^{1-1/p}},$$

which is asymptotically better than $x < (|y|/\alpha)^{1/p}$ since $1 - 2b^{1/p}|y|^{1-1/p} \to 1$ as $|y| \to \infty$. For $p = 0$ we obtain the optimal condition $b < |y|$.

For $p > 0$, the estimates of Markus [36, Lemma 5.2] lead to asymptotics which are even slightly better. Also note that simply taking the limit $\alpha \to b$ in Proposition 3.4.1 is not possible since then also $r_0 \to \infty$. ⌟

Recall that we denote by $N_+(r_1, r_2, G)$ the sum of the multiplicities of the eigenvalues of G in the interval $]r_1, r_2[$, see (3.16). Similarly, we write

$$N(r, G) = \sum_{\lambda \in \sigma_p(G) \cap \overline{B_r(0)}} \dim \mathcal{L}(\lambda) \tag{3.24}$$

for the sum of the multiplicities of all the eigenvalues λ with $|\lambda| \leq r$ and

$$N(K, G) = \sum_{\lambda \in \sigma_p(G) \cap K} \dim \mathcal{L}(\lambda) \quad \text{for every set} \quad K \subset \mathbb{C}. \tag{3.25}$$

Lemma 3.4.3 *If $n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a monotonically increasing function with*

$$\liminf_{r \to \infty} n(r) r^{p-1} < \infty \quad \textit{for some} \quad 0 \leq p < 1,$$

then

$$\liminf_{r \to \infty} \big(n(r + lr^p) - n(r - lr^p)\big) < \infty \quad \textit{for every} \quad l > 0.$$

Proof. Consider the case $p = 0$ first. If

$$\liminf_{r \to \infty} \big(n(r + l) - n(r - l)\big) = \infty \quad \text{for some} \quad l > 0,$$

then for every $a > 0$ there exists $r_0 \geq 0$ such that $n(r + 2l) - n(r) \geq a$ for all $r \geq r_0$. This implies $n(r_0 + 2kl) - n(r_0) \geq ka$ for $k \in \mathbb{N}$. Since for $r \geq r_0$ there exists $k \in \mathbb{N}$ such that $r - r_0 \in [2kl, 2(k+1)l]$, we deduce

$$\frac{n(r)}{r} \geq \frac{n(r_0 + 2kl)}{r_0 + 2(k+1)l} \geq \frac{ka}{r_0 + 2(k+1)l} \to \frac{a}{2l} \quad \text{as} \quad k \to \infty, \quad \text{i.e.} \quad r \to \infty.$$

Consequently $\liminf_{r\to\infty} n(r)r^{-1} = \infty$ since a was arbitrary.

For the case $p > 0$, we set $m(r) = n(r^{1/(1-p)})$ so that the assumption now reads $\liminf_{r\to\infty} m(r)r^{-1} < \infty$; therefore $\liminf_{r\to\infty}\big(m(r+2l) - m(r)\big) < \infty$ for every $l > 0$. Going back to n, this yields

$$\liminf_{r\to\infty}\Big(n\big((r^{1-p}+l)^{\frac{1}{1-p}}\big) - n(r)\Big) < \infty \quad \text{for every} \quad l > 0.$$

Since $1/(1-p) \geq 1$, we have

$$\big(r^{1-p}+l\big)^{\frac{1}{1-p}} = r\big(1+lr^{p-1}\big)^{\frac{1}{1-p}} \geq r\big(1+lr^{p-1}\big) = r + lr^p$$

and hence

$$\liminf_{r\to\infty}\big(n(r+lr^p) - n(r)\big) < \infty \quad \text{for every} \quad l > 0.$$

Now we set $s = r - lr^p$. Then $r + lr^p = s + 2lr^p \leq s + 3ls^p$ for r sufficiently large and thus

$$n(r+lr^p) - n(r-lr^p) \leq n(s+3ls^p) - n(s),$$

which proves the claim. □

We can now state the first perturbation theorem due to Markus and Matsaev [37], [36, Theorem 6.12].

Theorem 3.4.4 (Markus-Matsaev) *Let G be a normal operator with compact resolvent whose spectrum lies on a finite number of rays from the origin. Let S be p-subordinate to G with $0 \leq p < 1$. If*

$$\liminf_{r\to\infty}\frac{N(r,G)}{r^{1-p}} < \infty, \tag{3.26}$$

then $T = G + S$ admits a finitely spectral l^2-decomposition.

Proof. Let $e^{i\theta_j}\mathbb{R}_{\geq 0}$ with $0 \leq \theta_1 < \ldots < \theta_n < 2\pi$ be the rays containing the eigenvalues of G and let S be p-subordinate to G with bound b. From Proposition 3.4.1 we know that T has compact resolvent and that almost all of its eigenvalues lie inside sectors of the form

$$\Omega_j = \big\{z\in\mathbb{C}\,\big|\,|\arg z - \theta_j| < \psi_j\big\} \quad \text{with} \quad 0 < \psi_j \leq \frac{\pi}{4},$$

where the ψ_j can be chosen such that these sectors are disjoint. Lemma 3.3.2 shows that $\|(T-z)^{-1}\|$ is uniformly bounded for $z \notin \Omega_1\cup\ldots\cup\Omega_n$, $|z| \geq r_0$. Moreover, using the assumption on $N(r,G)$ and the previous lemma, for each sector Ω_j there is a sequence $(r_{jk})_{k\geq 1}$ of positive numbers tending monotonically to infinity such that

$$\sup_k N_+(r_{jk} - 2br_{jk}^p, r_{jk} + 2br_{jk}^p, e^{-i\theta_j}G) < \infty.$$

From Proposition 3.3.11 we thus obtain a corresponding sequence $(x_{jk})_{k\geq 1}$ such that $\|(T-z)^{-1}\|$ is uniformly bounded for $z \in \Omega_j$, $\mathrm{Re}(e^{-i\theta_j}z) = x_{jk}$. Let P be the Riesz projection associated with the set of those finitely many eigenvalues of T which are not contained in the sectors Ω_j. We can then apply Proposition 3.1.3 to the operator $T|_{\mathcal{R}(I-P)}$ and conclude that the system of root subspaces of T is dense in H.

Furthermore, if $(P_{jk})_{k\geq 1}$ are the Riesz projections from Proposition 3.3.11 corresponding to the eigenvalues $\lambda \in \Omega_j$ of T with $\mathrm{Re}(e^{-i\theta_j}\lambda) > x_{j1}$ and P_0 is the Riesz projection for the (finitely many) remaining ones, then

$$|(P_0u|v)| + \sum_{j=1}^{n}\sum_{k=1}^{\infty} |(P_{jk}u|v)| \leq C\|u\|\|v\|$$

with some constant $C \geq 0$. Now Proposition 2.2.7 shows that the family of projections P_0, $(P_{jk})_{j,k}$ generates an l^2-decomposition and the proof is complete in view of Proposition 2.3.8 and Definition 2.3.13. □

We apply Theorem 3.4.4 to a class of diagonally dominant block operator matrices. Let V_1, V_2 be Banach spaces and consider operators $A(V_1 \to V_1)$, $B(V_2 \to V_1)$, $C(V_1 \to V_2)$ and $D(V_2 \to V_2)$. Then the matrix

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3.27}$$

is called a *block operator matrix* on $V_1 \times V_2$. It induces an operator on $V_1 \times V_2$ which is also denoted by T:

$$\mathcal{D}(T) = (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(B) \cap \mathcal{D}(D)),$$
$$T\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Au + Bv \\ Cu + Dv \end{pmatrix} \quad \text{for} \quad \begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(T).$$

An arbitrary operator $T(V_1 \times V_2 \to V_1 \times V_2)$ can be represented by a block operator matrix if and only if its domain of definition is a Cartesian product $\mathcal{D}(T) = W_1 \times W_2$ with $W_j \subset V_j$. The representing matrix is in general not unique. For example, the operator A can be replaced by any extension of $A|_{\mathcal{D}(A)\cap\mathcal{D}(C)}$ without altering the operator induced by the matrix. Also note that if A, B, C and D are densely defined, this does not imply that T is densely defined too. For many results about the spectral theory of block operator matrices we refer the reader to the monograph of Tretter [49].

The concept of a *diagonally dominant* block operator matrix was introduced by Tretter [48]: The matrix from (3.27) with closable operators A, B, C, D is called diagonally dominant if C is relatively bounded with respect to A and B is relatively bounded with respect to D.

Proposition 3.4.5 *Let $A(H_1 \to H_1)$ and $D(H_2 \to H_2)$ be normal operators with compact resolvent on Hilbert spaces such that the spectra of A and D lie on finitely many rays from the origin and*

$$\liminf_{r\to\infty} \frac{N(r,A)}{r^{1-p}} < \infty, \quad \liminf_{r\to\infty} \frac{N(r,D)}{r^{1-p}} < \infty$$

with $0 \le p < 1$. Suppose that the operators $C(H_1 \to H_2)$ and $B(H_2 \to H_1)$ are p-subordinate[1] *to A and D, respectively,*

$$\|Cu\| \le b\|u\|^{1-p}\|Au\|^p \quad \text{for} \quad u \in \mathcal{D}(A) \subset \mathcal{D}(C),$$
$$\|Bv\| \le b\|v\|^{1-p}\|Dv\|^p \quad \text{for} \quad v \in \mathcal{D}(D) \subset \mathcal{D}(B).$$

Then the block operator matrix T from (3.27) has a compact resolvent, admits a finitely spectral l^2-decomposition, and for every $\alpha > b$ there is a constant $r_0 \ge 0$ such that

$$\sigma(T) \subset B_{r_0}(0) \cup \bigcup_{j=1}^{n} \big\{ e^{i\theta_j}(x+iy) \,\big|\, x \ge 0, |y| \le \alpha x^p \big\}.$$

Here $\theta_1, \dots, \theta_n$ with $0 \le \theta_j < 2\pi$ are the angles of the rays on which the spectra of A and D lie.

Proof. We decompose T as

$$T = G + S \quad \text{with} \quad G = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \quad S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

and want to apply Theorem 3.4.4 to this decomposition. First, it is clear from the assumptions on A and D that G is normal with compact resolvent. For its spectrum we have

$$\sigma(G) = \sigma(A) \cup \sigma(D) \quad \text{and} \quad N(r,G) = N(r,A) + N(r,D).$$

In particular, the spectrum of G lies on finitely many rays from the origin.

As a second step, we show that S is p-subordinate to G. Using Hölder's inequality and the p-subordination of C to A and B to D, we find

$$\left\| S \begin{pmatrix} u \\ v \end{pmatrix} \right\|^2 = \|Bv\|^2 + \|Cu\|^2 \le b^2\|v\|^{2(1-p)}\|Dv\|^{2p} + b^2\|u\|^{2(1-p)}\|Au\|^{2p}$$
$$\le b^2 \left(\|u\|^2 + \|v\|^2\right)^{1-p} \left(\|Au\|^2 + \|Dv\|^2\right)^p$$

for $u \in \mathcal{D}(A)$, $v \in \mathcal{D}(D)$. Consequently

$$\|Sw\| \le b\|w\|^{1-p}\|Gw\|^p \quad \text{for} \quad w \in \mathcal{D}(G) = \mathcal{D}(A) \times \mathcal{D}(D).$$

[1]This notion of p-subordination is more general than the one from Definition 3.2.1, since the operators B and C map from one Hilbert space into a (possibly) different one.

So all the conditions of Theorem 3.4.4 are fulfilled and the existence of the finitely spectral l^2-decomposition follows. Proposition 3.4.1 yields the compactness of the resolvent of T and the assertion about the shape of its spectrum. $\square$

Lemma 3.4.6 *Consider a sequence $(r_k)_{k\in\mathbb{N}}$ of positive numbers satisfying*

$$r_{k+1} - r_k \geq 2ar_k^p$$

with $a > 0$ and $0 \leq p < 1$. Then for $l > 0$ there exists $r_0 > 0$ such that $r \geq r_0$ with

$$r - lr^p \leq r_k < r_{k+1} < \ldots < r_{k+n} \leq r + lr^p$$

implies $n \leq 2l/a$.

Proof. By assumption on the sequence we have

$$r_{k+n} - r_k \geq 2nar_k^p, \quad \text{i.e.} \quad n \leq \frac{r_{k+n} - r_k}{2ar_k^p}.$$

Hence for r as in the assertion,

$$n \leq \frac{r + lr^p - (r - lr^p)}{2a\big(r - lr^p\big)^p} = \frac{l}{a\big(1 - lr^{p-1}\big)^p} \leq \frac{2l}{a},$$

provided r_0 is large enough. $\square$

Strengthening the assumptions on the spectrum of G, we obtain our second perturbation theorem. It extends results due to Kato [24, Theorem V.4.15a], Dunford and Schwartz [20, Theorem XIX.2.7], and Clark [11] since the case of multiple eigenvalues of G and clusters of eigenvalues is handled here too. Note that in [20] and [11], instead of the p-subordination of S to G the stronger assumption of the boundedness of SG^{-p} is made, compare Remark 3.2.5.

Theorem 3.4.7 *Let $G(H \to H)$ be a normal operator with compact resolvent and $S(H \to H)$ p-subordinate to G with bound b and $0 \leq p < 1$. Suppose that the spectrum of G lies on certain sequences of line segments on rays from the origin,*

$$\sigma(G) \subset \bigcup_{j=1}^{n} \bigcup_{k\geq 1} L_{jk}, \quad L_{jk} = \big\{e^{i\theta_j}x \,\big|\, x \geq 0,\, |x - r_{jk}| \leq \beta r_{jk}^p\big\}, \tag{3.28}$$

where $\beta \geq 0$, $0 \leq \theta_1 < \ldots < \theta_n < 2\pi$, and $(r_{jk})_{k\geq 1}$ are monotonically increasing sequences of nonnegative numbers such that

$$r_{jk} + (\beta + \delta b)r_{jk}^p \leq r_{j,k+1} - (\beta + \delta b)r_{j,k+1}^p \tag{3.29}$$

for almost all k, and δ is such that

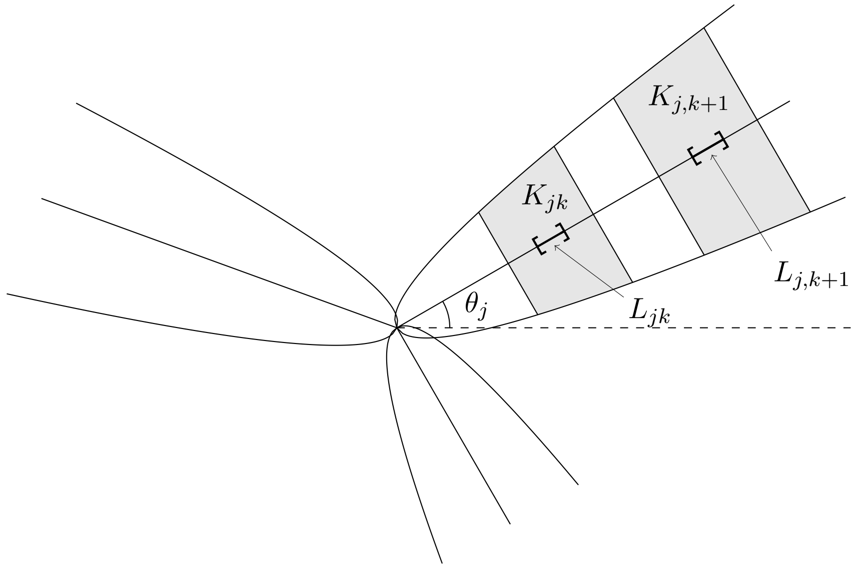

Figure 3.7: The situation of Theorem 3.4.7

$$\delta > \frac{4+\pi}{2\pi} + \sqrt{\frac{2\beta}{\pi b} + \Big(\frac{4+\pi}{2\pi}\Big)^2}. \tag{3.30}$$

Then $T = G+S$ has compact resolvent; for every $\alpha \in]b, \delta b]$ almost all eigenvalues of T lie inside the regions

$$K_{jk} = \big\{ e^{i\theta_j}(x+iy) \,\big|\, x \geq 0,\ |x - r_{jk}| \leq (\beta+\alpha) r_{jk}^p,\ |y| \leq \alpha x^p \big\},$$

$j = 1, \ldots, n$, $k \geq 1$ (cf. Figure 3.7); the spectral subspaces corresponding to the K_{jk} together with the subspace corresponding to $\sigma(T) \setminus \bigcup_{j,k} K_{jk}$ form a finitely spectral l^2-decomposition for T; and we have

$$N(L_{jk}, G) = N(K_{jk}, T) \quad \textit{for almost all pairs } (j,k).$$

Moreover, if there are constants $m, q > 0$ such that for almost all pairs (j,k) the assertions

(i) $N(L_{jk}, G) \leq m$ *and*

(ii) $\lambda_1, \lambda_2 \in \sigma(T) \cap K_{jk},\ \lambda_1 \neq \lambda_2 \quad \Rightarrow \quad |\lambda_1 - \lambda_2| > q r_{jk}^p$

hold, then the root subspaces of T form an l^2-decomposition of H.

Proof. We apply Propositions 3.4.1 and, for each ray, 3.3.12 with α replaced by $\widetilde{\alpha} = (\alpha + b)/2$ and $l = \beta + \alpha$. This shows that T has compact resolvent and that almost all eigenvalues of T lie inside regions

$$\left\{e^{i\theta_j}(x+iy)\,\middle|\, x\geq 0,\, |x-r_{jk}|<lr_{jk}^p,\, |y|<\widetilde{\alpha}x^p\right\}\subset K_{jk}.$$

As in the proof of Theorem 3.4.4 we have that $\|(T-z)^{-1}\|$ is uniformly bounded outside certain disjoint sectors Ω_j around the rays for $|z|$ large enough. For each ray, Proposition 3.3.12 yields a sequence $(x_{jk})_{k\in\mathbb{N}}$ tending monotonically to infinity such that $\|(T-z)^{-1}\|$ is bounded for $z\in\Omega_j$, $\operatorname{Re}(e^{-i\theta_j}z)=x_{jk}$. Consequently, Proposition 3.1.3 implies that the system of root subspaces of T is dense in H. Moreover, we have

$$|(P_0u|v)|+\sum_{j=1}^{n}\sum_{k=1}^{\infty}|(P_{jk}u|v)|\leq C\|u\|\|v\|$$

for some $C\geq 0$ where P_{jk} is the Riesz projection associated with K_{jk} and P_0 the one associated with $\sigma(T)\setminus\bigcup_{jk}K_{jk}$; Propositions 2.2.7 and 2.3.8 yield the finitely spectral l^2-decomposition. Finally, if Q_{jk} is the spectral projection of G associated with L_{jk}, then $\|P_{jk}-Q_{jk}\|<1$ and Lemma 3.3.14 implies the statement about the equality of the sums of the eigenvalue multiplicities.

Now suppose that with $m,q>0$ the additional assumptions (i) and (ii) hold for almost all pairs (j,k). We aim to show that the root subspaces corresponding to the eigenvalues of T in K_{jk} form an l^2-decomposition of $\mathcal{R}(P_{jk})$ with constant c independent of (j,k). Without loss of generality we may assume

$$\theta_j=0,\quad q\leq b,\quad\text{and}\quad \alpha\leq\min\{2,\delta-1\}b.$$

We want to apply Corollary 3.3.10 with $l_0=\beta+\alpha$ and set l accordingly. Due to the previous lemma, the number of elements r_{jk} in the interval $[r-lr^p,r+lr^p]$ is at most $2l/(\beta+\delta b)$ for r sufficiently large. Hence there is a constant m_0 such that

$$N_+(r-lr^p,r+lr^p,G)\leq m_0\quad\text{for } r \text{ sufficiently large.}$$

Let λ be an eigenvalue of T in K_{jk}. By Corollary 3.3.10 there exists $q_1\in\,]0,q[$ such that the points w on the circle around λ with radius $q_1r_{jk}^p$ satisfy $\|(T-w)^{-1}\|\leq C_0C_1^{m_0}r_{jk}^{-p}$. In addition, the circle lies inside the strip $|\operatorname{Re}z-r_{jk}|\leq(\beta+\delta b)r_{jk}^p$ and assumption (ii) thus implies that λ is the only possible eigenvalue of T inside that circle. Therefore, the Riesz projection P_λ for λ satisfies

$$\|P_\lambda\|\leq 2\pi q_1r_{jk}^p\frac{C_0C_1^{m_0}}{r_{jk}^p}\leq 2\pi qC_0C_1^{m_0}.$$

If $\lambda_1,\dots,\lambda_{m_1}$ are the eigenvalues of T in K_{jk}, we have $m_1\leq N(K_{jk},T)\leq m$ and conclude

$$\sum_{s=1}^{m_1}|(P_{\lambda_s}u|v)|\leq 2\pi mqC_0C_1^{m_0}\|u\|\|v\|.$$

According to Proposition 2.2.7, the subspaces $\mathcal{R}(P_{\lambda_s})$, $s = 1, \ldots, m_1$, form an l^2-decomposition of $\mathcal{R}(P_{jk})$ with constant c independent of k. This is true for each ray, and hence an application of Lemma 2.1.10 shows that the root subspaces of T form an l^2-decomposition. □

Remark 3.4.8 If almost all eigenvalues of G are simple and almost all line segments L_{jk} contain one eigenvalue only, then Theorem 3.4.7 yields a Riesz basis of eigenvectors and finitely many Jordan chains for T. Indeed in this case $N(K_{jk}, T) = 1$ for almost pairs (j, k). Hence, almost all subspaces of the finitely spectral l^2-decomposition for T are one-dimensional and Lemma 2.3.15 implies that T has a Riesz basis of eigenvectors and finitely many Jordan chains. ⌟

The next lemma implies that the spectral conditions in Theorem 3.4.7 are stronger than those of Theorem 3.4.4 if $N(L_{jk}, G)$ is bounded. Note that, in contrast to Theorem 3.4.7, the case of $\theta_{j_1} = \theta_{j_2}$ for $j_1 \neq j_2$ is allowed here.

Lemma 3.4.9 *Consider an operator G whose spectrum satisfies*

$$\sigma(G) \subset \bigcup_{j=1}^{n} \bigcup_{k\geq 1} L_{jk}, \quad L_{jk} = \{ e^{i\theta_j} x \,|\, x \geq 0,\, |x - r_{jk}| \leq a_j r_{jk}^p \},$$
$$r_{jk} + a_j r_{jk}^p \leq r_{j,k+1} - a_j r_{j,k+1}^p, \quad N(L_{jk}, G) \leq m,$$

with $0 \leq p < 1$, $m > 0$, $a_j > 0$, $0 \leq \theta_j < 2\pi$, $j = 1, \ldots, n$, and sequences of positive numbers $(r_{jk})_{k\geq 1}$. Then we have

$$\sup_{r\geq 1} \frac{N(r, G)}{r^{1-p}} < \infty.$$

Proof. It suffices to consider the case $n = 1$, $\theta_1 = 0$. We write r_k, L_k, a instead of r_{1k}, L_{1k}, a_1, choose $b \in]0, 2(1-p)a[$, and introduce the auxiliary sequence

$$s_k(r) = (r^{1-p} + kb)^{\frac{1}{1-p}}, \quad k \in \mathbb{N},\, r > 0.$$

Then we have the chain of equivalences

$$\begin{aligned}
& r + ar^p \geq s_1(r) - as_1(r)^p \\
\Leftrightarrow\ & r + ar^p \geq (r^{1-p} + b)^{\frac{1}{1-p}} - a(r^{1-p} + b)^{\frac{p}{1-p}} \\
\Leftrightarrow\ & ar^p\big((1 + br^{p-1})^{\frac{p}{1-p}} + 1\big) \geq r\big((1 + br^{p-1})^{\frac{1}{1-p}} - 1\big) \\
\Leftrightarrow\ & ar^p\big(1 + (1 + br^{p-1})^{\frac{-p}{1-p}}\big) \geq r\big(1 + br^{p-1} - (1 + br^{p-1})^{\frac{-p}{1-p}}\big) \\
\Leftrightarrow\ & a\big(1 + (1 + br^{p-1})^{\frac{-p}{1-p}}\big) \geq b + \frac{1 - (1 + br^{p-1})^{\frac{-p}{1-p}}}{r^{p-1}}.
\end{aligned} \tag{3.31}$$

L'Hospital's theorem implies

$$\lim_{r\to\infty} \frac{1-(1+br^{p-1})^{\frac{-p}{1-p}}}{r^{p-1}} = \lim_{r\to\infty} \frac{\frac{p}{1-p}(1+br^{p-1})^{\frac{-1}{1-p}} b(p-1)r^{p-2}}{(p-1)r^{p-2}} = \frac{p}{1-p}b.$$

Hence, the right-hand side of (3.31) converges to $b/(1-p)$ while the left-hand side tends to $2a$. Consequently (3.31) holds for r sufficiently large and we obtain

$$s_1(r_n) - as_1(r_n)^p \le r_n + ar_n^p \le r_{n+1} - ar_{n+1}^p$$

for large n. Since $r \mapsto r - ar^p$ is strictly increasing for large r, we conclude that $s_1(r_n) \le r_{n+1}$ for large n. Now we use induction with respect to k to show

$$s_k(r_n) \le r_{n+k} \quad \text{for all} \quad k \in \mathbb{N},\ n \text{ sufficiently large.}$$

Indeed $s_k(r_n) \le r_{n+k}$ implies

$$\begin{aligned} s_{k+1}(r_n) &= \left(r_n^{1-p} + (k+1)b\right)^{\frac{1}{1-p}} = \left(s_k(r_n)^{1-p} + b\right)^{\frac{1}{1-p}} \\ &\le \left(r_{n+k}^{1-p} + b\right)^{\frac{1}{1-p}} = s_1(r_{n+k}) \le r_{n+k+1}. \end{aligned}$$

Therefore the interval $[0, s_k(r_n)]$ intersects at most with the line segments up to L_{n+k}. Now for every $r \ge s_1(r_n)$ there exists $k \in \mathbb{N}$ such that $s_k(r_n) \le r < s_{k+1}(r_n)$ and we get the estimate

$$\frac{N(r,G)}{r^{1-p}} \le \frac{N(s_{k+1}(r_n),G)}{s_k(r_n)^{1-p}} \le \frac{(n+k+1)m}{r_n^{1-p} + kb},$$

where the right-hand side is bounded in k. □

The following lemma yields a connection between the asymptotic behaviour of a sequence of eigenvalues and the maximal possible value of p in Theorem 3.4.7.

Lemma 3.4.10 *Consider the sequence of nonnegative numbers given by*

$$r_k = ck^q + d_k k^{q-1}$$

with $c > 0$, $q \ge 1$ and a converging sequence $(d_k)_{k\in\mathbb{N}}$. Then for $a, p \ge 0$ the relation

$$r_k + ar_k^p \le r_{k+1} - ar_{k+1}^p$$

holds for almost all $k \in \mathbb{N}$ if

(i) $p < 1 - 1/q$, or

(ii) $p = 1 - 1/q$ and $a < qc^{1/q}/2$.

Proof. Using Taylor series expansion, we have

$$(k+1)^q = k^q + qk^{q-1} + f(k)k^{q-1} \quad \text{and} \quad (k+1)^{q-1} = k^{q-1} + g(k)k^{q-1}$$

with $\lim_{k\to\infty} f(k) = \lim_{k\to\infty} g(k) = 0$. This yields the equivalences

$$\begin{aligned}
& r_k + ar_k^p \le r_{k+1} - ar_{k+1}^p \\
\Leftrightarrow \quad & a(r_k^p + r_{k+1}^p) \le r_{k+1} - r_k \\
\Leftrightarrow \quad & a\Big(\big(ck^q + d_k k^{q-1}\big)^p + \big(c(k+1)^q + d_{k+1}(k+1)^{q-1}\big)^p\Big) \\
& \qquad \le c\big((k+1)^q - k^q\big) + d_{k+1}(k+1)^{q-1} - d_k k^{q-1} \\
& \qquad = \big(cq + cf(k) + d_{k+1} + d_{k+1}g(k) - d_k\big)k^{q-1} \\
\Leftrightarrow \quad & a\bigg(\Big(c + \frac{d_k}{k}\Big)^p + \Big(c\Big(1+\frac{1}{k}\Big)^q + \frac{d_{k+1}}{k}\Big(1+\frac{1}{k}\Big)^{q-1}\Big)^p\bigg) \\
& \qquad \le \big(cq + d_{k+1} - d_k + cf(k) + d_{k+1}g(k)\big)k^{q-1-qp}.
\end{aligned}$$

Now the left-hand side converges to $2ac^p$ while the right-hand side tends to

$$\begin{cases} 0 & \text{for} \quad q(1-p) < 1, \\ cq & \text{for} \quad q(1-p) = 1, \\ \infty & \text{for} \quad q(1-p) > 1 \end{cases} \qquad \text{as} \quad k \to \infty.$$

Therefore the above inequality holds for k sufficiently large if $q(1-p) > 1$, i.e. $p < 1 - 1/q$, or if $q(1-p) = 1$ and $2ac^p < cq$, i.e. $a < c^{1-p}q/2$. □

Next we establish sufficient conditions for the spectrum of an operator with compact resolvent and a Riesz basis of Jordan chains to be a p-subordinate perturbation of a normal operator. As a consequence, the assumptions on G in the previous theorems can be relaxed.

Lemma 3.4.11 *Consider $\lambda \in \mathbb{C}$ with $|\lambda| \ge 2$ and the $n \times n$ Jordan block*

$$A = \begin{pmatrix} \lambda & 1 & \\ & \ddots & \ddots \\ & & \lambda \end{pmatrix}.$$

Then we have $\|Ax\| \ge |\lambda|\|x\|/2$ for all $x \in \mathbb{C}^n$ where $\|\cdot\|$ denotes the Euclidean norm.

Proof. We have $\|Ax\|^2 = (Ax|Ax) = (A^*Ax|x)$ and

$$A^* = \begin{pmatrix} \overline{\lambda} & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & \overline{\lambda} \end{pmatrix}, \text{ i.e. } A^*A = \begin{pmatrix} |\lambda|^2 & \overline{\lambda} & & & \\ \lambda & |\lambda|^2+1 & \ddots & & \\ & \lambda & \ddots & \ddots & \\ & & \ddots & \ddots & \overline{\lambda} \\ & & & \lambda & |\lambda|^2+1 \end{pmatrix}.$$

Consider $x = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$. Then

$$\begin{aligned} \|Ax\|^2 &= |\lambda|^2 \sum_{j=1}^{n} |\alpha_j|^2 + \sum_{j=2}^{n} |\alpha_j|^2 + 2\sum_{j=1}^{n-1} \mathrm{Re}(\overline{\alpha}_j \overline{\lambda} \alpha_{j+1}) \\ &\geq |\lambda|^2 \sum_{j=1}^{n} |\alpha_j|^2 + \sum_{j=2}^{n} |\alpha_j|^2 - 2|\lambda| \sum_{j=1}^{n-1} |\alpha_j| \cdot |\alpha_{j+1}|. \end{aligned}$$

Without loss of generality, we may assume that $\lambda, \alpha_j \in \mathbb{R}_{\geq 0}$. Using $\lambda^2 \geq 4$, we further estimate

$$\begin{aligned} &\lambda^2 \sum_{j=1}^{n} \alpha_j^2 + \sum_{j=2}^{n} \alpha_j^2 - 2\lambda \sum_{j=1}^{n-1} \alpha_j \alpha_{j+1} \\ &= 2\left\| \begin{pmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} - \frac{\lambda}{2} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} \right\|^2 + \frac{1}{2}\lambda^2 \alpha_1^2 + \Big(\frac{1}{2}\lambda^2 - 1\Big)(\alpha_2^2 + \cdots + \alpha_{n-1}^2) + (\lambda^2 - 1)\alpha_n^2 \\ &\geq \Big(\frac{1}{2}\lambda^2 - 1\Big)(\alpha_1^2 + \cdots + \alpha_n^2) \geq \frac{1}{4}\lambda^2(\alpha_1^2 + \cdots + \alpha_n^2), \end{aligned}$$

which completes the proof. □

Lemma 3.4.12 *Let $G(H \to H)$ be an operator with compact resolvent and a Riesz basis of Jordan chains. Let λ_k, $k \in \mathbb{N}$, be the eigenvalues of G, $c \geq 0$, $0 \leq p < 1$, and $\mu_k \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{N}$, such that*

$$|\mu_k - \lambda_k| \leq c|\lambda_k|^p \quad \text{for almost all} \quad k.$$

Then there is an isomorphism $J : H \to H$, a normal operator $G_0(H \to H)$ with compact resolvent, and an operator $S_0(H \to H)$ p-subordinate to G_0 such that

$$J\mathcal{D}(G) = \mathcal{D}(G_0), \quad JGJ^{-1} = G_0 + S_0, \quad \sigma(G_0) = \{\mu_k \,|\, k \in \mathbb{N}\}.$$

In addition, J maps the Riesz basis of Jordan chains of G onto an orthonormal basis of eigenvectors of G_0 such that $x \in \mathcal{L}(\lambda_k, G)$ implies $Jx \in \mathcal{L}(\mu_k, G_0)$.

Proof. Since G has compact resolvent, we have $|\lambda_k| \to \infty$. From

$$|\lambda_k| \le |\mu_k| + |\mu_k - \lambda_k| \le |\mu_k| + c|\lambda_k|^p$$

we obtain

$$\left(1 - \frac{c}{|\lambda_k|^{1-p}}\right)|\lambda_k| \le |\mu_k|$$

for almost all k. Therefore $|\lambda_k|/2 \le |\mu_k|$ for almost all k and $|\mu_k| \to \infty$. We also have

$$|\mu_k| \le |\lambda_k| + c|\lambda_k|^p = \left(1 + \frac{c}{|\lambda_k|^{1-p}}\right)|\lambda_k|,$$

which implies $|\mu_k| \le 2|\lambda_k|$ for almost all k.

Now suppose that $(x_j)_{j\in\mathbb{N}}$ is a Riesz basis of Jordan chains of G and let J be an isomorphism such that $(Jx_j)_{j\in\mathbb{N}}$ is an orthonormal basis. Then $(Jx_j)_j$ consists of Jordan chains of JGJ^{-1}, and we may thus assume that $J = I$ and

$$G = \sum_{k=0}^{\infty}(\lambda_k + N_k)P_k$$

where P_k are orthogonal projections onto $\mathcal{L}(\lambda_k, G)$, $N_k : \mathcal{R}(P_k) \to \mathcal{R}(P_k)$ are nilpotent operators, and for every k there is an orthonormal basis of $\mathcal{R}(P_k)$ such that the matrix representing N_k in this basis is block diagonal with blocks of the form $\left(\begin{smallmatrix} 0 & 1 & \\ & \ddots & \ddots \\ & & 0 \end{smallmatrix}\right)$. We decompose G as

$$G = G_0 + S_1 + S_2 \quad \text{with} \quad G_0 = \sum_{k=0}^{\infty}\mu_k P_k,\ S_1 = \sum_{k=0}^{\infty}(\lambda_k - \mu_k)P_k,\ S_2 = \sum_{k=0}^{\infty}N_k P_k.$$

Then G_0 is a normal operator with compact resolvent, spectrum $\{\mu_k \mid k \in \mathbb{N}\}$, and $\mathcal{L}(\lambda_k, G) = \mathcal{R}(P_k) \subset \mathcal{L}(\mu_k, G_0)$. According to Proposition 2.3.3, we have

$$u \in \mathcal{D}(G) \quad \Longleftrightarrow \quad \sum_{k=0}^{\infty}\|(\lambda_k + N_k)P_k u\|^2 < \infty$$

and analogous characterisations hold for the domains of G_0 and S_1. We have

$$\begin{aligned}\|(\lambda_k + N_k)P_k u\| &\le (|\lambda_k| + 1)\|P_k u\| \le (|\mu_k| + c|\lambda_k|^p + 1)\|P_k u\| \\ &\le (|\mu_k| + 2^p c|\mu_k|^p + 1)\|P_k u\| \le (2 + 2^p c)|\mu_k|\|P_k u\|\end{aligned}$$

for almost all k, hence $\mathcal{D}(G_0) \subset \mathcal{D}(G)$. Using Lemma 3.4.11, we also have

$$\|(\lambda_k + N_k)P_k u\| \ge \frac{1}{2}|\lambda_k|\|P_k u\| \ge \frac{1}{4}|\mu_k|\|P_k u\|$$

for almost all k. This implies $\mathcal{D}(G) \subset \mathcal{D}(G_0)$ and hence $\mathcal{D}(G) = \mathcal{D}(G_0)$. Since $\mu_k \neq 0$, we have $|\lambda_k - \mu_k| \leq C|\mu_k|^p$ for all $k \in \mathbb{N}$ with some appropriate constant C. The estimate

$$\Big\| \sum_{k=0}^{\infty} (\lambda_k - \mu_k) P_k u \Big\|^2 = \sum_{k=0}^{\infty} |\lambda_k - \mu_k|^2 \|P_k u\|^2 \leq \sum_{k=0}^{\infty} C^2 |\mu_k|^{2p} \|P_k u\|^2$$
$$\leq C^2 \Big(\sum_{k=0}^{\infty} |\mu_k|^2 \|P_k u\|^2 \Big)^p \Big(\sum_{k=0}^{\infty} \|P_k u\|^2 \Big)^{1-p}$$

implies that $\mathcal{D}(G_0) \subset \mathcal{D}(S_1)$ and that S_1 is p-subordinate to G_0. Since S_2 is bounded and $0 \in \varrho(G_0)$, S_2 is also p-subordinate to G_0 and the proof is complete. □

Proposition 3.4.13 *Let $G(H \to H)$ be an operator with compact resolvent and a Riesz basis of Jordan chains. Suppose that $0 \leq p < 1$, $\alpha \geq 0$, $0 \leq \theta_j < 2\pi$, $j = 1, \dots, n$, such that either*

(i) there exists $r_0 > 0$ with

$$\sigma(G) \subset B_{r_0}(0) \cup \bigcup_{j=1}^{n} S_j, \quad S_j = \big\{ e^{i\theta_j}(x + iy) \,\big|\, x > 0,\ |y| \leq \alpha x^p \big\}, \quad \textit{or}$$

(ii) almost all eigenvalues of G lie inside regions

$$K_{jk} = \big\{ e^{i\theta_j}(x + iy) \,\big|\, r_{jk}^- \leq x \leq r_{jk}^+,\ |y| \leq \alpha x^p \big\},$$

$j = 1, \dots, n$, $k \geq 1$, where $(r_{jk}^{\pm})_{k \geq 1}$ are sequences of positive numbers satisfying

$$r_{jk}^- \leq r_{jk}^+ < r_{j,k+1}^-.$$

Then there is an isomorphism $J : H \to H$, a normal operator $G_0(H \to H)$ with compact resolvent, and an operator $S_0(H \to H)$ p-subordinate to G_0 such that

$$J\mathcal{D}(G) = \mathcal{D}(G_0), \quad JGJ^{-1} = G_0 + S_0.$$

In case (i), all eigenvalues of G_0 lie on the rays $e^{i\theta_j}\mathbb{R}_{\geq 0}$ and we have

$$N(r, G_0) = N(r, G) \quad \textit{for} \quad r \geq 1.$$

In case (ii), all eigenvalues of G_0 lie on the line segments

$$L_{jk} = \big\{ e^{i\theta_j} x \,\big|\, r_{jk}^- \leq x \leq r_{jk}^+ \big\},$$

and $N(L_{jk}, G_0) = N(K_{jk}, G)$ holds for almost all pairs (j, k).

Moreover, if $S(H \to H)$ is p-subordinate to G, then JSJ^{-1} is p-subordinate to G_0.

Proof. In order to apply the previous lemma, we need to properly choose the new eigenvalues $(\mu_l)_{l\in\mathbb{N}}$ of G_0 given the eigenvalues λ_l of G. For case (i), almost all λ_l lie inside $S_1 \cup \ldots \cup S_n$. For $\lambda_l \in S_j$ we set $\mu_l = e^{i\theta_j}|\lambda_l|$. With $\lambda_l = e^{i\theta_j}(x+iy)$, this implies

$$\begin{aligned} |\mu_l - \lambda_l| &\le \sqrt{x^2+y^2} - x + |y| = \frac{y^2}{\sqrt{x^2+y^2}+x} + |y| \\ &\le \frac{y^2}{2x} + |y| \le \frac{\alpha^2 x^{2p}}{2x} + \alpha x^p = \left(\frac{\alpha^2}{2x^{1-p}} + \alpha\right) x^p \le 2\alpha|\lambda_l|^p \end{aligned}$$

for $|\lambda_l|$ large enough. If $\lambda_l \notin S_j$ for every j and $\lambda_l \neq 0$, we set $\mu_l = e^{i\theta_1}|\lambda_l|$. If finally $\lambda_l = 0$, we take $\mu_l = e^{i\theta_1}$. In particular, our choice implies $|\lambda_l| = |\mu_l|$ (if $\lambda_l \neq 0$) and $N(r, G_0) = N(r, G)$ for $r \ge 1$.

For case (ii), if $\lambda_l = e^{i\theta_j}(x+iy)$ is an eigenvalue in K_{jk}, we set $\mu_l = e^{i\theta_j}x$. Then

$$|\mu_l - \lambda_l| = |y| \le \alpha x^p \le \alpha|\lambda_l|^p.$$

If $\lambda_l \notin K_{jk}$ for every (j,k), we set $\mu_l = e^{i\theta_1} r_{11}^-$. We thus get $N(L_{jk}, G_0) = N(K_{jk}, G)$ whenever $(j,k) \neq (1,1)$.

Finally suppose that S is p-subordinate to G with bound b. For $u \in \mathcal{D}(G_0)$ we have

$$\begin{aligned} \|JGJ^{-1}u\| \le \|G_0u\| + \|S_0u\| &\le \|G_0u\| + b_0\|u\|^{1-p}\|G_0u\|^p \\ &\le \big(1 + b_0\|G_0^{-1}\|^{1-p}\big)\|G_0u\| \end{aligned}$$

since $0 \in \varrho(G_0)$. Therefore

$$\begin{aligned} \|JSJ^{-1}u\| &\le \|J\|\|SJ^{-1}u\| \le b\|J\|\|J^{-1}u\|^{1-p}\|GJ^{-1}u\|^p \\ &\le b\|J\|\|J^{-1}\| \cdot \|u\|^{1-p}\|JGJ^{-1}u\|^p \\ &\le b\|J\|\|J^{-1}\|\big(1 + b_0\|G_0^{-1}\|^{1-p}\big)^p\|u\|^{1-p}\|G_0u\|^p. \end{aligned}$$

□

Remark 3.4.14 Let G and S satisfy the assumptions of the previous proposition and let $T = G + S$. Then we have

$$JTJ^{-1} = JGJ^{-1} + JSJ^{-1} = G_0 + S_0 + JSJ^{-1}.$$

If G satisfies the condition 3.4.13(i), then Proposition 3.4.1 and Theorem 3.4.4 may be applied to JTJ^{-1}; if G satisfies 3.4.13(ii), Theorem 3.4.7 may be applied. Therefore, these theorems also hold if G is as in Proposition 3.4.13 and b is the p-subordination bound of $S_0 + JSJ^{-1}$ to G_0. ⌟

3.5 Examples

We apply Theorems 3.4.4 and 3.4.7 to ordinary differential operators on a compact interval and to the Laplace operator on the unit disc.

For ordinary differential operators with possibly unbounded coefficient functions and appropriate boundary conditions we obtain finitely spectral l^2-decompositions; for certain boundary conditions we even show the existence of a Riesz basis of root vectors. For the case of bounded coefficients and regular boundary conditions, the existence of a Riesz basis (possibly with parentheses, depending on the boundary conditions) of root vectors is well known [11], [20, Theorem XIX.4.16], [43]. The case of unbounded coefficients is treated in [44].

In the first example we obtain a finitely spectral l^2-decomposition for a differential operator with possibly unbounded coefficient functions.

Example 3.5.1 Let $g_0, \dots, g_{n-2} \in L^2([a_1, a_2])$, $g_{n-1} \in L^\infty([a_1, a_2])$, and consider the differential operator T on $L^2([a_1, a_2])$ given by

$$Tu = i^n u^{(n)} + \sum_{l=0}^{n-1} g_l u^{(l)}, \quad \mathcal{D}(T) = \{u \in W^{n,2}([a_1, a_2]) \,|\, V_1(u) = \cdots = V_n(u) = 0\},$$

where the boundary condition $V_1(u) = \cdots = V_n(u) = 0$ is regular in the sense of Naimark [40, §4.8] and such that the operator $Gu = i^n u^{(n)}$, $\mathcal{D}(G) = \mathcal{D}(T)$ becomes selfadjoint. We also write

$$T = G + S \quad \text{with} \quad Su = \sum_{l=0}^{n-1} g_l u^{(l)}, \quad \mathcal{D}(S) = \mathcal{D}(T).$$

Then the resolvent of G is compact [19, Theorem XIII.4.1], and the spectrum of G consists of at most two sequences of eigenvalues of the form

$$\lambda_{jk} = c_j k^n + d_{jk} k^{n-1}, \quad k \geq k_{j0},\ j = 1, 2,$$

with $c_j \neq 0$ and converging sequences $(d_{jk})_{k \geq k_{j0}}$, see [40, §4.9]. In fact $c_j, d_{jk} \in \mathbb{R}$ since G is selfadjoint. Lemma 3.4.10 thus implies that each sequence $(\lambda_{jk})_{k \geq k_{j0}}$ satisfies

$$|\lambda_{jk}| + a|\lambda_{jk}|^p \leq |\lambda_{j,k+1}| - a|\lambda_{j,k+1}|^p$$

for almost all k if $p = (n-1)/n$ and $0 \leq a < n c_j^{1/n}/2$. As the multiplicity of every eigenvalue of G is at most n, Lemma 3.4.9 yields

$$\sup_{r \geq 1} \frac{N(r, G)}{r^{1-p}} < \infty \qquad \text{with} \qquad p = \frac{n-1}{n}.$$

Consider now the case $0 \in \varrho(G)$. Then S is $(n-1)/n$-subordinate to G due to Propositions 3.2.15 and 3.2.16. Consequently, Proposition 3.4.1 and Theorem 3.4.4

apply to the decomposition $T = G + S$: The resolvent of T is compact, almost all eigenvalues of T lie inside regions of the form

$$\{\operatorname{sign} c_j \cdot (x + iy) \,|\, x \geq 0, |y| \leq \alpha x^p\}$$

with some $\alpha > 0$, and T admits a finitely spectral l^2-decomposition.

Otherwise, if $0 \in \sigma(G)$, we choose any $\tau \in \varrho(G) \cap \mathbb{R}$. Then $G - \tau$ is selfadjoint with compact resolvent and has the same eigenvalue asymptotics as G. Moreover $S + \tau$ is $(n-1)/n$-subordinate to $G - \tau$ by Propositions 3.2.15 and 3.2.16. We can thus apply Proposition 3.4.1 and Theorem 3.4.4 to $T = G - \tau + S + \tau$ and obtain the same results as before. ⌟

A Riesz basis of root vectors may be obtained under additional assumptions:

Remark 3.5.2 Theorem 3.4.7 with $p = (n-1)/n$ may be applied to the operator T from the previous example if two additional conditions are met: First, if $\sigma(G)$ consists of two sequences $(\lambda_{jk})_k$ lying on the same half-axis, i.e. $c_1 c_2 > 0$, then it must be possible to cover *both* sequences $(\lambda_{jk})_k$ by *one* sequence of line segments

$$L_k = \{\operatorname{sign} c_1 \cdot x \,|\, |x - r_k| \leq \beta r_k^p\}$$

with $r_k, \beta \geq 0$ appropriate; if $c_1 c_2 < 0$ or if there is only one sequence $(\lambda_{jk})_k$, then the line segments may be chosen as $L_{jk} = \{\lambda_{jk}\}$, i.e. $r_{jk} = \lambda_{jk}$, $\beta = 0$.

Second, (3.29) must hold. In view of Lemma 3.4.10 this means that $\beta + \delta b$ must be small enough; in particular the p-subordination bound b of S to G must be small enough which in turn is satisfied if the norms $\|g_0\|_{L_2}, \dots, \|g_{n-2}\|_{L_2}$, $\|g_{n-1}\|_\infty$ are sufficiently small.

If now the boundary conditions are such that almost all eigenvalues of G are simple and the line segments L_{jk} can be chosen such that almost all L_{jk} contain only one eigenvalue of G, then T has a Riesz basis of eigenvectors and finitely many Jordan chains, see Remark 3.4.8. ⌟

A concrete choice of boundary conditions allows us to specify precise conditions under which Theorem 3.4.7 is applicable.

Example 3.5.3 Consider the operator T on $L^2([0,1])$ defined by

$$Tu = -u'' + g_1 u' + g_0 u, \quad \mathcal{D}(T) = \{u \in W^{2,2}([0,1]) \,|\, u(0) = u(1) = 0\}$$

where $g_0 \in L^2([0,1])$, $g_1 \in L^\infty([0,1])$. Analogously to the previous example we consider the operators $Gu = -u''$, $Su = g_1 u' + g_0 u$ with $\mathcal{D}(G) = \mathcal{D}(S) = \mathcal{D}(T)$. Direct calculations show that G is selfadjoint with compact resolvent and eigenvalues $\pi^2 k^2$, $k = 1, 2, \dots$, which are all simple. We have

$$\|u'\|_{L^2} \leq \|u\|_{L^2}^{1/2} \|Gu\|_{L^2}^{1/2} \quad \text{for} \quad u \in \mathcal{D}(G)$$

by Example 3.2.6. Moreover, for $u \in \mathcal{D}(G)$ the identity

$$u(x) = \int_0^x u'(t)\,dt$$

yields

$$|u(x)| \le \int_0^1 |u'(t)|\,dt \le \Big(\int_0^1 dt\Big)^{1/2} \Big(\int_0^1 |u'(t)|^2\,dt\Big)^{1/2} = \|u'\|_{L^2}$$

and thus $\|u\|_\infty \le \|u'\|_{L^2}$. We obtain the estimate

$$\begin{aligned}\|Su\|_{L^2} &\le \|g_1\|_\infty \|u'\|_{L^2} + \|g_0\|_{L^2}\|u\|_\infty \\ &\le \big(\|g_1\|_\infty + \|g_0\|_{L^2}\big)\|u'\|_{L^2} \le \big(\|g_1\|_\infty + \|g_0\|_{L^2}\big)\|u\|_{L^2}^{1/2}\|Gu\|_{L^2}^{1/2}\end{aligned}$$

for $u \in \mathcal{D}(G)$; S is $1/2$-subordinate to G with bound $b \le \|g_1\|_\infty + \|g_0\|_{L^2}$.

We want to apply Theorem 3.4.7 with $p = 1/2$, $\theta_1 = 0$, $r_{1k} = r_k = \pi^2 k^2$ and $\beta = 0$. The condition (3.29) then reads

$$r_k + \delta b r_k^{1/2} \le r_{k+1} - \delta b r_{k+1}^{1/2} \quad \text{with some} \quad \delta > \frac{4+\pi}{\pi}. \tag{3.32}$$

By Lemma 3.4.10, (3.32) holds for almost all k if $\delta b < \pi$. So if

$$\frac{4+\pi}{\pi}\big(\|g_1\|_\infty + \|g_0\|_{L^2}\big) < \pi, \quad \text{i.e.} \quad \|g_1\|_\infty + \|g_0\|_{L^2} < \frac{\pi^2}{4+\pi},$$

then we can find $\delta > (4+\pi)/\pi$ such that $\delta b < \pi$. Consequently (3.29) is satisfied and Theorem 3.4.7 yields that for every $\alpha > \|g_1\|_\infty + \|g_0\|_{L^2}$ almost all eigenvalues of T lie inside regions

$$K_k = \big\{x + iy \,\big|\, |x - \pi^2 k^2| \le \alpha\pi k,\ |y| \le \alpha x^{1/2}\big\}$$

and in fact $N(K_k, T) = 1$ for almost all k. In view of Remark 3.4.8, T has a Riesz basis of eigenvectors and finitely many Jordan chains.

If $g_1 = 0$ and $g_0 \in L^\infty([0,1])$, no condition on the norm of g_0 is necessary: For in this case, S is bounded with $\|S\| = \|g_0\|_\infty$ and we have

$$r_k + \delta\|S\| \le r_{k+1} - \delta\|S\| \quad \text{for almost all } k$$

by Lemma 3.4.10. We can thus apply Theorem 3.4.7 with $p = 0$ and obtain a Riesz basis of eigenvectors and finitely many Jordan chains of T and, for every $\alpha > \|g_0\|_\infty$, the localisation of almost all eigenvalues of T inside the rectangles

$$K_k = \big\{x + iy \,\big|\, |x - \pi^2 k^2| \le \alpha,\ |y| \le \alpha\big\}.$$

┘

For an elliptic differential operator of even order on a domain $\Omega \subset \mathbb{R}^m$, the existence of a Riesz basis with parentheses of root vectors was shown by Markus in [36, §10]. We consider the Laplacian on the unit disc.

Example 3.5.4 Consider the Laplace operator on the unit disc $B_1(0) \subset \mathbb{R}^2$ with Dirichlet boundary condition,

$$G\big(L^2(B_1(0)) \to L^2(B_1(0))\big), \quad \mathcal{D}(G) = W^{2,2}(B_1(0)) \cap W_0^{1,2}(B_1(0)),$$
$$Gu = -\Delta u = -\partial_1^2 u - \partial_2^2 u.$$

Then G is positive selfadjoint with compact resolvent [19, Theorem XIV.6.25], and the asymptotic behaviour of its spectrum is such that

$$\lim_{r\to\infty} \frac{N(r,G)}{r} = \frac{1}{4},$$

see [12, Theorem VI.16]. If S is a bounded operator on $L^2(B_1(0))$ and $T = G + S$, then Proposition 3.4.1 and Theorem 3.4.4 apply with $p = 0$: The operator T has a compact resolvent,

$$\sigma(G) \subset \overline{B_{\|S\|}(0)} \cup \big\{x + iy \,\big|\, x \geq 0,\ |y| \leq \|S\|\big\}$$

(cf. Remark 3.4.2 and Lemma 3.3.1), and T admits a finitely spectral l^2-decomposition. ⌟

Chapter 4

Hamiltonian operators and Riccati equations

We apply the results from the previous chapters to Hamiltonian operator matrices and the associated Riccati equation

$$A^*X + XA + XQ_1X - Q_2 = 0.$$

Riccati equations are generally hard to solve because they are quadratic operator equations and the involved operators need not commute. The known existence results yield a nonnegative and a nonpositive solution for the case that Q_1 and Q_2 are bounded, cf. Curtain and Zwart [14], Langer, Ran and van de Rotten [31], and Bubák, van der Mee and Ran [10].

In Theorem 4.4.1 we prove the existence of infinitely many selfadjoint solutions of the Riccati equation for unbounded Q_1, Q_2. In particular, we obtain a nonnegative solution X_+ and a nonpositive solution X_-. Under stronger assumptions we show the existence of bounded, boundedly invertible solutions and that every bounded solution can be represented as $X = X_+P + X_-(I - P)$ with some projection P, see Theorem 4.4.5. A similar representation was obtained by Curtain, Iftime and Zwart [13] under the assumption that X_- exists and is bounded and boundedly invertible.

In the first section we study basic properties of Hamiltonian operators and their relation to two Krein space inner products given by fundamental symmetries J_1, J_2. The existence of invariant graph subspaces and their relation to the symmetries J_1, J_2 is investigated in Section 4.2. Different notions of solutions of the Riccati equation and its connection to invariant graph subspaces are the topic of Section 4.3. In the last section, the existence of finitely spectral l^2-decompositions for Hamiltonians is shown, and the resulting invariant graph subspaces are used to obtain the existence and characterisations of solutions of the Riccati equation.

4.1 Hamiltonian operators and associated Krein spaces

We investigate properties of Hamiltonian operators with the help of the fundamental symmetries J_1 and J_2 from the introduction. We obtain results about the symmetry and separation of the spectrum with respect to the imaginary axis. The connection of J_1 to the Hamiltonian has been known for a long time; it was used for example by Potter [41] in 1966 and Mårtensson [38] in 1971. By contrast, the relation of J_2 to the Hamiltonian was first exploited by Langer, Ran and Temme [32] in 1997, followed by Langer, Ran and van de Rotten [31] in 2001, Azizov, Dijksma and Gridneva [4] in 2003, and Bubák, van der Mee and Ran [10] in 2005.

Our notion of a nonnegative Hamiltonian operator matrix is taken from [4]. For some remarks about the concept of block operator matrices see page 97.

Definition 4.1.1 Let H be a Hilbert space. A *Hamiltonian operator matrix* is a block operator matrix

$$T = \begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix} \tag{4.1}$$

acting on $H \times H$ with densely defined linear operators $A, Q_1, Q_2(H \to H)$ such that Q_1 and Q_2 are symmetric and T is densely defined.

If Q_1 and Q_2 are both nonnegative (positive, uniformly positive), then T is called a *nonnegative* (*positive, uniformly positive,* respectively) Hamiltonian operator matrix; accordingly for *nonpositive, negative,* and *uniformly negative.* ⌟

The condition that T is densely defined implies that A^* is densely defined; hence A is closable.

Connected to Hamiltonian operator matrices are two Krein space inner products on the Hilbert space $H \times H$,

$$\langle f|g\rangle = (J_1 f|g) \quad \text{and} \quad [f|g] = (J_2 f|g), \tag{4.2}$$

where $(\cdot|\cdot)$ is the natural scalar product on $H \times H$, and the fundamental symmetries J_1, J_2 are given by

$$J_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}. \tag{4.3}$$

In other words,

$$\left\langle \begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = i(u|y) - i(v|x) \quad \text{and}$$

$$\left[\begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{pmatrix} x \\ y \end{pmatrix} \right] = (u|y) + (v|x) \qquad \text{for} \quad \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \in H \times H.$$

The next proposition shows that a (nonnegative) Hamiltonian operator matrix is J_1-skew-symmetric (and J_2-accretive). Azizov, Dijksma and Gridneva [4] called an operator $T(H \times H \to H \times H)$, which is not necessarily represented by a block operator matrix, a (nonnegative) Hamiltonian operator if it is J_1-skew-adjoint (and J_2-accretive).

Proposition 4.1.2

(i) A Hamiltonian operator matrix is J_1-skew-symmetric.

(ii) If a J_1-skew-symmetric operator $T(H \times H \to H \times H)$ satisfies $\mathcal{D}(T) = S_1 \times S_2$, then it can be represented by a Hamiltonian operator matrix.

(iii) A Hamiltonian operator matrix is nonnegative (positive, uniformly positive) if and only if it is J_2-accretive (strictly accretive, uniformly accretive, respectively). In fact we have

$$\operatorname{Re}\Big[\begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\Big|\begin{pmatrix} u \\ v \end{pmatrix}\Big] = (Q_1 v|v) + (Q_2 u|u) \tag{4.4}$$

for $u \in \mathcal{D}(A) \cap \mathcal{D}(Q_2)$, $v \in \mathcal{D}(Q_1) \cap \mathcal{D}(A^*)$.

Proof. (i): Direct computation yields

$$\begin{aligned}\Big\langle\begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\Big|\begin{pmatrix} x \\ y \end{pmatrix}\Big\rangle &= i(Au + Q_1 v|y) - i(Q_2 u - A^* v|x) \\ &= i(u|A^* y - Q_2 x) - i(v| - Q_1 y - Ax) \\ &= \Big\langle\begin{pmatrix} u \\ v \end{pmatrix}\Big|\begin{pmatrix} -A & -Q_1 \\ -Q_2 & A^* \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\Big\rangle\end{aligned}$$

for $u, x \in \mathcal{D}(A) \cap \mathcal{D}(Q_2)$, $v, y \in \mathcal{D}(Q_1) \cap \mathcal{D}(A^*)$.

(ii): The assumption $\mathcal{D}(T) = S_1 \times S_2$ implies that T can be written as a block operator matrix

$$T = \begin{pmatrix} A & Q_1 \\ Q_2 & D \end{pmatrix}$$

with operators $A, Q_1, Q_2, D(H \to H)$. Without loss of generality, we may assume

$$\mathcal{D}(A) = \mathcal{D}(Q_2), \qquad \mathcal{D}(Q_1) = \mathcal{D}(D).$$

Since T is densely defined, $\mathcal{D}(A)$ and $\mathcal{D}(Q_1)$ are dense in H. The J_1-skew-symmetry yields

$$\begin{aligned} 0 &= \Big\langle\begin{pmatrix} A & Q_1 \\ Q_2 & D \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}\Big|\begin{pmatrix} x \\ y \end{pmatrix}\Big\rangle + \Big\langle\begin{pmatrix} u \\ v \end{pmatrix}\Big|\begin{pmatrix} A & Q_1 \\ Q_2 & D \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\Big\rangle \\ &= i(Au + Q_1 v|y) - i(Q_2 u + Dv|x) + i(u|Q_2 x + Dy) - i(v|Ax + Q_1 y) \\ &= -i(Q_2 u|x) + i(u|Q_2 x) + i(Au|y) + i(u|Dy) \\ &\quad - i(Dv|x) - i(v|Ax) + i(Q_1 v|y) - i(v|Q_1 y)\end{aligned}$$

for all $u, x \in \mathcal{D}(A)$ and $v, y \in \mathcal{D}(Q_1)$. Using this result for $v = y = 0$, $x = u = 0$, and $v = x = 0$, respectively, we find

$$\begin{aligned} (Q_2 u|x) &= (u|Q_2 x) \quad \text{for all} \quad u, x \in \mathcal{D}(Q_2), \\ (Q_1 v|y) &= (v|Q_1 y) \quad \text{for all} \quad v, y \in \mathcal{D}(Q_1), \\ (Au|y) &= -(u|Dy) \quad \text{for all} \quad u \in \mathcal{D}(A), y \in \mathcal{D}(D); \end{aligned}$$

hence Q_1 and Q_2 are symmetric and $D \subset -A^*$. Consequently, T is represented by a Hamiltonian operator matrix.

(iii): The claim follows from the relation

$$\begin{aligned} \left[\begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \middle| \begin{pmatrix} u \\ v \end{pmatrix}\right] &= (Au + Q_1 v|v) + (Q_2 u - A^* v|u) \\ &= \underbrace{(Q_1 v|v) + (Q_2 u|u)}_{\in \mathbb{R}} + \underbrace{(Au|v) - (v|Au)}_{\in i\mathbb{R}} \end{aligned}$$

for $u \in \mathcal{D}(A) \cap \mathcal{D}(Q_2)$, $v \in \mathcal{D}(Q_1) \cap \mathcal{D}(A^*)$. □

As a consequence of the skew-symmetry, a Hamiltonian operator matrix is always closable. However we will not compute the closure in the general case since all Hamiltonian operators from the perturbation theorems in Section 4.4 will be closed automatically.

Another consequence of the J_1-skew-symmetry is the symmetry of the spectrum of T with respect to the imaginary axis:

Corollary 4.1.3 *Let T be a Hamiltonian operator matrix.*

(i) If T has a dense system of root subspaces, then the point spectrum $\sigma_p(T)$ is symmetric with respect to the imaginary axis.

(ii) If $\lambda, -\overline{\lambda} \in \varrho(T)$ for some $\lambda \in \mathbb{C}$, then T is J_1-skew-adjoint and the spectrum $\sigma(T)$ is symmetric with respect to the imaginary axis.

Proof. Since iT is J_1-symmetric, the claims are a direct consequence of Theorem 2.5.12, Remark 2.5.8 and Proposition 2.5.9. □

The J_2-accretivity of a nonnegative Hamiltonian operator leads to a characterisation of the point spectrum on the imaginary axis. First, we prove a lemma about nonnegative operators in Hilbert spaces.

Lemma 4.1.4 *Let $Q(H \to H)$ be a nonnegative symmetric operator on a Hilbert space and $u \in \mathcal{D}(Q)$. Then $(Qu|u) = 0$ implies $Qu = 0$.*

Proof. Since Q is symmetric and nonnegative, it has a nonnegative selfadjoint extension which in turn has a square root. That is, there exists a nonnegative selfadjoint operator B such that $Q \subset B^2$. Then

$$0 = (Qu|u) = (Bu|Bu) = \|Bu\|^2$$

implies $Bu = 0$ and thus $Qu = 0$. □

Lemma 4.1.5 *For a nonnegative Hamiltonian operator matrix T, $(u, v) \in \mathcal{D}(T)$ and $t \in \mathbb{R}$ we have*

$$T\begin{pmatrix} u \\ v \end{pmatrix} = it\begin{pmatrix} u \\ v \end{pmatrix} \quad \iff \quad \begin{cases} u \in \ker(A - it) \cap \ker Q_2 \quad \textit{and} \\ v \in \ker(A^* + it) \cap \ker Q_1. \end{cases} \tag{4.5}$$

Proof. Suppose we have $x = (u, v) \in \mathcal{D}(T)$ with $(T - it)x = 0$. Then

$$(A - it)u + Q_1 v = 0, \quad Q_2 u - (A^* + it)v = 0$$

and

$$0 = \operatorname{Re}(it[x|x]) = \operatorname{Re}[Tx|x] = (Q_1 v|v) + (Q_2 u|u).$$

Then $(Q_1 v|v) = (Q_2 u|u) = 0$ since Q_1 and Q_2 are nonnegative. Using Lemma 4.1.4, we obtain $Q_1 v = Q_2 u = 0$, which in turn implies $(A - it)u = (A^* + it)v = 0$. The other implication is immediate. □

We can now give some conditions which yield a separation of the spectrum at the imaginary axis.

Proposition 4.1.6 *Let T be a nonnegative Hamiltonian operator matrix.*

(i) We have $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ if and only if

$$\ker(A - it) \cap \ker Q_2 = \ker(A^* + it) \cap \ker Q_1 = \{0\} \quad \textit{for all} \quad t \in \mathbb{R}. \tag{4.6}$$

In particular, $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ for positive Hamiltonians.

(ii) If T is uniformly positive, then a strip around the imaginary axis belongs to the set of points of regular type for T. More precisely, $Q_1, Q_2 \geq \gamma$ with $\gamma > 0$ implies that

$$\left\{\lambda \in \mathbb{C} \,\middle|\, |\operatorname{Re}\lambda| < \gamma\right\} \subset r(T).$$

If T is also closed with a dense system of root subspaces, then

$$\left\{\lambda \in \mathbb{C} \,\middle|\, |\operatorname{Re}\lambda| < \gamma\right\} \subset \varrho(T).$$

Proof. (i) is an immediate consequence of the previous lemma. (ii) follows from Proposition 2.6.2 with the Krein inner product $[\cdot|\cdot]$ and because

$$\operatorname{Re}[Tx|x] = (Q_1 v|v) + (Q_2 u|u) \geq \gamma\big(\|u\|^2 + \|v\|^2\big) \quad \text{for} \quad x = (u,v) \in \mathcal{D}(T).$$

□

If T is a nonnegative Hamiltonian operator which satisfies (4.6), Proposition 2.6.6 implies that the root subspaces corresponding to eigenvalues in the right and left half-plane are J_2-nonnegative and J_2-nonpositive, respectively. Sharpening the condition (4.6), we can even show that they are J_2-positive/-negative:

Proposition 4.1.7 *Let T be a nonnegative Hamiltonian operator matrix with*

$$Q_2 > 0 \quad \textit{and} \quad \ker(A^* - \lambda) \cap \ker Q_1 = \{0\} \quad \textit{for all} \quad \lambda \in \mathbb{C}, \tag{4.7}$$

or

$$Q_1 > 0 \quad \textit{and} \quad \ker(A - \lambda) \cap \ker Q_2 = \{0\} \quad \textit{for all} \quad \lambda \in \mathbb{C}. \tag{4.8}$$

Then the root subspaces $\mathcal{L}(\lambda)$ of T are J_2-positive if $\operatorname{Re}\lambda > 0$ and J_2-negative if $\operatorname{Re}\lambda < 0$.

Proof. Suppose that (4.7) holds and that $\operatorname{Re}\lambda > 0$; the proofs for the other cases are analogous. From the previous proposition we know that T has no purely imaginary eigenvalues. Proposition 2.6.6 thus implies that $\mathcal{L}(\lambda)$ is J_2-nonnegative. Take $(x,y) \in \mathcal{L}(\lambda)\setminus\{0\}$ and let n be the first natural number such that $(T-\lambda)^n(x,y) = 0$. We use induction on n to show that (x,y) can not be J_2-neutral and is thus J_2-positive.

For $n = 1$ we have

$$\operatorname{Re}\lambda \cdot \left[\begin{pmatrix} x \\ y \end{pmatrix} \,\Big|\, \begin{pmatrix} x \\ y \end{pmatrix}\right] = \operatorname{Re}\left[T\begin{pmatrix} x \\ y \end{pmatrix} \,\Big|\, \begin{pmatrix} x \\ y \end{pmatrix}\right] = (Q_1 y|y) + (Q_2 x|x).$$

If (x,y) was J_2-neutral, then $(Q_1 y|y) + (Q_2 x|x) = 0$. Since Q_1 is nonnegative and Q_2 positive, it follows that $x = 0$. Hence

$$T\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} Q_1 y \\ -A^* y \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ y \end{pmatrix},$$

i.e., $y \in \ker Q_1$ and $A^*y = -\lambda y$. Assumption (4.7) yields $y = 0$, a contradiction.

For $n > 1$ we set

$$\begin{pmatrix} u \\ v \end{pmatrix} = (T - \lambda)\begin{pmatrix} x \\ y \end{pmatrix}.$$

Then (u,v) is J_2-positive by the induction hypothesis. If (x,y) was J_2-neutral, we would have

$$0 = \operatorname{Re}\lambda \cdot \left[\begin{pmatrix} x \\ y \end{pmatrix} \,\Big|\, \begin{pmatrix} x \\ y \end{pmatrix}\right] = \operatorname{Re}\left[T\begin{pmatrix} x \\ y \end{pmatrix} \,\Big|\, \begin{pmatrix} x \\ y \end{pmatrix}\right] - \operatorname{Re}\left[\begin{pmatrix} u \\ v \end{pmatrix} \,\Big|\, \begin{pmatrix} x \\ y \end{pmatrix}\right],$$

i.e.,

$$\mathrm{Re}\Big[\begin{pmatrix}u\\v\end{pmatrix}\Big|\begin{pmatrix}x\\y\end{pmatrix}\Big] = (Q_1y|y) + (Q_2x|x) \geq 0.$$

For $r \in \mathbb{R}$ let

$$w = \begin{pmatrix}u\\v\end{pmatrix} + r\begin{pmatrix}x\\y\end{pmatrix}.$$

Then

$$[w|w] = 2r\,\mathrm{Re}\Big[\begin{pmatrix}u\\v\end{pmatrix}\Big|\begin{pmatrix}x\\y\end{pmatrix}\Big] + \Big[\begin{pmatrix}u\\v\end{pmatrix}\Big|\begin{pmatrix}u\\v\end{pmatrix}\Big].$$

Therefore, if $\mathrm{Re}[(u,v)|(x,y)] > 0$, then $[w|w] < 0$ for r sufficiently small, which is a contradiction to $w \in \mathcal{L}(\lambda)$ J_2-nonnegative. Consequently we have

$$\mathrm{Re}\Big[\begin{pmatrix}u\\v\end{pmatrix}\Big|\begin{pmatrix}x\\y\end{pmatrix}\Big] = (Q_1y|y) + (Q_2x|x) = 0$$

and hence $x = 0$ and $Q_1y = 0$ (Lemma 4.1.4). But this implies that $u = 0$ and hence (u, v) is J_2-neutral, again a contradiction. □

4.2 Invariant graph subspaces in Krein spaces

In this section we derive conditions for a subspace invariant under the Hamiltonian to be the graph of an operator X. We will also see that certain properties of X such as its selfadjointness are equivalent to properties of the graph subspace with respect to the fundamental symmetries J_1 and J_2. These equivalences have also been studied by Dijksma and de Snoo [16] and Langer, Ran and van de Rotten [31].

Finally, for a Hamiltonian operator T with a finitely spectral l^2-decomposition we show that the compatible subspaces associated with a partition of $\sigma_p(T)$ which separates skew-conjugate points are the graphs of selfadjoint operators. The corresponding result for dichotomous Hamiltonian operators and the spectral subspaces associated with the right and left half-plane was obtained in [31].

To an operator $X(H \to H)$, two kinds of *graph subspaces* in $H \times H$ may be associated. We use the notation

$$\Gamma(X) = \Big\{\begin{pmatrix}u\\Xu\end{pmatrix}\Big|\,u \in \mathcal{D}(X)\Big\}, \qquad \mathrm{L}(X) = \Big\{\begin{pmatrix}Xv\\v\end{pmatrix}\Big|\,v \in \mathcal{D}(X)\Big\}. \tag{4.9}$$

Observe that if X is injective, then $\Gamma(X) = \mathrm{L}(X^{-1})$. Furthermore, an arbitrary subspace $U \subset H \times H$ is of the form $U = \Gamma(X)$ if and only if $(0, v) \in U$ implies $v = 0$; in this case

$$\mathcal{D}(X) = \Big\{u \in H\,\Big|\,\exists\, v \in H : \begin{pmatrix}u\\v\end{pmatrix} \in U\Big\} \quad \text{and} \quad Xu = v \;\Leftrightarrow\; \begin{pmatrix}u\\v\end{pmatrix} \in U.$$

Analogously we have $U = \mathrm{L}(X)$ if and only if $(u, 0) \in U$ implies $u = 0$.

Proposition 4.2.1 *Consider an operator* $X(H \to H)$ *on a Hilbert space and let* $U = \Gamma(X)$ *or* $\mathrm{L}(X)$ *be one of its graph subspaces. Then*

(i) U *J_1-neutral (i.e.* $U \subset U^{\langle\perp\rangle}$*)* $\iff$ X *Hermitian;*

(ii) $U = U^{\langle\perp\rangle} \iff X$ *selfadjoint.*

If U *is J_1-neutral, then also*

(iii) U *J_2-nonnegative (-positive)* $\iff$ X *nonnegative (-positive);*

(iv) U *J_2-positive (-negative)* $\iff$ X *positive (negative);*

(v) U *J_2-uniformly positive (negative)* $\iff$ X *bounded and uniformly positive (negative).*

Proof. We consider $U = \Gamma(X)$; the case $U = \mathrm{L}(X)$ is analogous.

(i): U is J_1-neutral if and only if

$$0 = \left\langle \begin{pmatrix} u \\ Xu \end{pmatrix} \middle| \begin{pmatrix} v \\ Xv \end{pmatrix} \right\rangle = i(u|Xv) - i(Xu|v) \quad \text{for all} \quad u, v \in \mathcal{D}(X),$$

that is, X is Hermitian.

(ii): Using (i), we may assume $U \subset U^{\langle\perp\rangle}$ and that X is Hermitian. Therefore

$$\begin{aligned} U = U^{\langle\perp\rangle} &\iff U^{\langle\perp\rangle} \subset U \\ &\iff \Big(\forall g \in U : \langle f|g\rangle = 0 \;\Rightarrow\; f \in U\Big) \\ &\iff \left(\forall u \in \mathcal{D}(X) : \left\langle \begin{pmatrix} x \\ y \end{pmatrix} \middle| \begin{pmatrix} u \\ Xu \end{pmatrix} \right\rangle = 0 \;\Rightarrow\; \begin{pmatrix} x \\ y \end{pmatrix} \in U\right) \\ &\iff \Big(\forall u \in \mathcal{D}(X) : (x|Xu) = (y|u) \;\Rightarrow\; x \in \mathcal{D}(X),\, y = Xx\Big). \end{aligned} \tag{4.10}$$

On the other hand, since X is Hermitian, it is selfadjoint if and only if

$$\mathcal{D}(X) \subset H \text{ dense} \quad \text{and} \quad \mathcal{D}(X^*) \subset \mathcal{D}(X). \tag{4.11}$$

To prove that (4.10) and (4.11) are equivalent, let us first assume that (4.10) holds. Then, if $y \in \mathcal{D}(X)^\perp$, we find $(y|u) = 0 = (0|Xu)$ for all $u \in \mathcal{D}(X)$ and (4.10) implies $y = X0 = 0$; $\mathcal{D}(X)$ is dense. If $x \in \mathcal{D}(X^*)$, we have $(x|Xu) = (X^*x|u)$ for all u and (4.10) yields $x \in \mathcal{D}(X)$. Now suppose (4.11) holds and let $(x|Xu) = (y|u)$ for all $u \in \mathcal{D}(X)$. This implies $x \in \mathcal{D}(X^*)$; so $x \in \mathcal{D}(X)$ and thus $(Xx|u) = (y|u)$ for all u by the Hermiticity of X. Therefore $Xx = y$.

(iii) and (iv): For Hermitian X we have

$$\left[\begin{pmatrix} u \\ Xu \end{pmatrix} \middle| \begin{pmatrix} u \\ Xu \end{pmatrix} \right] = (u|Xu) + (Xu|u) = 2(Xu|u)$$

and the assertions follow immediately.

(v): First suppose that U is J_2-uniformly positive, i.e.,

$$\left[\begin{pmatrix} u \\ Xu \end{pmatrix} \,\middle|\, \begin{pmatrix} u \\ Xu \end{pmatrix}\right] \geq \alpha \left\| \begin{pmatrix} u \\ Xu \end{pmatrix} \right\|^2 = \alpha\|u\|^2 + \alpha\|Xu\|^2$$

for all $u \in \mathcal{D}(X)$. Therefore

$$2\|Xu\|\|u\| \geq 2(Xu|u) \geq \alpha\|u\|^2 + \alpha\|Xu\|^2,$$

which implies

$$(Xu|u) \geq \frac{\alpha}{2}\|u\|^2 \quad \text{and} \quad \frac{2}{\alpha}\|u\| \geq \|Xu\|$$

for all $u \in \mathcal{D}(X)$; X is bounded and uniformly positive. If on the other hand X is bounded and uniformly positive, we can estimate

$$\left[\begin{pmatrix} u \\ Xu \end{pmatrix} \,\middle|\, \begin{pmatrix} u \\ Xu \end{pmatrix}\right] = 2(Xu|u) \geq 2\beta\|u\|^2 \geq \beta\|u\|^2 + \frac{\beta}{\|X\|^2}\|Xu\|^2,$$

and consequently U is uniformly positive. The negative case is analogous. □

The next lemma in conjunction with Proposition 4.2.1(v) is crucial to prove the boundedness of solutions of Riccati equations.

Lemma 4.2.2 *Let X_+, X_- be bounded selfadjoint operators on a Hilbert space H with X_+ uniformly positive and X_- nonpositive. If $X(H \to H)$ is a Hermitian operator satisfying*

$$\mathcal{D}(X) = D_+ \dotplus D_-, \qquad Xu = \begin{cases} X_+u & \text{if } u \in D_+, \\ X_-u & \text{if } u \in D_-, \end{cases}$$

then X is bounded.

Proof. First consider $u \in D_+$, $v \in D_-$ with $\|u\| = \|v\| = 1$. Then

$$\begin{aligned}(u - v|X_+u + X_-v) &= (u|X_+u) - (v|X_+u) + (u|X_-v) - (v|X_-v) \\ &= (u|X_+u) - (v|Xu) + (u|Xv) - (v|X_-v).\end{aligned}$$

Using the Hermiticity of X and the assumptions on $X_\pm$, we find that

$$\operatorname{Re}(u - v|X_+u + X_-v) = (u|X_+u) - (v|X_-v) \geq \gamma$$

with some constant $\gamma > 0$ and hence

$$\gamma \leq |(u - v|X_+u + X_-v)| \leq \|u - v\| \cdot \big(\|X_+\| + \|X_-\|\big).$$

This implies

$$\left(\frac{\gamma}{\|X_+\| + \|X_-\|}\right)^2 \le \|u - v\|^2 = 2 - 2\,\mathrm{Re}(u|v)$$

and

$$\mathrm{Re}(u|v) \le 1 - \delta \quad \text{with} \quad \delta = \frac{1}{2}\left(\frac{\gamma}{\|X_+\| + \|X_-\|}\right)^2 > 0.$$

Consequently

$$|(u|v)| \le 1 - \delta \quad \text{for all} \quad u \in D_+, v \in D_- \text{ with } \|u\| = \|v\| = 1.$$

Now for arbitrary $u \in D_+$, $v \in D_-$ we have the estimates

$$\begin{aligned}
&\|X(u+v)\| = \|X_+u + X_-v\| \le \max\{\|X_+\|, \|X_-\|\}\big(\|u\| + \|v\|\big),\\
&\big(\|u\| + \|v\|\big)^2 \le 2\big(\|u\|^2 + \|v\|^2\big),\\
&\|u+v\|^2 \ge \|u\|^2 + \|v\|^2 - 2|(u|v)| \ge \|u\|^2 + \|v\|^2 - 2(1-\delta)\|u\|\|v\|\\
&\qquad\quad \ge \|u\|^2 + \|v\|^2 - (1-\delta)\big(\|u\|^2 + \|v\|^2\big) = \delta\big(\|u\|^2 + \|v\|^2\big).
\end{aligned}$$

Therefore

$$\|X(u+v)\| \le \sqrt{\frac{2}{\delta}}\max\big\{\|X_+\|, \|X_-\|\big\}\|u+v\|$$

and X is bounded. □

The first component of an l^2-decomposition of the graph subspace of a bounded operator is again an l^2-decomposition:

Lemma 4.2.3 *Consider a bounded operator $X : H \to H$ whose graph admits an l^2-decomposition*

$$\Gamma(X) = \left\{\begin{pmatrix} u \\ Xu \end{pmatrix} \,\middle|\, u \in H\right\} = \bigoplus_{k\in\mathbb{N}}^2 U_k.$$

If D_k are the subspaces obtained by projection of U_k onto the first component, i.e.

$$U_k = \left\{\begin{pmatrix} u \\ Xu \end{pmatrix} \,\middle|\, u \in D_k\right\},$$

then $(D_k)_{k\in\mathbb{N}}$ forms an l^2-decomposition for H.

Proof. With c the constant corresponding to the decomposition $\bigoplus_k^2 U_k$, $u_k \in D_k$ and $n \in \mathbb{N}$, we have the estimates

$$\begin{gathered}
c^{-1}\sum_{k=0}^{n}\|u_k\|^2 \le c^{-1}\sum_{k=0}^{n}\left\|\begin{pmatrix} u_k \\ Xu_k \end{pmatrix}\right\|^2 \le \left\|\sum_{k=0}^{n}\begin{pmatrix} u_k \\ Xu_k \end{pmatrix}\right\|^2 \le (1 + \|X\|^2)\left\|\sum_{k=0}^{n} u_k\right\|^2,\\
\left\|\sum_{k=0}^{n} u_k\right\|^2 \le \left\|\sum_{k=0}^{n}\begin{pmatrix} u_k \\ Xu_k \end{pmatrix}\right\|^2 \le c\sum_{k=0}^{n}\left\|\begin{pmatrix} u_k \\ Xu_k \end{pmatrix}\right\|^2 \le c(1 + \|X\|^2)\sum_{k=0}^{n}\|u_k\|^2.
\end{gathered}$$

For arbitrary $u \in H$ we can expand $(u, Xu) \in U$ according to the l^2-decomposition $\bigoplus_k^2 U_k$ as

$$\begin{pmatrix} u \\ Xu \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} u_k \\ Xu_k \end{pmatrix} \quad \text{with} \quad u_k \in D_k.$$

Consequently $u = \sum_{k=0}^{\infty} u_k$ and $\sum_{k \in \mathbb{N}} D_k \subset H$ is dense. □

In some cases, the boundedness of X can be characterised via Riesz bases as follows, cf. Kuiper and Zwart [29, Theorem 5.6]:

Remark 4.2.4 If the graph $\Gamma(X)$ of a closed densely defined operator $X(H \to H)$ has a Riesz basis of the form $(e_k, Xe_k)_{k\in\mathbb{N}}$, then X is bounded if and only if $(e_k)_{k\in\mathbb{N}}$ is a Riesz basis for H. The proof of the implication from left to right is completely analogous to the previous lemma. For the other direction, the estimate

$$\Big\| X \sum_{k=0}^{n} \alpha_k e_k \Big\|^2 \leq \Big\| \sum_{k=0}^{n} \alpha_k \begin{pmatrix} e_k \\ Xe_k \end{pmatrix} \Big\|^2 \leq M \sum_{k=0}^{n} |\alpha_k|^2 \leq \frac{M}{m_1} \Big\| \sum_{k=0}^{n} \alpha_k e_k \Big\|^2$$

shows the boundedness of X.

Note however, if the graph of X has an l^2-decomposition $\Gamma(X) = \bigoplus_{k\in\mathbb{N}}^2 U_k$ and, with the notation from the lemma, $(D_k)_k$ forms an l^2-decomposition of H, then X need not be bounded in general. As a counter example, consider a selfadjoint X with an orthonormal basis $(e_k)_k$ of eigenvectors such that $Xe_k = ke_k$, and $D_k = \mathbb{C}e_k$. ⌟

The next two propositions show that under appropriate assumptions on the Hamiltonian all neutral invariant subspaces are graph subspaces.

Proposition 4.2.5 *Consider a nonnegative Hamiltonian operator matrix T with $it \in \varrho(T)$ for some $t \in \mathbb{R}$ and a J_1-neutral subspace U that is $(T - it)^{-1}$-invariant.*

(i) If Q_1 is positive and $it \notin \sigma_p(A)$, then $U = \Gamma(X)$ for a Hermitian operator $X(H \to H)$.

(ii) If Q_2 is positive and $-it \notin \sigma_p(A^*)$, then $U = \mathrm{L}(Y)$ for a Hermitian operator $Y(H \to H)$.

(iii) If Q_1 and Q_2 are positive (i.e., T is positive), then $U = \Gamma(X)$ with $X(H \to H)$ Hermitian and injective.

Proof. (i): It suffices to show that $(0, v) \in U$ implies $v = 0$. The Hermiticity of X is then immediate from Proposition 4.2.1. For $(0, v) \in U$ we set

$$\begin{pmatrix} x \\ y \end{pmatrix} = (T - it)^{-1} \begin{pmatrix} 0 \\ v \end{pmatrix},$$

i.e.,

$$(A - it)x + Q_1 y = 0, \quad Q_2 x - (A^* + it)y = v.$$

Using the neutrality and $(T - it)^{-1}$-invariance of U, we have

$$0 = \left\langle \begin{pmatrix} 0 \\ v \end{pmatrix} \,\middle|\, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = -i(v|x)$$

and thus

$$0 = (v|x) = (Q_2 x|x) - (y|(A - it)x) = (Q_2 x|x) + (y|Q_1 y).$$

As Q_1 and Q_2 are nonnegative, this implies $(Q_2 x|x) = (Q_1 y|y) = 0$. The positivity of Q_1 yields $y = 0$ and consequently $(A - it)x = 0$. Since $it \notin \sigma_p(A)$, we obtain $x = 0$ and $v = 0$.

(ii): It suffices to show that $(u, 0) \in U$ implies $u = 0$. The proof is then completely analogous to (i).

(iii): Let $(0, v) \in U$ and choose (x, y) as above. Thus $(Q_2 x|x) = (Q_1 y|y) = 0$ and from the positivity of Q_1 and Q_2 we conclude that $x = y = 0$ and hence $v = 0$. The proof that $(u, 0) \in U$ implies $u = 0$ is analogous. □

The following proposition uses a method due to Langer, Ran and van de Rotten [31, Theorem 5.1]. There, Q_1 and Q_2 are assumed to be bounded and the conditions analogous to (4.12) and (4.13) are referred to as *approximate controllability* and *observability*, respectively.

Proposition 4.2.6 *Consider a nonnegative Hamiltonian operator matrix and a domain $M \subset \varrho(A)$ such that $i\mathbb{R} \cap \varrho(T) \cap M$ has an accumulation point in M and*

$$\operatorname{span}\big\{(A - z)^{-1} Q_1^* u \,\big|\, z \in M,\, u \in \mathcal{D}(Q_1^*)\big\} \subset H \quad \textit{is dense.} \tag{4.12}$$

If U is a J_1-neutral subspace that is $(T - it)^{-1}$-invariant for all $it \in i\mathbb{R} \cap \varrho(T) \cap M$, then $U = \Gamma(X)$ is the graph of a Hermitian operator $X(H \to H)$.

If instead of (4.12) we have

$$\operatorname{span}\big\{(A^* - \bar{z})^{-1} Q_2^* v \,\big|\, z \in M,\, v \in \mathcal{D}(Q_2^*)\big\} \subset H \quad \textit{dense,} \tag{4.13}$$

then $U = \mathrm{L}(Y)$ for a Hermitian operator $Y(H \to H)$.

Proof. As in the proof of Proposition 4.2.5 we consider an element $(0, v) \in U$ and set $(x, y) = (T - it)^{-1}(0, v)$ for $it \in i\mathbb{R} \cap \varrho(T) \cap M$. We thus have

$$0 = (v|x) = (Q_2 x|x) + (Q_1 y|y).$$

Since Q_1 and Q_2 are nonnegative, Lemma 4.1.4 implies $Q_2 x = Q_1 y = 0$ and hence $-(A^* + it)y = v$. From $it \in \varrho(A)$ we get $-it \in \varrho(A^*)$ and

$$0 = Q_1 y = -Q_1 (A^* + it)^{-1} v.$$

For $u \in \mathcal{D}(Q_1^*)$ this implies

$$0 = \big(Q_1(A^* + it)^{-1}v\big|u\big) = \big((A^* + it)^{-1}v\big|Q_1^*u\big).$$

Consequently, the function $f(\zeta) = ((A^* - \zeta)^{-1}v|Q_1^*u)$, which is holomorphic on the complex conjugate of M, satisfies $f(-it) = 0$ for all $it \in i\mathbb{R} \cap \varrho(T) \cap M$. From the identity theorem we obtain

$$0 = \big((A^* - \bar{z})^{-1}v\big|Q_1^*u\big) = \big(v\,\big|\,(A - z)^{-1}Q_1^*u\big) \quad \text{for all } z \in M.$$

Since $u \in \mathcal{D}(Q_1^*)$ was arbitrary, the density assumption (4.12) now implies $v = 0$. The proof for the case of (4.13) is analogous. □

The density conditions (4.12) and (4.13) are closely related to the spectral conditions (4.6), (4.7) and (4.8).

Lemma 4.2.7 *Let $A(H \to H)$ be a normal operator with compact resolvent and $M \subset \varrho(A)$ a set with accumulation point in $\varrho(A)$. If the closed densely defined operator $Q(H \to H)$ is such that $\ker Q$ contains no eigenvectors of A, then*

$$\operatorname{span}\big\{(A - z)^{-1}Q^*v\,\big|\,z \in M,\, v \in \mathcal{D}(Q^*)\big\} \subset H \quad \textit{dense.}$$

Proof. Let $(\lambda_k)_{k\in\mathbb{N}}$ be an enumeration of the eigenvalues of A and P_k the corresponding orthogonal projections onto the eigenspaces. Let $u \in H$ be such that

$$u \perp \operatorname{span}\big\{(A - z)^{-1}Q^*v\,\big|\,z \in M,\, v \in \mathcal{D}(Q^*)\big\},$$

i.e., $((A - z)^{-1}Q^*v|u) = 0$ for all $z \in M$, $v \in \mathcal{D}(Q^*)$. Due to the estimate $\sum_k |(P_kQ^*v|u)| \le \|Q^*v\|\|u\|$, the series

$$f(z) = \big((A - z)^{-1}Q^*v\big|u\big) = \sum_{k\in\mathbb{N}} \frac{1}{\lambda_k - z}(P_kQ^*v|u)$$

converges absolutely and uniformly on every compact subset of $\varrho(A)$ and is a holomorphic function on $\varrho(A)$. We have $f(z) = 0$ for $z \in M$ and hence $f = 0$ by the identity theorem. If we integrate the series along a circle in $\varrho(A)$ enclosing exactly one λ_k, we obtain

$$0 = -2\pi i(P_kQ^*v|u).$$

Consequently $(Q^*v|P_ku) = 0$ for all $k \in \mathbb{N}$, $v \in \mathcal{D}(Q^*)$, i.e.

$$P_ku \in \mathcal{R}(Q^*)^\perp = \ker Q^{**} = \ker Q.$$

The assumption now implies $P_ku = 0$ for all $k \in \mathbb{N}$ and thus $u = 0$. □

Proposition 4.2.8 *Consider densely defined operators $A, Q(H \to H)$ and a set $M \subset \varrho(A)$ with accumulation point in $\varrho(A)$. For the assertions*

(i) $\ker(A^* - it) \cap \ker Q = \{0\}$ *for all* $t \in \mathbb{R}$,

(ii) $\ker(A^* - \lambda) \cap \ker Q = \{0\}$ *for all* $\lambda \in \mathbb{C}$,

(iii) $\operatorname{span}\{(A - z)^{-1} Q^* v \mid z \in M, v \in \mathcal{D}(Q^*)\} \subset H$ *dense,*

we have the implications (iii) ⇒ (ii) ⇒ (i). If A is normal with compact resolvent and furthermore Q is closed or $\mathcal{D}(A) \subset \mathcal{D}(Q)$, then (ii) ⇔ (iii).

Proof. (ii)⇒(i) is trivial. For (iii)⇒(ii) consider $A^*u = \lambda u$, $Qu = 0$. Then for every $z \in M$, $v \in \mathcal{D}(Q^*)$ we have

$$\big((A - z)^{-1} Q^* v \big| u\big) = \big(v \big| Q(A^* - \bar{z})^{-1} u\big) = \big(v \big| Q(\lambda - \bar{z})^{-1} u\big) = 0$$

and the density assumption implies $u = 0$. Under the additional conditions the eigenvectors of A and A^* coincide, and we have $\mathcal{D}(A) \cap \ker Q = \mathcal{D}(A) \cap \ker \overline{Q}$. Hence, (ii)⇒(iii) is a consequence of the previous lemma. □

For Hamiltonian operator matrices with a finitely spectral l^2-decomposition, we now prove the existence of invariant subspaces which are the graph of selfadjoint operators. Note that by Corollary 4.1.3 the point spectrum of a Hamiltonian with a finitely determining l^2-decomposition is symmetric with respect to the imaginary axis.

Definition 4.2.9 For an operator T whose point spectrum is symmetric with respect to the imaginary axis and satisfies $\sigma_p(T) \cap i\mathbb{R} = \varnothing$, we say that a partition $\sigma_p(T) = \sigma \mathbin{\dot\cup} \tau$ *separates skew-conjugate points* if

$$\lambda \in \sigma \quad \Longleftrightarrow \quad -\overline{\lambda} \in \tau.$$

⌟

Lemma 4.2.10 *Let T be a Hamiltonian operator matrix with $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ and a finitely spectral l^2-decomposition $H \times H = \bigoplus^2_{k\in\mathbb{N}} V_k$. If $\sigma_p(T) = \sigma \mathbin{\dot\cup} \tau$ is a partition which separates skew-conjugate points and the compatible subspace associated with σ is the graph $\Gamma(X)$ or $\mathrm{L}(X)$ of an operator $X(H \to H)$, then X is selfadjoint.*

If T is nonnegative and $\Gamma(X)$ or $\mathrm{L}(X)$ is the compatible subspace associated with $\sigma_p^+(T)$ and $\sigma_p^-(T)$, then X is nonnegative and nonpositive, respectively.

Proof. If $\Gamma(X)$ is the compatible subspace associated with σ, then $\Gamma(X) = \Gamma(X)^{\langle\perp\rangle}$ by Theorem 2.5.16; Proposition 4.2.1 yields the selfadjointness of X. If T is nonnegative and $\sigma = \sigma_p^{\pm}(T)$, then T is J_2-accretive and $\Gamma(X)$ J_2-nonnegative/-nonpositive by Proposition 2.6.6. Proposition 4.2.1 now implies that X is nonnegative/nonpositive. □

Proposition 4.2.11 *Consider a nonnegative Hamiltonian operator matrix T with $\sigma_p(T)\cap i\mathbb{R} = \varnothing$ and a finitely spectral l^2-decomposition $\bigoplus_{k\in\mathbb{N}}^2 V_k$. Let $\sigma_p(T) = \sigma \mathbin{\dot\cup} \tau$ be a partition which separates skew-conjugate points and U the compatible subspace associated with σ. If*

(a1) Q_1 is positive and $it \in \varrho(T) \setminus \sigma_p(A)$ for some $t \in \mathbb{R}$, or

(a2) there is a domain $M \subset \varrho(A)$ such that $i\mathbb{R} \cap \varrho(T) \cap M$ has an accumulation point in M and

$$\operatorname{span}\bigl\{(A - z)^{-1}Q_1^* u \,\big|\, z \in M,\, u \in \mathcal{D}(Q_1^*)\bigr\} \subset H \quad \textit{dense},$$

then U is the graph $U = \Gamma(X)$ of a selfadjoint operator $X(H \to H)$. If

(b1) Q_2 is positive and $it \in \varrho(T) \setminus \sigma_p(-A^*)$ for some $t \in \mathbb{R}$, or

(b2) there is a domain $M \subset \varrho(A)$ such that $i\mathbb{R} \cap \varrho(T) \cap M$ has an accumulation point in M and

$$\operatorname{span}\bigl\{(A^* - \bar z)^{-1}Q_2^* v \,\big|\, z \in M,\, v \in \mathcal{D}(Q_2^*)\bigr\} \subset H \quad \textit{dense},$$

then U is the graph $U = \mathrm{L}(Y)$ of a selfadjoint operator $Y(H \to H)$.

If the conditions (a1) or (a2) as well as (b1) or (b2) are satisfied, or if T is positive with $\varrho(T) \cap i\mathbb{R} \neq \varnothing$, then $U = \Gamma(X)$ with X selfadjoint and injective.

Proof. The subspace U is J_1-neutral and $(T - \lambda)^{-1}$-invariant for all $\lambda \in \varrho(T)$, see Theorem 2.5.16. The representation as a graph subspace is thus a direct consequence of Propositions 4.2.5 and 4.2.6. The selfadjointness of X and Y follows from Lemma 4.2.10. □

Strengthening the assumptions on T, we obtain the boundedness and bounded invertibility of the operator X:

Proposition 4.2.12 *Consider a closed, uniformly positive Hamiltonian operator T with $Q_1, Q_2 \geq \gamma$, $\dim \mathcal{L}(\lambda) < \infty$ for all $\lambda \in \sigma_p(T)$, and*

$$\sigma_p(T) \subset \bigl\{z \in \mathbb{C} \,\big|\, |\operatorname{Re} z| \leq a\bigr\}$$

for some $a > 0$. Suppose that T has a Riesz basis of Jordan chains. Then

$$\bigl\{z \in \mathbb{C} \,\big|\, |\operatorname{Re} z| < \gamma\bigr\} \subset \varrho(T)$$

and for every partition $\sigma_p(T) = \sigma \mathbin{\dot\cup} \tau$ which separates skew-conjugate points the compatible subspace associated with σ is the graph $\Gamma(X)$ of a selfadjoint isomorphism $X : H \to H$. The operators $X_\pm$ corresponding to $\sigma = \sigma_p^\pm(T)$ are uniformly positive and negative, respectively.

Proof. By Proposition 4.1.6 we have $z \in \varrho(T)$ for $|\operatorname{Re} z| < \gamma$. From Lemma 2.3.15(i) it follows that the root subspaces form a finitely spectral l^2-decomposition. Proposition 4.2.11 implies that the compatible subspace associated with σ is the graph $\Gamma(X)$ of a selfadjoint injective operator. By Proposition 2.6.6, the compatible subspaces $\Gamma(X_\pm)$ associated with $\sigma_p^\pm(T)$ are uniformly J_2-positive and -negative, respectively; $X_\pm$ are then bounded and uniformly positive/negative due to Proposition 4.2.1. Since the root subspaces form an l^2-decomposition, we have

$$\Gamma(X) = W_+ \oplus W_- \quad \text{with} \quad W_+ = \bigoplus_{\substack{\lambda\in\sigma \\ \operatorname{Re}\lambda>0}}^2 \mathcal{L}(\lambda), \quad W_- = \bigoplus_{\substack{\lambda\in\sigma \\ \operatorname{Re}\lambda<0}}^2 \mathcal{L}(\lambda).$$

If we denote by $D_\pm$ the subspaces obtained by projecting $W_\pm$ onto the first component, then $\mathcal{D}(X) = D_+ \dotplus D_-$ and $X|_{D_\pm} = X_\pm|_{D_\pm}$. Therefore X is bounded by Lemma 4.2.2. With $\Gamma(X) = \mathrm{L}(X^{-1})$ and $\Gamma(X_\pm) = \mathrm{L}(X_\pm^{-1})$, an analogous argument yields that X^{-1} is also bounded. □

4.3 Invariant graph subspaces and the Riccati equation

We study the correspondence between graph subspaces $\Gamma(X)$ invariant under the Hamiltonian and solutions X of the associated Riccati equation. There are several notions of strong and weak solutions of the Riccati equation depending on the boundedness of A, Q_1, Q_2 and X. For the case that all these operators are unbounded, we introduce the concept of a core solution which ensures that the Riccati equation holds on a core of X.

In control theory, the case of bounded Q_1, Q_2 and X typically occurs and the weak form of the Riccati equation is widely used, see e.g. [14, 29]. Langer, Ran and van de Rotten [31] considered bounded as well as unbounded solutions of the strong and weak Riccati equation for bounded Q_1, Q_2. For a bounded selfadjoint block operator matrix, Kostrykin, Makarov and Motovilov [26] explicitly defined the notion of unbounded strong and weak solutions of the associated Riccati equation. They showed the equivalence of strong solutions, weak solutions and invariant graph subspaces in their setting. The corresponding result for bounded Hamiltonians is contained in Proposition 4.3.1.

In this section we consider Hamiltonian operators that are diagonally dominant (cf. page 97) and not necessarily nonnegative. In fact, analogous results hold for general densely defined block operator matrices

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $\mathcal{D}(A) \subset \mathcal{D}(C)$ and $\mathcal{D}(D) \subset \mathcal{D}(B)$.

Proposition 4.3.1 *Let T be a diagonally dominant Hamiltonian operator matrix. Then for the operator $X(H \to H)$ the following two statements are equivalent:*

(i) The graph $\Gamma(X)$ of X is a T-invariant subspace.

(ii) X is a solution of the Riccati equation

$$X(A + Q_1 X) = Q_2 - A^* X \quad \text{on} \quad \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*). \tag{4.14}$$

In particular, $u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ implies $Au + Q_1 Xu \in \mathcal{D}(X)$.

For densely defined X, (i) or (ii) imply

(iii) X is a solution of the weak Riccati equation

$$\begin{aligned} &(Xu|Av) + (Au|X^*v) + (Q_1 Xu|X^*v) - (Q_2 u|v) = 0, \\ &\qquad\qquad u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*), \quad v \in \mathcal{D}(A) \cap \mathcal{D}(X^*). \end{aligned} \tag{4.15}$$

If moreover X is closed, densely defined, and $\mathcal{D}(A) \cap \mathcal{D}(X^*)$ is a core for X^*, then both (i) and (ii) are equivalent to (iii).

Proof. The graph of X is T-invariant if and only if for all $u \in \mathcal{D}(A) \cap \mathcal{D}(X)$ with $Xu \in \mathcal{D}(A^*)$ there exists $v \in \mathcal{D}(X)$ such that

$$\begin{pmatrix} Au + Q_1 Xu \\ Q_2 u - A^* Xu \end{pmatrix} = \begin{pmatrix} v \\ Xv \end{pmatrix}.$$

This is obviously equivalent to (ii). For densely defined X, (4.15) is easily obtained from (4.14) by taking the scalar product with $v \in \mathcal{D}(A) \cap \mathcal{D}(X^*)$. If we finally assume (4.15), we can rewrite it as

$$(Au + Q_1 Xu|X^*v) = (Q_2 u - A^* Xu|v).$$

If $\mathcal{D}(A) \cap \mathcal{D}(X^*)$ is a core for X^*, this equation holds for all $v \in \mathcal{D}(X^*)$. Furthermore, the right-hand side is continuous in v, and we have $X^{**} = X$ if X is closed. This implies $Au + Q_1 Xu \in \mathcal{D}(X^{**}) = \mathcal{D}(X)$ and

$$X(Au + Q_1 Xu) = Q_2 u - A^* Xu.$$

□

Note that (4.14) always has the trivial solution $X(H \to H)$ with $\mathcal{D}(X) = \{0\}$. And even if X is densely defined, $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*) = \{0\}$ is still possible in general. With the following definition we exclude such trivial solutions.

Definition 4.3.2 Let T be a diagonally dominant Hamiltonian operator matrix. The operator $X(H \to H)$ is called a *core solution* of the Riccati equation

$$X(A + Q_1 X) = Q_2 - A^* X \tag{4.16}$$

if X is densely defined, $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for X, and X satisfies (4.16) on $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$. ⌟

Corollary 4.3.3 *The selfadjoint operator X is a core solution of (4.16) if and only if $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for X and X solves the weak Riccati equation*

$$(Xu|Av)+(Au|Xv)+(Q_1Xu|Xv)-(Q_2u|v) = 0, \quad u,v \in \mathcal{D}(A)\cap X^{-1}\mathcal{D}(A^*). \tag{4.17}$$

Proof. If $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for X, (4.17) implies that

$$(A^*Xu|v) + (Au|Xv) + (Q_1Xu|Xv) - (Q_2u|v) = 0$$

for all $u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$, $v \in \mathcal{D}(X)$; in particular, X is a solution of (4.15). The claim is thus a direct consequence of Proposition 4.3.1. □

Proposition 4.3.4 *Consider a diagonally dominant Hamiltonian operator matrix T with a finitely determining l^2-decomposition $\bigoplus^2_{k\in\mathbb{N}} V_k$. If $X(H \to H)$ is a densely defined operator whose graph $\Gamma(X)$ is a T-invariant subspace compatible with $\bigoplus^2_k V_k$, then X is a core solution of (4.16).*

Proof. From $\Gamma(X) = \bigoplus^2_k U_k$ with $U_k \subset V_k$ T-invariant, it follows that $\sum_k U_k$ is dense in $\Gamma(X)$ and hence the subspace obtained by projection of $\sum_k U_k$ onto the first component is a core for X. This subspace is also a subset of $\mathcal{D}(A)\cap X^{-1}\mathcal{D}(A^*)$ since $\sum_k U_k \subset \mathcal{D}(T)$; hence $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for X. Finally, Proposition 4.3.1 shows that X solves the Riccati equation. □

If Q_1 and X are bounded, we obtain Riccati equations on larger domains. For the case that Q_2 is bounded too, this result is well known in control theory, compare [14, Exercise 6.25] and [29, Lemma 5.1].

Proposition 4.3.5 *Let T be a diagonally dominant Hamiltonian operator matrix with $Q_1 : H \to H$ bounded. Then for the bounded operator $X : H \to H$ the following statements are equivalent:*

(i) The graph $\Gamma(X)$ is T-invariant and $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for A.

(ii) X is a solution of the Riccati equation

$$A^*X + XA + XQ_1X - Q_2 = 0 \quad \text{on} \quad \mathcal{D}(A); \tag{4.18}$$

in particular $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$.

(iii) X is a solution of the weak Riccati equation

$$(Xu|Av) + (Au|X^*v) + (Q_1Xu|X^*v) - (Q_2u|v) = 0, \quad u, v \in \mathcal{D}(A). \quad (4.19)$$

Proof. The implication (ii)⇒(i) follows from Proposition 4.3.1 and the fact that

$$X\mathcal{D}(A) \subset \mathcal{D}(A^*) \quad \Leftrightarrow \quad \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*) = \mathcal{D}(A).$$

For (iii)⇒(ii), we rewrite (4.19) as

$$(Xu|Av) = (-XAu - XQ_1Xu + Q_2u|v), \qquad u, v \in \mathcal{D}(A).$$

Since the right-hand side is continuous in v, we obtain $Xu \in \mathcal{D}(A^*)$ and (4.18). If we finally assume (i), Proposition 4.3.1 yields that for $u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ and $v \in \mathcal{D}(A)$

$$(Xu|Av) + (Au|X^*v) + (Q_1Xu|X^*v) - (u|Q_2v) = 0.$$

Since this equation is valid for u in a core for A, and Q_1 and X are bounded, the equation also holds for $u \in \mathcal{D}(A)$; (iii) is proved. □

Note that all bounded solutions of (4.18) are core solutions of (4.16).

Remark 4.3.6 In a completely analogous way, T-invariant graph subspaces $\mathrm{L}(Y)$ are related to the Riccati equation

$$AY + YA^* - YQ_2Y + Q_1 = 0.$$

⌟

A solution X of a Riccati equation leads to a transformation of the Hamiltonian to upper block triangular form. The transformation is given by the block operator

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix} : \mathcal{D}(X) \times H \to \mathcal{D}(X) \times H,$$

which is bijective with inverse

$$\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -X & I \end{pmatrix}.$$

This transformation was also studied by Kuiper and Zwart [29, Lemma 5.5] for Q_1, Q_2, X bounded and A the generator of a C_0-semigroup.

Proposition 4.3.7 *Consider a diagonally dominant Hamiltonian T and a solution $X(H \to H)$ of the associated Riccati equation (4.14), i.e., the graph $\Gamma(X)$ of X is T-invariant. Then we have*

$$\begin{pmatrix} I & 0 \\ -X & I \end{pmatrix} \begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} = \begin{pmatrix} A + Q_1X & Q_1 \\ 0 & -A^* - XQ_1 \end{pmatrix} \tag{4.20}$$

on $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*) \times \mathcal{D}(A^*) \cap Q_1^{-1}\mathcal{D}(X)$. For $\lambda \in \mathbb{C}$ and $u \in \mathcal{D}(X)$ we obtain

$$u \in \ker\big((A + Q_1X - \lambda)|_{\mathcal{D}(A)\cap X^{-1}\mathcal{D}(A^*)}\big)^k \quad \Leftrightarrow \quad \begin{pmatrix} u \\ Xu \end{pmatrix} \in \ker(T-\lambda)^k, \tag{4.21}$$

in particular $\sigma_p(A + Q_1X|_{\mathcal{D}(A)\cap X^{-1}\mathcal{D}(A^*)}) = \sigma_p(T|_{\Gamma(X)})$.

If $X : H \to H$ is bounded, then

$$\varrho\big((A + Q_1X)|_{\mathcal{D}(A)\cap X^{-1}\mathcal{D}(A^*)}\big) = \varrho(T|_{\Gamma(X)}).$$

If moreover $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$, then $\Gamma(X)$ is also $(T-\lambda)^{-1}$-invariant for every $\lambda \in \varrho(T) \cap \varrho(A + Q_1X)$.

Proof. Let $u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$, $v \in \mathcal{D}(A^*)$. Then, using the Riccati equation,

$$\begin{aligned} \begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix} \begin{pmatrix} u \\ Xu + v \end{pmatrix} \\ &= \begin{pmatrix} Au + Q_1Xu + Q_1v \\ Q_2u - A^*Xu - A^*v \end{pmatrix} = \begin{pmatrix} Au + Q_1Xu + Q_1v \\ X(Au + Q_1Xu) - A^*v \end{pmatrix}. \end{aligned}$$

If $Q_1v \in \mathcal{D}(X)$, we can rewrite this as

$$\begin{aligned} \begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} Au + Q_1Xu + Q_1v \\ X(Au + Q_1Xu + Q_1v) - A^*v - XQ_1v \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ X & I \end{pmatrix} \begin{pmatrix} A + Q_1X & Q_1 \\ 0 & -A^* - XQ_1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \end{aligned}$$

and obtain (4.20). Now we consider the mapping

$$\varphi \,:\, \mathcal{D}(X) \to \Gamma(X), \quad u \mapsto \begin{pmatrix} u \\ Xu \end{pmatrix},$$

which is bijective and maps $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ onto $\Gamma(X) \cap \mathcal{D}(T)$. This implies

$$\varphi^{-1}T|_{\Gamma(X)}\varphi \,:\, \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*) \to \mathcal{D}(X), \quad u \mapsto Au + Q_1Xu.$$

Consequently

$$\varphi^{-1}(T|_{\Gamma(X)} - \lambda)^k\varphi = \big((A + Q_1X - \lambda)|_{\mathcal{D}(A)\cap X^{-1}\mathcal{D}(A^*)}\big)^k; \tag{4.22}$$

hence (4.21) and the equality of the point spectra.

If X is bounded, then $\varphi : H \to \Gamma(X)$ is an isomorphism, and (4.22) with $k = 1$ implies the equality of the resolvent sets. Now suppose that $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ too. Then (4.20) holds on $\mathcal{D}(A) \times \mathcal{D}(A^*)$. Let $E = A + Q_1X$, $F = -A^* - XQ_1$,

$$\widetilde{T} = \begin{pmatrix} E & Q_1 \\ 0 & F \end{pmatrix},$$

and $\lambda \in \varrho(T) \cap \varrho(E)$. Then $\lambda \in \varrho(\widetilde{T})$. In particular, $\widetilde{T} - \lambda$ is surjective and so $F - \lambda$ must be surjective. From the surjectivity of $E - \lambda$ and the injectivity of $\widetilde{T} - \lambda$ it follows that $F - \lambda$ is also injective. Consequently

$$(\widetilde{T} - \lambda)^{-1} = \begin{pmatrix} (E-\lambda)^{-1} & -(E-\lambda)^{-1}Q_1(F-\lambda)^{-1} \\ 0 & (F-\lambda)^{-1} \end{pmatrix}.$$

Therefore $H \times \{0\}$ is $(\widetilde{T} - \lambda)^{-1}$-invariant. Since $\left(\begin{smallmatrix} I & 0 \\ X & I \end{smallmatrix}\right)$ maps $H \times \{0\}$ onto $\Gamma(X)$, we conclude that $\Gamma(X)$ is $(T - \lambda)^{-1}$-invariant. □

Remark 4.3.8 For a diagonally dominant Hamiltonian operator matrix and solutions $X, Y : H \to H$ of the Riccati equations

$$\begin{aligned} A^*X + XA + XQ_1X - Q_2 &= 0 \quad \text{on} \quad \mathcal{D}(A), \\ AY + YA^* - YQ_2Y + Q_1 &= 0 \quad \text{on} \quad \mathcal{D}(A^*) \end{aligned}$$

such that $\left(\begin{smallmatrix} I & Y \\ X & I \end{smallmatrix}\right)$ is invertible, we obtain the block diagonalisation

$$\begin{pmatrix} I & Y \\ X & I \end{pmatrix}^{-1} \begin{pmatrix} A & Q_1 \\ Q_2 & -A^* \end{pmatrix} \begin{pmatrix} I & Y \\ X & I \end{pmatrix} = \begin{pmatrix} A + Q_1X & 0 \\ 0 & -A^* + Q_2Y \end{pmatrix}.$$ ⌟

With the following proposition we establish a one-to-one correspondence between bounded solutions of the Riccati equation and invariant graph subspaces of bounded operators compatible with a spectral l^2-decomposition of the Hamiltonian.

Proposition 4.3.9 *Consider a diagonally dominant Hamiltonian operator T with $Q_1 : H \to H$ bounded and a finitely determining l^2-decomposition $H \times H = \bigoplus_{k\in\mathbb{N}}^2 V_k$. Suppose that there is a sequence $(z_k)_{k\in\mathbb{N}}$ in $\varrho(A)$ with $\|(A - z_k)^{-1}\| \to 0$.*

(i) If the graph $\Gamma(X)$ of a bounded operator $X : H \to H$ is T-invariant compatible with $\bigoplus_k^2 V_k$, i.e.

$$\Gamma(X) = \bigoplus_{k\in\mathbb{N}}^2 U_k \quad \textit{with} \quad U_k \subset V_k \quad T\textit{-invariant},$$

then X satisfies the Riccati equation (4.18) and we have

$$\sigma(T|_{\Gamma(X)}) = \sigma(A + Q_1X).$$

The subspaces D_k obtained by projection of U_k onto the first component form a finitely determining l^2-decomposition for $A + Q_1X$.

(ii) If A and T have compact resolvents, the decomposition $\bigoplus_k^2 V_k$ is finitely spectral, and $X : H \to H$ is a bounded solution of the Riccati equation (4.18), then the graph $\Gamma(X)$ of X is T-invariant compatible with $\bigoplus_k^2 V_k$.

Proof. (i): The family $(D_k)_{k\in\mathbb{N}}$ forms an l^2-decomposition by Lemma 4.2.3. From the identity (4.22) in the previous proposition, we see that since each U_k is the span of certain root vectors of T, D_k is the span of the corresponding root vectors of $A+Q_1X|_{\mathcal{D}(A)\cap X^{-1}\mathcal{D}(A^*)}$. Since Q_1 and X are bounded, we have $\|Q_1X(A-z_k)^{-1}\| < 1$ for k sufficiently large. Then

$$A + Q_1X - z_k = (I + Q_1X(A - z_k)^{-1})(A - z_k)$$

implies $z_k \in \varrho(A + Q_1X)$. Applying Proposition 2.3.8, we deduce that $(D_k)_{k\in\mathbb{N}}$ forms a finitely determining l^2-decomposition for $A + Q_1X$. In particular, $\sum_k D_k$ is a core for $A + Q_1X$ and hence (since Q_1X is bounded) for A. Since moreover $\sum_k D_k \subset \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$, we can apply Proposition 4.3.5 to obtain (4.18). Now Proposition 4.3.7 yields $\varrho(A + Q_1X) = \varrho(T|_{\Gamma(X)})$.

(ii): Since A has compact resolvent and

$$(A + Q_1X - z_k)^{-1} = (A - z_k)^{-1}\big(I + Q_1X(A - z_k)^{-1}\big)^{-1}$$

for k sufficiently large, $A + Q_1X$ has compact resolvent too. By Proposition 4.3.7, $\Gamma(X)$ is $(T - \lambda)^{-1}$-invariant for all $\lambda \in \varrho(T) \setminus \sigma(A + Q_1X)$, where $\sigma(A + Q_1X)$ has only finitely many points in any bounded subset of $\mathbb{C}$. We can thus apply the reasoning from the proof of Proposition 2.4.5 to get $\Gamma(X) = \bigoplus_k^2 \Gamma(X) \cap V_k$. □

4.4 Hamiltonian operators with spectral l^2-decompositions

Now we use the perturbation theory from Chapter 3 to obtain finitely spectral l^2-decompositions for Hamiltonian operator matrices where A is normal with compact resolvent and Q_1, Q_2 are p-subordinate to A. For a nonnegative Hamiltonian, the l^2-decomposition enables us to prove the existence of infinitely many selfadjoint solutions of the Riccati equation, see Theorem 4.4.1; in particular, we obtain a nonnegative and a nonpositive solution $X_\pm$. For a Hamiltonian such that Q_1 and Q_2 are bounded, Theorem 4.4.4 yields a representation of all bounded solutions of the Riccati equation in terms of invariant subspaces. In Theorem 4.4.5 we finally show the existence of bounded, boundedly invertible, selfadjoint solutions for a uniformly positive Hamiltonian with Q_1, Q_2 bounded and A skew-adjoint. We also obtain a representation of every bounded solution as $X = X_+P + X_-(I - P)$ where P is a projection.

For a dichotomous Hamiltonian operator with bounded Q_1, Q_2, the existence of a selfadjoint nonnegative and nonpositive solution of the Riccati equation was shown by Langer, Ran and van de Rotten [31]. Under the additional assumption that $-A$ is maximal uniformly sectorial, which implies that the spectrum of A is contained in a sector in the right half-plane strictly separated from the imaginary axis, the boundedness of the nonnegative and bounded invertibility of the nonpositive solution was shown. A similar result was obtained by Bubák, van der Mee and Ran [10] for a Hamiltonian that is exponentially dichotomous with Q_1 compact. By contrast, Theorem 4.4.1 also holds for unbounded operators Q_1, Q_2 and non-dichotomous Hamiltonians, compare Example 5.1.1. In Theorem 4.4.5, the operator A has its spectrum on the imaginary axis.

For a Riesz-spectral Hamiltonian T, Kuiper and Zwart [29, Theorem 5.6] obtained a representation of all bounded solutions of the Riccati equation in terms of eigenvectors of the Hamiltonian. Under the assumption that all eigenvalues of T are simple, the authors gave conditions such that T is Riesz-spectral. Theorem 4.4.4 applies to the more general class of Hamiltonians with a finitely spectral l^2-decomposition and requires no assumption on the eigenvalue multiplicities.

For the Riccati equation from optimal control, the existence of a bounded nonnegative solution is usually proved via a semigroup based approach, see e.g. [14]. Curtain, Iftime, and Zwart [13] obtained the representation $X = X_+P + X_-(I - P)$ for all bounded selfadjoint solutions without requiring that the Hamiltonian is uniformly positive. However, they had to assume the existence of a bounded, boundedly invertible, negative solution X_- of the Riccati equation.

Recall that the point spectrum of a Hamiltonian with finitely determining l^2-decomposition is symmetric with respect to the imaginary axis by Corollary 4.1.3. Also recall the notation $N(r, A)$ for the sum of the multiplicities of all eigenvalues λ of an operator A with $|\lambda| \le r$, see (3.24).

Theorem 4.4.1 *Let T be a nonnegative Hamiltonian operator matrix such that A is normal with compact resolvent, Q_1, Q_2 are p-subordinate to A with $0 \le p < 1$, and*

$$\ker(A - it) \cap \ker Q_1 = \ker(A - it) \cap \ker Q_2 = \{0\} \quad \textit{for all} \quad t \in \mathbb{R}.$$

Suppose that the spectrum of A lies on finitely many rays from the origin and that

$$\liminf_{r\to\infty} \frac{N(r, A)}{r^{1-p}} < \infty.$$

Then $\sigma(T) \cap i\mathbb{R} = \varnothing$ and T has a compact resolvent and a finitely spectral l^2-decomposition $\bigoplus_{k\in\mathbb{N}}^2 V_k$.

Let $\sigma(T) = \sigma \,\dot\cup\, \tau$ be a partition of the spectrum of T which separates skew-conjugate points. If

$$(a) \qquad \ker(A - \lambda) \cap \ker Q_1 = \{0\} \quad \textit{for all} \quad \lambda \in \mathbb{C},$$

then the compatible subspace associated with σ is the graph $\Gamma(X)$ of a selfadjoint core solution $X(H \to H)$ of the Riccati equation

$$X(A + Q_1 X) = Q_2 - A^* X. \tag{4.23}$$

The solutions $X_\pm$ corresponding to $\sigma = \sigma_p^\pm(T)$ are nonnegative and nonpositive, respectively. If

(b) $\ker(A - \lambda) \cap \ker Q_2 = \{0\}$ *for all* $\lambda \in \mathbb{C}$,

then the compatible subspace associated with σ is the graph $\mathrm{L}(Y)$ of a selfadjoint core solution $Y(H \to H)$ of

$$Y(Q_2 Y - A^*) = AY + Q_1. \tag{4.24}$$

The solutions $Y_\pm$ corresponding to $\sigma = \sigma_p^\pm(T)$ are nonnegative and nonpositive, respectively.

Proof. Since A is a normal operator, we have $\ker(A - \lambda) = \ker(A^* - \overline{\lambda})$ for $\lambda \in \mathbb{C}$ and $\|Au\| = \|A^*u\|$ for $u \in \mathcal{D}(A) = \mathcal{D}(A^*)$. Hence $N(r, A) = N(r, A^*)$ and Q_1 is p-subordinate to A^*. Proposition 3.4.5 thus shows that T has a compact resolvent and a finitely spectral l^2-decomposition; Proposition 4.1.6 implies $\sigma(T) \cap i\mathbb{R} = \varnothing$.

We can now find an open disc $M \subset \varrho(A) \cap \varrho(T)$ with centre on the imaginary axis. By Proposition 4.2.8, property (a) implies property (a2) from Proposition 4.2.11; similarly, (b) implies (b2). Propositions 4.2.11, 4.3.4 and Remark 4.3.6 thus yield the existence of the core solutions. The solutions $X_\pm$ and $Y_\pm$ corresponding to $\sigma_p^\pm(T)$ are nonnegative and nonpositive by Lemma 4.2.10. □

Remark 4.4.2 Since T has a compact resolvent, $\sigma(T)$ consists of countably many skew-conjugate pairs of eigenvalues (for $\dim H = \infty$). A partition which separates skew-conjugate points then amounts to the choice of one eigenvalue from each skew-conjugate pair. There are thus uncountably many such partitions and we obtain uncountably many corresponding core solutions of (4.23) and (4.24), respectively.

In contrast to the discrete nature of the choices from the eigenvalue pairs, a family of solutions depending on a continuous parameter is also possible, see Example 5.1.3. ⌟

Corollary 4.4.3 *Let the assumptions of Theorem 4.4.1 be satisfied.*

(i) *If X is a selfadjoint core solution of (4.23) such that $\Gamma(X)$ is compatible with $\bigoplus_k^2 V_k$ and the condition (b) from Theorem 4.4.1 holds, then X is injective. Similarly, if Y is a selfadjoint core solution of (4.24) such that $\mathrm{L}(Y)$ is compatible with $\bigoplus_k^2 V_k$ and (a) holds, then Y is injective.*

(ii) *Let both (a) and (b) be satisfied, $\sigma(T) = \sigma \,\dot\cup\, \tau$ a partition which separates skew-conjugate points, and X the core solution of (4.23) corresponding to σ. Then X is injective and $Y = X^{-1}$ is the core solution of (4.24) corresponding to σ.*

(iii) *Suppose that (a) and $Q_2 > 0$ or that (b) and $Q_1 > 0$ holds. Then the solutions $X_\pm$ of (4.23) corresponding to $\sigma_p^\pm(T)$ are positive and negative, respectively; they are the uniquely determined nonnegative and nonpositive selfadjoint core solutions of (4.23) whose graph is compatible with $\bigoplus_k^2 V_k$.*

Proof. (i): From the proof of Theorem 4.4.1 we know that there exists an open disc $M \subset \varrho(A) \cap \varrho(T)$ and that the properties (a) and (b) imply (4.12) and (4.13), respectively. Suppose that X is selfadjoint, $\Gamma(X)$ is T-invariant compatible with $\bigoplus_k^2 V_k$, and (b) holds. Then $\Gamma(X)$ is J_1-neutral, $(T-\lambda)^{-1}$-invariant and Proposition 4.2.6 implies that $\Gamma(X) = \mathrm{L}(Y_0)$ with some operator Y_0. Hence X is injective. The proof for Y is analogous.

(ii): This is a direct consequence of (i).

(iii): $X_\pm$ are injective by (ii) and thus positive and negative by Lemma 4.1.4. Let X be nonnegative selfadjoint with

$$\Gamma(X) = \bigoplus_{k \in \mathbb{N}}^2 U_k, \quad U_k \subset V_k \quad T\text{-invariant.}$$

Then $\Gamma(X)$ is J_2-nonnegative and each U_k is the span of certain root vectors of T. By Proposition 4.1.7, the root subspaces of T for eigenvalues in the right/left half-plane are J_2-positive/-negative. Therefore, U_k is spanned by root vectors corresponding to the right half-plane and we obtain $\Gamma(X) \subset U_+$ where $U_+ = \Gamma(X_+)$ is the compatible subspace associated with the spectrum in the right half-plane. So $X \subset X_+$ and hence $X = X_+$ since both operators are selfadjoint. The proof of the uniqueness of X_- is analogous. □

Hamiltonian operators with bounded Q_1 and Q_2 typically occur in the theory of optimal control. For this class of Hamiltonians the next theorem establishes a one-to-one correspondence between bounded solutions of the Riccati equation and compatible T-invariant graph subspaces. Note that we do not need the nonnegativity of T here.

Theorem 4.4.4 *Consider a Hamiltonian operator matrix T with $Q_1, Q_2 : H \to H$ bounded. Suppose that A is normal with compact resolvent, $\sigma(A)$ lies on finitely many rays from the origin, and*

$$\liminf_{r \to \infty} \frac{N(r, A)}{r} < \infty.$$

Then T has a compact resolvent and a finitely spectral l^2-decomposition $\bigoplus_{k \in \mathbb{N}}^2 V_k$. The bounded operator $X : H \to H$ is a solution of the Riccati equation

$$A^* X + XA + XQ_1X - Q_2 = 0 \quad \text{on} \quad \mathcal{D}(A) \tag{4.25}$$

if and only if its graph $\Gamma(X)$ is T-invariant compatible with $\bigoplus_k^2 V_k$. In this case we have $\sigma(T|_{\Gamma(X)}) = \sigma(A + Q_1 X)$.

Proof. As in the proof of Theorem 4.4.1 we can use Proposition 3.4.5, now with $p = 0$, to deduce the compactness of the resolvent of T and the existence of the l^2-decomposition. Since the spectrum of A lies on a finite number of rays, there is a sequence $(z_k)_k$ in $\varrho(A)$ with $\|(A - z_k)^{-1}\| \to 0$. Hence all assumptions of Proposition 4.3.9 are fulfilled and the assertion follows. □

For uniformly positive Hamiltonians, i.e. uniformly positive Q_1 and Q_2, we will now prove the existence of bounded, boundedly invertible solutions of the Riccati equation and derive the representation $X = X_+P + X_-(I - P)$.

Theorem 4.4.5 *Consider a uniformly positive Hamiltonian operator matrix with $A(H \to H)$ skew-adjoint with compact resolvent, $Q_1, Q_2 : H \to H$ bounded and $Q_1, Q_2 \geq \gamma$. Suppose that almost all eigenvalues of A are simple and*

$$\sigma(A) \subset \left\{\pm i r_k^\pm \,\middle|\, k \in \mathbb{N}\right\}$$

where $(r_k^\pm)_{k\in\mathbb{N}}$ are monotonically increasing sequences of nonnegative numbers such that

$$r_{k+1}^\pm - r_k^\pm \geq 2\delta b \quad \textit{for almost all } k, \quad b = \max\left\{\|Q_1\|, \|Q_2\|\right\}, \quad \delta > \frac{4+\pi}{\pi}.$$

Then T has a compact resolvent, almost all of its eigenvalues are simple,

$$\sigma(T) \subset \left\{z \in \mathbb{C} \,\middle|\, \gamma \leq |\operatorname{Re} z| \leq b\right\},$$

and T admits a Riesz basis of eigenvectors and finitely many Jordan chains.

For every partition $\sigma_p(T) = \sigma \,\dot\cup\, \tau$ which separates skew-conjugate points, the compatible subspace associated with σ is the graph $\Gamma(X)$ of a selfadjoint, bounded, boundedly invertible solution $X : H \to H$ of the Riccati equation

$$-AX + XA + XQ_1X - Q_2 = 0 \quad \textit{on} \quad \mathcal{D}(A); \tag{4.26}$$

in particular $X\mathcal{D}(A) \subset \mathcal{D}(A)$. The solutions $X_\pm$ corresponding to $\sigma_p^\pm(T)$ are uniformly positive and negative, respectively; they are the uniquely determined nonnegative and nonpositive bounded solutions of (4.26).

A bounded operator $X : H \to H$ is a solution of (4.26) if and only if its graph $\Gamma(X)$ is T-invariant compatible with the l^2-decomposition of root subspaces $\bigoplus^2_{\lambda\in\sigma(T)} \mathcal{L}(\lambda)$. In this case there is a projection $P : H \to H$ such that

$$X = X_+P + X_-(I - P).$$

Finally, every bounded selfadjoint solution X of (4.26) is boundedly invertible and satisfies

$$X\mathcal{D}(A) = \mathcal{D}(A), \quad X_- \leq X \leq X_+, \quad X_-^{-1} \leq X^{-1} \leq X_+^{-1}.$$

Proof. We apply Theorem 3.4.7 with $p = \beta = 0$, $\alpha = \delta b$ to the decomposition

$$T = G + S, \quad G = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \quad S = \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix}.$$

G is skew-adjoint with compact resolvent, $\sigma(G) \subset \{\pm i r_k^\pm \mid k \in \mathbb{N}\}$, and almost all of its eigenvalues have multiplicity 2. S is bounded with $\|S\| = b$. Consequently, T has a compact resolvent, almost all eigenvalues lie inside rectangular regions

$$K_k^\pm = \big\{\pm ix + y \,\big|\, |x - r_k^\pm| \le \delta b,\ |y| \le \delta b\big\},$$

and $N(K_k^\pm, T) = 2$ for almost all k. Since the spectrum of T is symmetric with respect to the imaginary axis and due to Proposition 4.1.6, almost all $K_k^\pm$ contain only one skew-conjugate pair of simple eigenvalues λ, $-\overline{\lambda}$ with $|\operatorname{Re}\lambda| \ge \gamma$. Therefore, Theorem 3.4.7 implies that the root subspaces of T form an l^2-decomposition of $H \times H$ and that almost all of them have dimension one. Lemma 2.3.15 then yields the existence of the Riesz basis of eigenvectors and finitely many Jordan chains. In view of Remark 3.4.2 we have $\sigma(T) \subset \{|\operatorname{Re} z| \le b\}$ and obtain the asserted shape of the spectrum.

With $z_k = k$, $k \ge 1$, Proposition 4.3.9 yields the correspondence between arbitrary bounded solutions of (4.26) and invariant graph subspaces compatible with $\bigoplus_\lambda^2 \mathcal{L}(\lambda)$. By Proposition 4.2.12, the compatible subspace associated with σ is the graph of a selfadjoint isomorphism X. In particular, X solves (4.26). The solutions $X_\pm$ are the unique nonnegative/nonpositive solutions by Corollary 4.4.3. Moreover, the graph of any bounded solution may be written as

$$\Gamma(X) = \bigoplus_{\operatorname{Re}\lambda_k > 0}^2 U_k \oplus \bigoplus_{\operatorname{Re}\lambda_k < 0}^2 U_k \quad \text{with} \quad U_k \subset \mathcal{L}(\lambda_k) \quad T\text{-invariant},$$

where $(\lambda_k)_{k\in\mathbb{N}}$ are the eigenvalues of T. If D_k is the subspace obtained by projection of U_k onto the first component, we have

$$H = \bigoplus_{\operatorname{Re}\lambda_k > 0}^2 D_k \oplus \bigoplus_{\operatorname{Re}\lambda_k < 0}^2 D_k$$

by Lemma 4.2.3. Let $P : H \to H$ be the projection onto $\bigoplus_{\operatorname{Re}\lambda_k > 0}^2 D_k$ corresponding to this decomposition. We obtain $X = X_+ P + X_-(I - P)$ since $X|_{D_k} = X_\pm|_{D_k}$ for $\operatorname{Re}\lambda_k \gtrless 0$.

Now let X be a bounded selfadjoint solution of (4.26). Taking the difference of the Riccati equations for X and X_+, we obtain

$$\begin{aligned} 0 &= (Au|(X_+ - X)u) + ((X_+ - X)u|Au) + (Q_1 X_+ u|X_+ u) - (Q_1 X u|X u) \\ &= ((A + Q_1 X_+)u|(X_+ - X)u) + ((X_+ - X)u|(A + Q_1 X_+)u) \\ &\qquad - (Q_1(X_+ - X)u|(X_+ - X)u) \end{aligned}$$

for $u \in \mathcal{D}(A)$. With $\Delta = X_+ - X$ and $t \in \mathbb{R}$ we deduce

$$2\,\mathrm{Re}\big((A + Q_1X_+ - it)u\big|\Delta u\big) = (Q_1\Delta u|\Delta u) \geq 0.$$

Proposition 4.3.9 implies that

$$\sigma(A + Q_1X_+) = \sigma(T|_{U_+}) \subset \{z \in \mathbb{C} \mid \mathrm{Re}\, z \geq \gamma\}.$$

Thus

$$\mathrm{Re}\big(\Delta u\big|(A + Q_1X_+ - it)^{-1}u\big) \geq 0 \quad \text{for} \quad u \in H.$$

Since all eigenvalues of $A+Q_1X_+$ lie in the right half-plane, Proposition 2.6.4 yields

$$\frac{1}{\pi}\int_{\mathbb{R}}' (A + Q_1X_+ - it)^{-1}u\,dt = u \qquad \text{for} \quad u \in \sum_{\lambda\in\sigma_p(A+Q_1X_+)} \mathcal{L}(\lambda);$$

hence

$$\pi(\Delta u|u) = \int_{\mathbb{R}}' \mathrm{Re}\big(\Delta u\big|(A + Q_1X_+ - it)^{-1}u\big)\,dt \geq 0 \quad \text{for} \quad u \in \sum_{\lambda\in\sigma_p(A+Q_1X_+)} \mathcal{L}(\lambda).$$

By Propositions 4.3.7 and 4.3.9 the root subspaces of $A + Q_1X_+$ form an l^2-decomposition of H. Thus $(\Delta u|u) \geq 0$ for all $u \in H$, that is $X \leq X_+$. An analogous reasoning yields $X_- \leq X$. From Proposition 4.2.5 we see that X is injective and we have the decomposition

$$\mathrm{L}(X^{-1}) = W_+ \oplus W_-, \quad W_\pm = \bigoplus_{\mathrm{Re}\,\lambda_k \gtrless 0}^{2} U_k, \quad U_k \subset \mathcal{L}(\lambda_k) \quad T\text{-invariant.}$$

As in the proof of Proposition 4.2.12, this implies that X^{-1} is bounded. Using the fundamental symmetry $J_2 : (u, v) \mapsto (v, u)$ and setting

$$\widetilde{T} = J_2TJ_2 = \begin{pmatrix} A & Q_2 \\ Q_1 & A \end{pmatrix}, \quad \widetilde{U}_k = J_2U_k, \quad \widetilde{V}_k = J_2\mathcal{L}(\lambda_k),$$

we have that $\bigoplus_{k\in\mathbb{N}}^2 \widetilde{V}_k$ is an l^2-decomposition of root subspaces for $\widetilde{T}$ and

$$\Gamma(X^{-1}) = J_2\mathrm{L}(X^{-1}) = \bigoplus_{k\in\mathbb{N}}^{2} \widetilde{U}_k, \quad \widetilde{U}_k \subset \widetilde{V}_k \quad \widetilde{T}\text{-invariant.}$$

Proposition 4.3.9 applied to the Hamiltonian $\widetilde{T}$ then yields $X^{-1}\mathcal{D}(A) \subset \mathcal{D}(A)$ and thus $X\mathcal{D}(A) = \mathcal{D}(A)$. Finally, the same calculations as above for X^{-1}, $X_\pm^{-1}$ and $\widetilde{T}$ yield the relation $X_-^{-1} \leq X^{-1} \leq X_+^{-1}$. □

In view of Remark 3.4.14, the assumptions on A in Theorems 4.4.1, 4.4.4 and 4.4.5 to be normal with spectrum on rays from the origin can be relaxed:

Remark 4.4.6 Let A be an operator with compact resolvent and a Riesz basis of Jordan chains, let Q_1 be p-subordinate to A^*, Q_2 p-subordinate to A, and consider the decomposition

$$T = G + S \quad \text{with} \quad G = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad S = \begin{pmatrix} 0 & Q_1 \\ Q_2 & 0 \end{pmatrix}.$$

Then A^* and G also have a compact resolvent and a Riesz basis of Jordan chains. Furthermore, S is p-subordinate to G, cf. Proposition 3.4.5. If A satisfies the condition (i) from Proposition 3.4.13, then so does G and that proposition implies

$$JTJ^{-1} = JGJ^{-1} + JSJ^{-1} = G_0 + S_0 + JSJ^{-1}$$

where $S_0 + JSJ^{-1}$ is p-subordinate to G_0. By Theorem 3.4.4, JTJ^{-1} and hence also T have a finitely spectral l^2-decomposition, and all conclusions of Theorem 4.4.1 and Corollary 4.4.3 hold if some assumptions are adapted: We need (4.6) to obtain $\sigma(T) \cap i\mathbb{R} = \varnothing$ and conditions (a2) and (b2) of Proposition 4.2.11 to show the existence of core solutions of (4.23) and (4.24), respectively.

Analogously, Theorem 4.4.4 continues to hold if the spectrum of A is located in strips around rays from the origin. For Theorem 4.4.5 we use case (ii) of Proposition 3.4.13 and obtain the condition

$$\sigma(A) \subset \left\{ \pm i r_k^{\pm} + y \,\middle|\, k \in \mathbb{N},\, y \in [-\alpha, \alpha] \right\}$$

and $b = \|S_0 + JSJ^{-1}\|$. ⌟

Chapter 5

Examples and applications

In this chapter we present examples and applications for the theorems from the previous chapter on Hamiltonian operator matrices and solutions of Riccati equations. In Section 5.1 we consider explicitly solvable examples as well as non-trivial Riccati equations involving differential and multiplication operators.

In Section 5.2 the theory is applied to the Riccati equation from optimal control. We prove the existence of infinitely many selfadjoint solutions. So far, only the existence of a nonnegative and a nonpositive solution has been shown [14, 31, 10]. Moreover, we study the heat equation with an unbounded control operator.

5.1 Examples for Hamiltonians with spectral l^2-decompositions

To illustrate the conditions and results from Section 4.4, we consider some examples in which determining l^2-decompositions of the Hamiltonian and solutions of the Riccati equation can be explicitly calculated. The examples include cases with unbounded solutions, invertible solutions with unbounded inverse, solutions that are not invertible, non-selfadjoint solutions, a family of solutions depending on a continuous parameter, and a Hamiltonian having Jordan chains of arbitrary length. After this, we apply the theory to non-trivial examples of Riccati equations whose coefficients are differential and multiplication operators.

Let T be a nonnegative Hamiltonian with compact resolvent such that A is normal and the operators $A, Q_1, Q_2 (H \to H)$ have a common finitely determining orthogonal decomposition

$$H = \bigoplus_{k \geq 1} H_k.$$

Then the subspaces $V_k = H_k \times H_k$ constitute a finitely determining orthogonal decomposition for T (cf. Proposition 2.3.8).

The first two examples show the existence of solutions of the Riccati equation that are unbounded, bounded and not boundedly invertible, and unbounded and not boundedly invertible, respectively.

Example 5.1.1 Let $\dim H_k = 1$, $H_k = \mathbb{C}e_k$ where $(e_k)_{k\geq 1}$ is an orthonormal basis of H. Let $Q_1 = I$, $Ae_k = ia_k e_k$, $Q_2 e_k = q_k^2 e_k$ with $a_k, q_k \in \mathbb{R}_{>0}$; so $T|_{V_k}$ is represented by the matrix

$$T|_{V_k} \cong \begin{pmatrix} ia_k & 1 \\ q_k^2 & ia_k \end{pmatrix}. \tag{5.1}$$

Consequently, $T|_{V_k}$ has the eigenvalues and corresponding eigenvectors

$$\lambda_k^{\pm} = ia_k \pm q_k, \quad v_k^{\pm} = \begin{pmatrix} e_k \\ \pm q_k e_k \end{pmatrix}.$$

We choose $a_k = k^2$, $q_k = \sqrt{k}$ for $k \geq 1$ so that Q_2 is unbounded, 1/2-subordinate to A, and T is positive: Theorem 4.4.1 can be applied. In particular, T has indeed a compact resolvent and $H \times H = \bigoplus_k V_k$ is a finitely spectral decomposition for T.

The selfadjoint core solution X_σ corresponding to σ from a partition $\sigma(T) = \sigma \,\dot\cup\, \tau$ which separates skew-conjugate points is given by

$$\Gamma(X_\sigma) = \bigoplus_{k\geq 1} U_k \quad \text{with} \quad U_k = \begin{cases} \mathbb{C}v_k^+ & \text{if } \lambda_k^+ \in \sigma, \\ \mathbb{C}v_k^- & \text{if } \lambda_k^- \in \sigma. \end{cases} \tag{5.2}$$

Hence

$$X_\sigma e_k = \begin{cases} q_k e_k = \sqrt{k}\, e_k & \text{if } \lambda_k^+ \in \sigma, \\ -q_k e_k = -\sqrt{k}\, e_k & \text{if } \lambda_k^- \in \sigma; \end{cases} \tag{5.3}$$

in particular, X_σ is unbounded. The positive and negative solutions are given by $X_\pm e_k = \pm q_k e_k$. Moreover, if a densely defined solution X satisfies $\Gamma(X) = \bigoplus_k U_k$ with $U_k \subset V_k$ T-invariant, then for every k either $U_k = \mathbb{C}v_k^+$ or $U_k = \mathbb{C}v_k^-$; hence $X = X_\sigma$ with σ appropriate. Every densely defined solution with $\Gamma(X)$ compatible with $\bigoplus_k V_k$ is thus selfadjoint, unbounded and there are infinitely many of these.

Consider the sequence

$$x_k = \begin{pmatrix} \frac{2}{\sqrt{k}} e_k \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{k}} e_k \\ e_k \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{k}} e_k \\ -e_k \end{pmatrix} = \sqrt{\frac{1+k}{k}}\, \frac{v_k^+}{\|v_k^+\|} + \sqrt{\frac{1+k}{k}}\, \frac{v_k^-}{\|v_k^-\|}.$$

We have $\lim x_k = 0$ while the components $(1 + k^{-1})^{1/2} v_k^{\pm}/\|v_k^{\pm}\|$ with respect to $V_k = \mathbb{C}v_k^+ \oplus \mathbb{C}v_k^-$ do not converge to zero. Consequently the algebraic direct sum

$$\bigoplus_{k\geq 1} \mathbb{C}v_k^+ \dotplus \bigoplus_{k\geq 1} \mathbb{C}v_k^-$$

is not topological direct, the system of root subspaces $(\mathbb{C}v_k^+, \mathbb{C}v_k^-)_{k\ge 1}$ does not form an l^2-decomposition, and the operator T is neither spectral nor dichotomous; yet a strip around the imaginary axis belongs to $\varrho(T)$; compare Remark 2.1.9, Theorem 2.3.17, and Definition 2.4.8. ⌟

Example 5.1.2 We modify Example 5.1.1 by setting $q_k = 1/k$ for $k \ge 1$. So Q_2 is now bounded and Theorems 4.4.1 and 4.4.4 can be applied. The solutions X_σ,

$$X_\sigma e_k = \pm\frac{1}{k}e_k \quad \text{if} \quad \lambda_k^\pm \in \sigma,$$

are bounded, selfadjoint, injective, yet not boundedly invertible. Just as in Example 5.1.1, the solutions X_σ cover all possible densely defined solutions whose graph is compatible with $\bigoplus_k V_k$, and there are infinitely many of these. Again, the direct sum $\bigoplus_k \mathbb{C}v_k^+ \dotplus \bigoplus_k \mathbb{C}v_k^-$ is not topological direct and the system of root subspaces does not form an l^2-decomposition.

We can further modify the example by setting

$$q_k = \begin{cases} \sqrt{k} & \text{if } k \text{ odd}, \\ k^{-1} & \text{if } k \text{ even}. \end{cases}$$

The solutions X_σ are then unbounded and not boundedly invertible. ⌟

Now we illustrate how multiple eigenvalues of the Hamiltonian lead to families of selfadjoint and non-selfadjoint solutions of the Riccati equation which depend on a continuous parameter.

Example 5.1.3 Suppose that $\dim H_1 = 2$, $\dim H_k = 1$ for $k \ge 2$, $Q_1 = Q_2 = I$, and $A|_{H_k} = ik^2 I_{H_k}$ for all k. So we are in the situation of Theorem 4.4.5. Let (e_1, e_2) be an orthonormal basis of H_1. Then $T|_{V_1}$ has the double eigenvalues $i \pm 1$ with a corresponding basis of eigenvectors

$$v_1^\pm = \begin{pmatrix} e_1 \\ \pm e_1 \end{pmatrix}, \quad v_2^\pm = \begin{pmatrix} e_2 \\ \pm e_2 \end{pmatrix}.$$

Consider the invariant subspace

$$U_1 = \operatorname{span}\{v_1^+ + rv_2^+, -rv_1^- + v_2^-\} \subset V_1 \quad \text{with} \quad |r| < 1. \tag{5.4}$$

Then

$$\begin{pmatrix} 0 \\ x \end{pmatrix} = \alpha(v_1^+ + rv_2^+) + \beta(-rv_1^- + v_2^-) = \begin{pmatrix} (\alpha - r\beta)e_1 + (r\alpha + \beta)e_2 \\ (\alpha + r\beta)e_1 + (r\alpha - \beta)e_2 \end{pmatrix}$$

implies

$$\alpha - r\beta = r\alpha + \beta = 0 \quad \Rightarrow \quad (r^2 + 1)\beta = 0 \quad \Rightarrow \quad \beta = 0 \quad \Rightarrow \quad \alpha = 0,\ x = 0.$$

Hence $U_1 = \Gamma(X_1)$ with

$$X_1(e_1 + re_2) = e_1 + re_2, \quad X_1(-re_1 + e_2) = re_1 - e_2.$$

From $(e_1 + re_2 | -re_1 + e_2) = 0$ it follows that X_1 is selfadjoint. Together with appropriate choices of invariant subspaces $U_k \subset V_k$, $k \geq 2$, this leads to bounded selfadjoint solutions X of the Riccati equation associated with T which depend on the parameter r.

For the invariant subspace

$$\widetilde{U}_1 = \operatorname{span}\{v_1^+, rv_1^- + v_2^-\} \subset V_1 \quad \text{with} \quad r \in \mathbb{C}, \tag{5.5}$$

we have the implication

$$\begin{pmatrix} 0 \\ x \end{pmatrix} = \alpha v_1^+ + \beta(rv_1^- + v_2^-) = \begin{pmatrix} (\alpha + r\beta)e_1 + \beta e_2 \\ (\alpha - r\beta)e_1 - \beta e_2 \end{pmatrix}$$
$$\Rightarrow \quad \alpha + r\beta = \beta = 0 \quad \Rightarrow \quad \alpha = 0 \quad \Rightarrow \quad x = 0.$$

So now $\widetilde{U}_1 = \Gamma(\widetilde{X}_1)$ with $\widetilde{X}_1 e_1 = e_1$ and

$$\widetilde{X}_1 e_2 = \widetilde{X}_1(re_1 + e_2) - r\widetilde{X}_1 e_1 = -2re_1 - e_2.$$

With respect to the orthonormal basis (e_1, e_2), $\widetilde{X}_1$ is thus represented by the matrix

$$\widetilde{X}_1 \cong \begin{pmatrix} 1 & -2r \\ 0 & -1 \end{pmatrix},$$

i.e., $\widetilde{X}_1$ is not selfadjoint for $r \neq 0$. We obtain bounded non-selfadjoint solutions of the Riccati equation which depend on r. ⌟

This example features solutions that are not invertible:

Example 5.1.4 Let $\dim H_k = 1$ for all k, A selfadjoint, $Q_1 = I$, and $A|_{H_1} = 1$, $Q_2|_{H_1} = 0$, i.e.

$$T|_{V_1} \cong \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

So in Theorem 4.4.1, assumption (a) is fulfilled while (b) is not. Eigenvectors corresponding to the eigenvalues 1 and -1 of $T|_{V_1}$ are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

respectively. Hence, for the solution X_σ corresponding to σ such that $1 \in \sigma$, we have $H_1 \subset \ker X_\sigma$; equivalently, $\Gamma(X_\sigma)$ can not be written as a graph subspace $\mathrm{L}(Y_\sigma)$. ⌟

The following example shows that in the setting of Theorems 4.4.1 and 4.4.4 Hamiltonians with Jordan chains of arbitrary length are possible.

Example 5.1.5 Suppose that $\dim H_k = k$, $Q_1 = Q_2$ and

$$A|_{H_k} = A_k = \begin{pmatrix} ik^2 & 1 & & \\ -1 & ik^2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & -1 & ik^2 \end{pmatrix}, \quad Q_1|_{H_k} = B_k = \begin{pmatrix} \alpha & 1 & & \\ 1 & \alpha & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & \alpha \end{pmatrix}$$

with $\alpha > 2$. So A is skew-adjoint and $T|_{V_k} = \begin{pmatrix} A_k & B_k \\ B_k & A_k \end{pmatrix}$. Straightforward calculations show that

$$\begin{aligned} (k^2-2)\|x\|^2 &\le (iA_k x|x) \le (k^2+2)\|x\|^2, \\ (\alpha-2)\|x\|^2 &\le (B_k x|x) \ \le (\alpha+2)\|x\|^2 \end{aligned}$$

for all $x \in H_k$, $k \ge 1$. From this it follows that A has a compact resolvent and satisfies $\lim_{r\to\infty} N(r,A)r^{-1} < \infty$. Furthermore, Q_1 is bounded and positive, and Theorems 4.4.1 and 4.4.4 are thus applicable. Now

$$A_k+B_k = \begin{pmatrix} ik^2+\alpha & 2 & & \\ & \ddots & \ddots & \\ & & \ddots & 2 \\ & & & ik^2+\alpha \end{pmatrix}, \quad A_k-B_k = \begin{pmatrix} ik^2-\alpha & & & \\ -2 & \ddots & & \\ & \ddots & \ddots & \\ & & -2 & ik^2-\alpha \end{pmatrix}$$

and

$$\begin{aligned} \big(T|_{V_k} - (ik^2+\alpha)\big)\begin{pmatrix} x \\ x \end{pmatrix} &= \begin{pmatrix} \big(A_k + B_k - (ik^2+\alpha)\big)x \\ \big(A_k + B_k - (ik^2+\alpha)\big)x \end{pmatrix}, \\ \big(T|_{V_k} - (ik^2-\alpha)\big)\begin{pmatrix} x \\ -x \end{pmatrix} &= \begin{pmatrix} \big(A_k - B_k - (ik^2-\alpha)\big)x \\ -\big(A_k - B_k - (ik^2-\alpha)\big)x \end{pmatrix}. \end{aligned}$$

Hence T has Jordan chains of arbitrary length. ⌟

We apply the theory from Chapter 4 to Riccati equations whose coefficients are ordinary differential operators. In the first example, we allow Q_1 and Q_2 to be unbounded.

Example 5.1.6 Let $H = L^2([a,b])$ and consider the operators A, Q_1, Q_2 on H given by

$$\begin{aligned} &Au = u''', \quad Q_1 u = -(g_1 u')' + h_1 u, \quad Q_2 u = -(g_2 u')' + h_2 u, \\ &\mathcal{D}(A) = \big\{u \in W^{3,2}([a,b]) \,\big|\, u(a) = u(b) = 0,\ u'(a) = u'(b)\big\}, \\ &\mathcal{D}(Q_1) = \mathcal{D}(Q_2) = \big\{u \in C^2([a,b]) \,\big|\, u(a) = u(b) = 0\big\} \end{aligned}$$

where $g_1, g_2 \in C^1([a,b])$, $h_1, h_2 \in L^2([a,b])$, $g_1, g_2, h_1, h_2 \geq 0$. Then A is skew-adjoint with compact resolvent, $0 \in \varrho(A)$, and

$$\sup_{r>0} \frac{N(r,A)}{r^{1/3}} < \infty$$

(compare Example 3.5.1; the boundary conditions of A are regular). The operators Q_1 and Q_2 are symmetric, nonnegative and 2/3-subordinate to A (see Propositions 3.2.15 and 3.2.16). Moreover Q_1 is positive if $g_1 > 0$ or $h_1 > 0$, analogously for Q_2. If Q_1 and Q_2 are positive, then the Hamiltonian operator T corresponding to A, Q_1, Q_2 satisfies $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ and Theorem 4.4.1 yields the existence of infinitely many selfadjoint injective core solutions of

$$X(A + Q_1 X) = Q_2 - A^* X.$$

All conclusions still hold if we replace A with $e^{i\varphi}A$, $\varphi \in [0, 2\pi]$. ⌟

For a skew-adjoint differential operator A and bounded, boundedly invertible multiplication operators Q_1, Q_2, we prove the existence of bounded, boundedly invertible solutions of the Riccati equation:

Example 5.1.7 Let $H = L^2([0,1])$ and consider the operators

$$\begin{aligned} &Au = iu'', \quad \mathcal{D}(A) = \{u \in W^{2,2}([0,1]) \,|\, u(0) = u(1) = 0\}, \\ &Q_1 u = f_1 u, \quad Q_2 u = f_2 u, \quad \mathcal{D}(Q_1) = \mathcal{D}(Q_2) = H \end{aligned}$$

with $f_1, f_2 \in L^\infty([0,1])$, $f_1, f_2 \geq \varepsilon > 0$. A is skew-adjoint with compact resolvent and simple eigenvalues. Q_1 and Q_2 are bounded and uniformly positive. The eigenvalues of A are $\lambda_k = -i\pi^2 k^2$, $k \geq 1$, which implies $|\lambda_{k+1}| - |\lambda_k| \to \infty$ as $k \to \infty$. Hence, all conditions of Theorem 4.4.5 are fulfilled, and in particular we obtain the existence of infinitely many selfadjoint, bounded, boundedly invertible solutions of

$$-AX + XA + XQ_1X - Q_2 = 0 \quad \text{on} \quad \mathcal{D}(A).$$

We can also apply the theorem if A is the operator of first derivation $u \mapsto u'$ with boundary condition $u(0) = u(1)$. In this case the eigenvalues of A are $\lambda_k = 2\pi i k$ with $k \in \mathbb{Z}$, i.e. $\lambda_{k+1} - \lambda_k = 2\pi i$, and we need the additional assumption

$$\max\{\|f_1\|_\infty, \|f_2\|_\infty\} < \frac{\pi^2}{4+\pi}$$

to guarantee the spectral condition of Theorem 4.4.5. ⌟

5.2 Hamiltonian operators in optimal control

We apply the results from Section 4.4 to the linear quadratic optimal control of infinite-dimensional systems. In Theorem 5.2.3 we prove the existence of infinitely many selfadjoint solutions of the Riccati equation and obtain a representation of all bounded solutions in terms of invariant subspaces of the Hamiltonian. As examples, we consider heat and wave equations with distributed control; the final example features an unbounded control operator B.

The only known methods to prove the existence of solutions of the Riccati equation for infinite-dimensional control systems seem to be the semigroup based approach from control theory, see Theorem 5.2.2, and the methods due to Langer, Ran and van de Rotten [31], and Bubák, van der Mee and Ran [10] for the case of dichotomous Hamiltonians. In both cases, only the existence of a nonnegative and a nonpositive solution has been shown.

A characterisation of all bounded solutions of the Riccati equation in terms of eigenvectors of the Hamiltonian was obtained by Kuiper and Zwart [29, Theorem 5.6] for the case of a Riesz-spectral Hamiltonian. Under the assumption of the existence of a bounded, boundedly invertible, negative solution of the Riccati equation, Curtain, Iftime and Zwart [13] derived a representation of all bounded selfadjoint solutions in terms of invariant subspaces of the semigroup generated by $A - BB^*X_+$; here X_+ is the minimal nonnegative solution of the Riccati equation. Theorem 5.2.3 allows for the more general class of Hamiltonians with a finitely spectral l^2-decomposition and has no a priori assumption about the existence of a solution of the Riccati equation.

We start by briefly reviewing the concepts of linear quadratic optimal control. For more details we refer to the book of Curtain and Zwart [14] and to the introduction.

Definition 5.2.1 A *control system* or *state linear system* is a system

$$\begin{aligned} \dot z(t) &= Az(t) + Bu(t) \quad \text{for} \quad t \geq 0, \qquad z(0) = z_0, \\ y(t) &= Cz(t) \end{aligned} \tag{5.6}$$

with operators on Hilbert spaces $A(Z \to Z)$, $B : U \to Z$, $C : Z \to Y$, where A is the generator of a strongly continuous semigroup $T(t)$ and B and C are bounded. The function $z : [0, \infty[\to Z$ is called the *state* of the system, $z_0 \in Z$ is the initial state, and $\dot z$ denotes the derivative with respect to the time t. $u : [0, \infty[\to U$ is the *input* or *control* and $y : [0, \infty[\to Y$ the *output*. ⌟

For $z_0 \in \mathcal{D}(A)$ and $u \in C^1([0, \infty[\, , U)$ the control system has a classical solution $z \in C^1([0, \infty[\, , \mathcal{D}(A))$ given by the variation of constants formula

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)\, ds. \tag{5.7}$$

For arbitrary $z_0 \in Z$ and $u \in L^2([0,\infty[\,,U)$, (5.7) yields a function $z \in C^0([0,\infty[\,,Z)$, which is then called a *mild solution* of the state linear system.

The problem of *linear quadratic optimal control* on the infinite-time horizon is now for given initial state $z_0 \in Z$ to minimise the so-called *cost functional*

$$J(z_0,u) = \int_0^\infty \left(\|y(t)\|^2 + \|u(t)\|^2\right) dt \tag{5.8}$$

among all controls $u \in L^2([0,\infty[\,,U)$, where z is the mild solution corresponding to z_0 and u.

For optimisable systems, this problem can indeed be solved [14, Theorem 6.2.4]:

Theorem 5.2.2 *If the control system is* optimisable, *i.e., for every $z_0 \in Z$ there exists $u \in L^2([0,\infty[\,,U)$ such that $J(z_0,u) < \infty$, then the cost functional has a minimum for every $z_0 \in Z$ and there is a nonnegative selfadjoint operator $X \in L(Z)$ such that*

$$\min_{u \in L^2([0,\infty[\,,U)} J(z_0,u) = (Xz_0|z_0) \quad \textit{for all} \quad z_0 \in Z.$$

The operator X is the minimal bounded nonnegative solution of the weak algebraic Riccati equation

$$(Az_1|Xz_2) + (Xz_1|Az_2) - (B^*Xz_1|B^*Xz_2) + (Cz_1|Cz_2) = 0, \quad z_1, z_2 \in \mathcal{D}(A), \tag{5.9}$$

and the optimal control is given by

$$u(t) = -B^*Xz(t).$$

□

The Hamiltonian operator matrix related to the control system has the form

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}. \tag{5.10}$$

From Proposition 4.3.5 it follows that the bounded selfadjoint operator $X : Z \to Z$ is a solution of (5.9) if and only if $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and the graph $\Gamma(X)$ is T-invariant. By Definition 4.1.1, the Hamiltonian T is nonpositive. Since in Section 4.4 nonnegative Hamiltonian operators were considered, we apply the respective theorems to

$$-T = \begin{pmatrix} -A & BB^* \\ C^*C & A^* \end{pmatrix}.$$

As a consequence, the compatible subspace associated with the spectrum of T in the right half-plane is J_2-nonpositive and the graph of a nonpositive solution X_- of the Riccati equation; the compatible subspace associated with the spectrum in the left half-plane yields a nonnegative solution X_+.

Theorem 5.2.3 *Consider operators on Hilbert spaces $A(Z \to Z)$, $B : U \to Z$ and $C : Z \to Y$ such that A is normal with compact resolvent and B and C are bounded. Suppose that the spectrum of A lies on finitely many rays from the origin,*

$$\liminf_{r\to\infty} \frac{N(r,A)}{r} < \infty,$$

and that $\ker(A-\lambda)\cap\mathcal{R}(B)^\perp = \ker(A-\lambda)\cap\ker C = \{0\}$ *for all $\lambda \in \mathbb{C}$. Then the Hamiltonian operator*

$$T = \begin{pmatrix} A & -BB^* \\ -C^*C & -A^* \end{pmatrix}$$

has a compact resolvent, a finitely spectral l^2-decomposition $Z\times Z = \bigoplus^2_{k\in\mathbb{N}} V_k$, and its spectrum satisfies $\sigma(T)\cap i\mathbb{R} = \varnothing$.

For every partition $\sigma(T) = \sigma \mathbin{\dot\cup} \tau$ which separates skew-conjugate points, the compatible subspace associated with σ is the graph $\Gamma(X)$ of an injective selfadjoint operator $X(Z\to Z)$ that is a core solution of the Riccati equation

$$X(A - BB^*X) = -C^*C - A^*X. \tag{5.11}$$

The operator $X_\pm$ obtained for the compatible subspace associated with the spectrum in the left and right half-plane is positive and negative, respectively. Moreover, every selfadjoint core solution X of (5.11) such that $\Gamma(X)$ is compatible with $\bigoplus^2_k V_k$ is injective. If also $\mathcal{R}(B)\subset Z$ is dense or $\ker C = \{0\}$, *then X nonnegative/nonpositive implies $X = X_\pm$.*

Finally, a bounded operator $X : Z \to Z$ is a solution of

$$A^*X + XA - XBB^*X + C^*C = 0 \quad \text{on} \quad \mathcal{D}(A) \tag{5.12}$$

if and only if its graph $\Gamma(X)$ is compatible with $\bigoplus^2_k V_k$.

Proof. We want to apply Theorems 4.4.1, 4.4.4 and Corollary 4.4.3 to the operator $-T$ and have to show that the conditions (a) and (b) in Theorem 4.4.1 are satisfied. Indeed by Lemma 4.1.4,

$$z\in\ker(BB^*) \;\Leftrightarrow\; (BB^*z|z) = 0 \;\Leftrightarrow\; \|B^*z\|^2 = 0 \;\Leftrightarrow\; B^*z = 0 \;\Leftrightarrow\; z\in\mathcal{R}(B)^\perp$$

and analogously $\ker(C^*C) = \ker C$. Moreover, BB^* is injective if and only if $\mathcal{R}(B)^\perp = \{0\}$ and C^*C is injective if and only if $\ker C = \{0\}$. This yields the uniqueness result for $X_\pm$. □

Remark 5.2.4 In order to obtain selfadjoint core solutions of (5.11), it is sufficient that $\ker(A-\lambda)\cap\ker C = \{0\}$ holds for all $\lambda\in i\mathbb{R}$ instead of $\lambda\in\mathbb{C}$.

To show the existence of bounded solutions, we could apply Theorem 4.4.5 to $-T$. Then we would have to assume that A is skew-adjoint and B, C are boundedly invertible. However, these assumptions appear to be unnatural in control theory. ⌟

Motivated by examples in Curtain and Zwart [14] and Kuiper and Zwart [29], we apply Theorem 5.2.3 to controlled heat and wave equations.

Example 5.2.5 Consider the two-dimensional heat equation on the unit disc $B_1(0)$ with distributed control and Dirichlet boundary condition,

$$\begin{aligned} \partial_t z(t,x) &= \Delta z(t,x) + b(x)u(t,x) && \text{for} \quad (t,x) \in \mathbb{R}_{\geq 0} \times B_1(0), \\ z(0,x) &= z_0(x) && \text{for} \quad x \in B_1(0), \\ z(t,x) &= 0 && \text{for} \quad (t,x) \in \mathbb{R}_{\geq 0} \times \partial B_1(0), \end{aligned}$$

where $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ is the Laplacian, $\partial B_1(0)$ the boundary of the unit disc, and $b \in L^\infty(B_1(0))$, $b \geq 0$, $b \neq 0$. We choose $Z = U = L^2(B_1(0))$ as the state and input spaces and define A and B by

$$\begin{aligned} Av &= \Delta v, \quad \mathcal{D}(A) = W^{2,2}(B_1(0)) \cap W_0^{1,2}(B_1(0)), \\ Bu &= bu. \end{aligned}$$

In addition, we take $Y = Z$, $C = I$, that is, we consider the cost functional

$$J(z_0, u) = \int_0^\infty \left(\|z(t)\|^2 + \|u(t)\|^2 \right) dt.$$

Then A is selfadjoint with compact resolvent and the asymptotic behaviour of its spectrum is such that

$$\lim_{r\to\infty} \frac{N(r,A)}{r} = \frac{1}{4},$$

see Example 3.5.4. An orthonormal basis of eigenfunctions for A in polar coordinates is given by

$$v_{kl}(r,\varphi) = \beta_{k,|l|} J_{|l|}(\alpha_{k,|l|} r) e^{il\varphi} \quad \text{with} \quad k \in \mathbb{N} \setminus \{0\},\ l \in \mathbb{Z} \tag{5.13}$$

where J_n are the Bessel functions, α_{kn} are the positive zeros of J_n, and β_{kn} are normalisation constants, see [12, §V.5.5]. In particular,

$$0 = (Bv_{kl}|v_{kl}) = (bv_{kl}|v_{kl}) = \left\|\sqrt{b}\, v_{kl}\right\|^2$$

implies $\sqrt{b}\, v_{kl} = 0$ and thus $b = 0$, since the set of zeros of v_{kl} has measure zero in $B_1(0)$. But $b \neq 0$ by assumption, and hence $(Bv_{kl}|v_{kl}) \neq 0$ and $v_{kl} \notin \mathcal{R}(B)^\perp$ for all k, l. The Hamiltonian of this control problem is

$$T = \begin{pmatrix} A & -BB^* \\ -I & -A \end{pmatrix}$$

and Theorem 5.2.3 can be applied. ⌟

Example 5.2.6 Consider the following wave equation with distributed control,

$$\begin{aligned} \partial_t^2 w(t,x) &= \partial_x^2 w(t,x) + b(x)u(t,x) \quad &&\text{for} \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\ \partial_t w(t,0) &= \partial_t w(t,1) = 0 \quad &&\text{for} \quad t \in \mathbb{R}_{\geq 0}, \end{aligned}$$

with $b \in L^\infty([0,1])$, $b \geq 0$, $b \neq 0$. As a first step we reformulate the problem as a system which is of first order in time. One possibility is to choose as new state variables the momentum p and the strain q,

$$p(t,x) = \partial_t w(t,x), \quad q(t,x) = \partial_x w(t,x).$$

The transformed system is then

$$\begin{aligned} &\partial_t \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} bu \\ 0 \end{pmatrix} \quad \text{on} \quad \mathbb{R}_{\geq 0} \times [0,1], \\ &p(t,0) = p(t,1) = 0. \end{aligned}$$

Let $Z = L^2([0,1])^2$ be the state space, $U = L^2([0,1])$ the input space, and define the operators $A(Z \to Z)$, $B : U \to Z$ by

$$\begin{aligned} A \begin{pmatrix} p \\ q \end{pmatrix} &= \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \quad \mathcal{D}(A) = \left\{ (p,q) \in W^{1,2}([0,1])^2 \,\middle|\, p(0) = p(1) = 0 \right\}, \\ Bu &= \begin{pmatrix} bu \\ 0 \end{pmatrix}. \end{aligned}$$

As cost functional we consider

$$J(z_0, u) = \int_0^\infty \left(\|z(t)\|^2_{L^2([0,1])^2} + \|u(t)\|^2_{L^2([0,1])} \right) dt,$$

i.e. $Y = Z$, $C = I$.

Straightforward calculations show that A is skew-adjoint with compact resolvent, $\sigma(A) = \{ i\pi k \mid k \in \mathbb{Z} \}$, and

$$v_k(x) = \begin{pmatrix} \sin(\pi k x) \\ -i \cos(\pi k x) \end{pmatrix} \quad \text{with} \quad k \in \mathbb{Z}$$

is an orthonormal basis of eigenvectors for A. This yields

$$\lim_{r \to \infty} \frac{N(r,A)}{r} = \frac{2}{\pi}$$

and $(Bv_{k,1} | v_k) \neq 0$ for all k, where $v_{k,1}$ denotes the first component of v_k. We can thus apply Theorem 5.2.3 to the Hamiltonian

$$T = \begin{pmatrix} A & -BB^* \\ -I & A \end{pmatrix}$$

of the system. ⌟

Our final example is a system with unbounded control operator B. Although the standard control theory from Theorem 5.2.2 and also Theorem 5.2.3 are not applicable, we can nevertheless put the system in the form (5.6) with B unbounded and apply Theorem 4.4.1 to the resulting Hamiltonian.

Example 5.2.7 We consider the one-dimensional heat equation with distributed control,

$$\begin{aligned} \partial_t z(t,x) &= \partial_x^2 z(t,x) + b(x)u(t,x) && \text{for} \quad (t,x) \in \mathbb{R}_{\geq 0} \times [0,1], \\ z(0,x) &= z_0(x) && \text{for} \quad x \in [0,1], \\ z(t,0) &= z(t,1) = 0 && \text{for} \quad t \in \mathbb{R}_{\geq 0}, \end{aligned}$$

with $b \in L^4([0,1])$, $b \geq 0$, $b \neq 0$. We choose $Z = U = L^2([0,1])$ as the state and input spaces and define the operators $A(Z \to Z)$ and $B(U \to Z)$ by

$$\begin{aligned} Av &= \partial_x^2 v, \quad \mathcal{D}(A) = \{v \in W^{2,2}([0,1]) \,|\, v(0) = v(1) = 0\}, \\ Bu &= bu, \quad \mathcal{D}(B) = \{u \in L^2([0,1]) \,|\, bu \in L^2([0,1])\}. \end{aligned}$$

Then A is selfadjoint with compact resolvent, $\sigma(A) = \{-\pi^2 k^2 \,|\, k = 1, 2, \ldots\}$, and

$$v_k(x) = \sqrt{2}\,\sin(\pi k x) \quad \text{with} \quad k \geq 1$$

is an orthonormal basis of eigenvectors. We have $N(\pi^2 k^2, A) = k$ and hence

$$\lim_{r\to\infty} \frac{N(r,A)}{r^{1/2}} = \frac{1}{\pi}.$$

The operator B is densely defined and symmetric, and for $u \in C^0([0,1])$ we have $u \in \mathcal{D}(BB^*)$ and $BB^*u = b^2 u$. From Proposition 3.2.16 it follows that BB^* is 1/2-subordinate to A. Since $b \geq 0$, $b \neq 0$, we have

$$(Bv_k|v_k) = 2\int_0^1 b(x)\sin^2(\pi k x)\,dx \neq 0,$$

i.e. $v_k \notin \mathcal{R}(B)^\perp$. As in the proof of Theorem 5.2.3 we have

$$z \in \ker(BB^*) \quad \Leftrightarrow \quad z \in \mathcal{R}(B)^\perp$$

and thus $v_k \notin \ker(BB^*)$ for all k.

Choosing $C = I$ as the output operator, the Hamiltonian of the system becomes

$$T = \begin{pmatrix} A & -BB^* \\ -I & -A \end{pmatrix}.$$

We have $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ and can apply Theorem 4.4.1 to $-T$. In particular, for every σ from a partition $\sigma(T) = \sigma \mathbin{\dot\cup} \tau$ which separates skew-conjugate points, this yields the existence of a selfadjoint injective core solution of

$$X(A - BB^*X) = -I - AX.$$

The solutions $X_\pm$ corresponding to left and right half-plane are positive and negative, respectively. ⌟

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Notation index

$L(V)$	space of bounded operators $T : V \to V$
I	identity operator
$\varrho(T)$	resolvent set, 13
$\sigma(T)$	$= \mathbb{C} \setminus \varrho(T)$, spectrum
$\sigma_p(T)$	point spectrum, the set of all eigenvalues
$\sigma_p^{\pm}(T)$	point spectrum in the open right/left half-plane
$r(T)$	set of points of regular type, 36
$\mathcal{L}(\lambda)$	root subspace, 13
$N(K, G)$	sum of multiplicities of all eigenvalues in $K \subset \mathbb{C}$, 95
$N(r, G)$	$= N(\overline{B_r(0)}, G)$, 95
$N_+(r_1, r_2, G)$	$= N(]r_1, r_2[\,, G)$, 84

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