Hamiltonians with Riesz Bases of Generalised Eigenvectors and Riccati Equations

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Abstract. An algebraic Riccati equation for linear operators is studied, which arises in systems theory. For the case that all involved operators are unbounded, the existence of infinitely many selfadjoint solutions is shown. The proof uses invariant graph subspaces of the associated Hamiltonian operator matrix, Riesz bases with parentheses of generalised eigenvectors and indefinite inner products. Under additional assumptions, the existence and a representation of all bounded solutions is obtained. The theory is applied to Riccati equations of differential operators.

Keywords. Riccati equation, Hamiltonian operator matrix, Riesz basis of generalised eigenvectors, invariant subspace, indefinite inner product.

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1 Introduction

We consider the algebraic Riccati equation

$$A^*X + XA + XBX - C = 0 \tag{1}$$

for linear operators on a Hilbert space H where B and C are selfadjoint and nonnegative. In particular, we study the case of unbounded B and C. Riccati equations of type (1) are a key tool in systems theory, see e.g. [9, 20] and the references therein. Unbounded B and C appear e.g. in [23, 28, 35].

It is well known that the solutions X of (1) are in one-to-one correspondence with graph subspaces that are invariant under the operator matrix

$$T = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix},$$

the so-called Hamiltonian. This correspondence was extensively studied in the finite-dimensional setting and led to a complete description of all solutions of the Riccati equation, see e.g. [20, 24, 27]. In the infinite-dimensional setting with B, C bounded, the invariant subspace approach was used by Kuiper and Zwart [19] for Riesz-spectral T and by Langer, Ran and van de Rotten [21] for dichotomous T (see also [7]).

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We extend these results to the case where B and C are unbounded: For Hamiltonians with a Riesz basis with parentheses of generalised eigenvectors we show the existence of infinitely many selfadjoint solutions of (1). Here the concept of a Riesz basis with parentheses of generalised eigenvectors includes Riesz-spectral operators, and it also allows for operators that are not dichotomous.

In systems theory, solutions of (1) that are bounded and nonnegative are of particular importance. For the case that T has a Riesz basis of generalised eigenvectors and B and C are uniformly positive, we prove the existence of infinitely many bounded selfadjoint solutions X, among them a nonnegative one X_+ and a nonpositive one X_-. Moreover we obtain the relations

$$X_- \leq X \leq X_+ \qquad \text{and} \qquad X = X_+ P + X_-(I-P), \tag{2}$$

where P is an appropriate projection.

Bounded nonnegative solutions of (1) were obtained in [19, 21] without the assumption of uniform positivity of B, C. However, in [21] the spectrum $\sigma(A)$ of A was restricted to a sector in the open left half-plane while here $\sigma(A)$ may also contain points in the right half-plane. In [19] conditions for the existence of solutions were formulated in terms of the eigenvectors of T while we impose conditions on the operators A, B, C only. In the system theoretic setting, the relations (2) were derived in [8, 26], yet under the explicit assumption of the existence of X_-.

The connection of invariant graph subspaces of block operator matrices to solutions of a corresponding Riccati equation is not limited to Hamiltonian matrices. It was exploited in [22, 29] for certain dichotomous operator matrices and in [17] for selfadjoint ones. We also mention that, in systems theory, nonnegative solutions of (1) are constructed by minimising a quadratic functional, see e.g. [9].

The structure of this article is as follows: In Sections 2 and 3 we study linear operators T on a Hilbert space which possess a Riesz basis consisting of finite-dimensional spectral subspaces V_k of T. We call such a Riesz basis finitely spectral for T. Up to certain technical details, it is equivalent to a Riesz basis with parentheses of generalised eigenvectors, see Proposition 3.3. A finitely spectral Riesz basis of subspaces yields the non-trivial T-invariant subspaces

$$\overline{\sum_{k\in\mathbb{N}} U_k}, \qquad U_k \subset V_k \ T\text{-invariant},$$

which we call compatible with the Riesz basis, see Definition 2.3 and Lemma 3.10. In particular, for every subset of the point spectrum there is an associated invariant compatible subspace.

In Theorem 3.9 we use perturbation theory from [37] to obtain a general existence result for finitely spectral Riesz bases of subspaces and apply it to Hamiltonian operators in Theorem 4.6; Theorem 4.7 even yields a Riesz basis of eigenvectors and finitely many generalised eigenvectors. On the other hand, there is a huge literature on Riesz bases of eigenvectors (with or without parentheses) for various types of operators, e.g. [15, 38, 39]; all these provide examples for finitely spectral Riesz bases of subspaces.

In Section 4 we study Hamiltonian operator matrices. We use ideas from [21] and consider the indefinite inner products

$$\langle x|y\rangle = (J_1 x|y) \qquad \text{and} \qquad [x|y] = (J_2 x|y)$$

on $H \times H$ associated with the fundamental symmetries

$$J_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}, \qquad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

The Hamiltonian T is J_1-skew-symmetric and J_2-accretive, i.e.

$$\langle Tx|y\rangle = -\langle x|Ty\rangle \quad \text{and} \quad \mathrm{Re}[Tx|x] \geq 0, \quad x, y \in \mathcal{D}(T).$$

This implies the symmetry of the spectrum of T with respect to the imaginary axis and also yields a characterisation of its purely imaginary eigenvalues.

In Section 5 we consider subsets σ of the point spectrum of T that are skew-conjugate, i.e., σ contains exactly one eigenvalue from each skew-conjugate pair of eigenvalues. With σ we then associate an invariant compatible subspace U that is hypermaximal J_1-neutral. This means that $U = U^{\langle\perp\rangle}$ where $U^{\langle\perp\rangle}$ is the orthogonal complement of U with respect to the J_1 inner product. The subspaces $U_\pm$ corresponding to the spectrum in the right and left half-plane are J_2-nonnegative and -nonpositive, respectively: we have $[x|x] \geq 0$ for $x \in U_+$ and $[x|x] \leq 0$ for $x \in U_-$.

Section 6 is devoted to the existence of solutions of the Riccati equation (1). In Theorem 6.3 we establish a density condition in terms of A and B which implies that every hypermaximal J_1-neutral invariant compatible subspace is the graph of a selfadjoint solution. If T has infinitely many skew-conjugate pairs of eigenvalues, for example if T has no purely imaginary eigenvalues, then there are infinitely many possible skew-conjugate subsets σ and we obtain infinitely many solutions. The solutions $X_\pm$ corresponding to $U_\pm$ are nonnegative and nonpositive. Since all these solutions are unbounded in general, the Riccati equation in fact takes the slightly different form

$$X(Au + BXu) = Cu - A^*Xu, \qquad u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*).$$

For the convenience of the reader, Theorem 6.7 contains a combination of the main results of Theorems 4.6 and 6.3; the density condition can then be expressed in terms of eigenspaces of A and the kernel of B.

Bounded solutions of (1) are the topic of Section 7. Theorem 7.6 shows that a bounded linear operator is a solution of (1) on $\mathcal{D}(A)$ if and only if its graph is an invariant compatible subspace. In Example 8.3 we construct a bounded solution such that (1) holds only on a proper subspace of $\mathcal{D}(A)$ and whose graph is invariant but not compatible. Finally we consider Hamiltonians with a Riesz basis of Jordan chains and uniformly positive B and C. Then Theorem 7.9 yields that the solutions corresponding to hypermaximal J_1-neutral compatible subspaces are in fact bounded and boundedly invertible, and that the relations (2) hold.

2 Riesz bases of subspaces

We study the concept of a Riesz basis of subspaces and define the notion of a compatible subspace with respect to such a basis. We also recall the closely related concepts of a Riesz basis (of vectors) and a Riesz basis with parentheses; see [13, Chapter VI], [30, §15], [33, §1] and [36, §2] for more details about Riesz bases.

Let V be a separable Hilbert space. We denote the subspace generated by a system $(V_\lambda)_{\lambda\in\Lambda}$ of subspaces $V_\lambda \subset V$ by

$$\sum_{\lambda\in\Lambda} V_\lambda = \{x_{\lambda_1} + \cdots + x_{\lambda_n} \mid x_{\lambda_j} \in V_{\lambda_j}, \lambda_j \in \Lambda, n \in \mathbb{N}\}.$$

The system is said to be *complete* if $\sum_{\lambda\in\Lambda} V_\lambda \subset V$ is dense.

Definition 2.1 Let V be a separable Hilbert space.

(i) A sequence $(v_k)_{k\in\mathbb{N}}$ in V is called a *Riesz basis* of V if there is an isomorphism $\Phi : V \to V$ such that $(\Phi v_k)_{k\in\mathbb{N}}$ is an orthonormal basis of V.

(ii) A sequence of closed subspaces $(V_k)_{k\in\mathbb{N}}$ of V is called a *Riesz basis of subspaces* of V if there is an isomorphism $\Phi : V \to V$ such that $(\Phi(V_k))_{k\in\mathbb{N}}$ is a complete system of pairwise orthogonal subspaces.

(iii) The sequence $(v_k)_{k\in\mathbb{N}}$ in V is called a *Riesz basis with parentheses* of V if there exists a Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$ of V and a subsequence $(n_k)_k$ of $\mathbb{N}$ with $n_0 = 0$ such that $(v_{n_k}, \dots, v_{n_{k+1}-1})$ is a basis of V_k.

The sequence $(v_k)_{k\in\mathbb{N}}$ is a Riesz basis if and only if $\operatorname{span}\{v_k\} \subset V$ is dense and there are constants $m, M > 0$ such that

$$m\sum_{k=0}^{n} |\alpha_k|^2 \le \left\| \sum_{k=0}^{n} \alpha_k v_k \right\|^2 \le M \sum_{k=0}^{n} |\alpha_k|^2, \quad \alpha_k \in \mathbb{C},\ n \in \mathbb{N}. \tag{3}$$

The sequence of closed subspaces $(V_k)_{k\in\mathbb{N}}$ is a Riesz basis of subspaces of V if and only if $(V_k)_{k\in\mathbb{N}}$ is complete and there exists a constant $c \ge 1$ such that

$$c^{-1}\sum_{k=0}^{n} \|x_k\|^2 \le \left\| \sum_{k=0}^{n} x_k \right\|^2 \le c\sum_{k=0}^{n} \|x_k\|^2, \quad x_k \in V_k,\ n \in \mathbb{N}. \tag{4}$$

If $(v_k)_{k\in\mathbb{N}}$ is a Riesz basis of V, then every $x \in V$ has a unique representation $x = \sum_{k=0}^{\infty} \alpha_k v_k$, $\alpha_k \in \mathbb{C}$, and the convergence of the series is unconditional. For a Riesz basis with parentheses we have the unique representation

$$x = \sum_{k=0}^{\infty} \left(\sum_{j=n_k}^{n_{k+1}-1} \alpha_j v_j \right), \qquad \alpha_j \in \mathbb{C},$$

where the series over k converges unconditionally. These expansions are special cases of the situation for a Riesz basis of subspaces:

Proposition 2.2 *A Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$ has the following properties:*

(i) There are projections $P_k \in L(V)$ onto V_k satisfying $P_jP_k = 0$ for $j \ne k$ and a constant $c \ge 1$ such that

$$c^{-1}\sum_{k=0}^{\infty} \|P_k x\|^2 \le \|x\|^2 \le c\sum_{k=0}^{\infty} \|P_k x\|^2 \quad \textit{for all} \quad x \in V. \tag{5}$$

(ii) If $x_k \in V_k$ with $\sum_{k=0}^\infty \|x_k\|^2 < \infty$, then the series $\sum_{k=0}^\infty x_k$ converges unconditionally.

(iii) Every $x \in V$ has a unique expansion $x = \sum_{k=0}^\infty x_k$ with $x_k \in V_k$, and we have $x_k = P_k x$.

Proof. The proof is immediate since all assertions hold (with $c = 1$) if the V_k are pairwise orthogonal, and they continue to hold (with some $c \geq 1$ now) if we apply the isomorphism Φ from Definition 2.1. □

For a Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$, the unique expansion from (iii) yields a decomposition of the space V into the subspaces V_k, which we denote by

$$V = \bigoplus_{k\in\mathbb{N}}^{2} V_k. \tag{6}$$

Here, the superscript 2 indicates that, due to (5), the original norm on V is equivalent to the l^2-type norm $(\sum_{k\in\mathbb{N}} \|P_k x\|^2)^{1/2}$.

Definition 2.3 Let $(V_k)_{k\in\mathbb{N}}$ be a Riesz basis of subspaces of V. We say that a subspace $U \subset V$ is *compatible* with $(V_k)_{k\in\mathbb{N}}$ if

$$U = \overline{\sum_{k\in\mathbb{N}} U_k} \quad \text{with closed subspaces } U_k \subset V_k.$$

Evidently, $(U_k)_{k\in\mathbb{N}}$ is then a Riesz basis of subspaces of U. We thus have the decomposition

$$U = \bigoplus_{k\in\mathbb{N}}^{2} U_k.$$

As a special case, for every $J \subset \mathbb{N}$ we obtain the compatible subspace $\bigoplus_{k\in J}^2 V_k$.[1]

Remark 2.4 It is easy to see that, with P_k as above, U is compatible with (V_k) if and only if $P_k(U) \subset U$; in this case $U = \bigoplus_k^2 P_k(U)$.

Definition 2.5 If U and W are two subspaces of V satisfying $U \cap W = \{0\}$, we say that their sum is *algebraically direct*, denoted by $U \dotplus W$. We say that the sum is *topologically direct* and write $U \oplus W$ if moreover the associated projection from $U \dotplus W$ onto U is bounded.

As a consequence of the closed graph theorem, if $U \cap W = \{0\}$ and U, W and $U \dotplus W$ are closed, then in fact $U \oplus W$ is topologically direct.

Proposition 2.6 *Let $(V_k)_{k\in\mathbb{N}}$ be a Riesz basis of subspaces of V.*

(i) If $V_k = U_k \oplus W_k$ for all k, then the sum

$$\bigoplus_{k\in\mathbb{N}}^{2} U_k \dotplus \bigoplus_{k\in\mathbb{N}}^{2} W_k \subset V$$

is algebraically direct and dense.

[1]Note here that Definition 2.1 implicitly covers the case of systems with arbitrary index set $J \subset \mathbb{N}$ since $V_k = \{0\}$ is possible.

(ii) For $J \subset \mathbb{N}$ we have the topologically direct sum

$$V = \bigoplus_{k\in J}^{2} V_k \oplus \bigoplus_{k\in\mathbb{N}\setminus J}^{2} V_k.$$

The associated projection onto the first summand is given by

$$P_J : \sum_{k\in\mathbb{N}} x_k \mapsto \sum_{k\in J} x_k, \quad x_k \in V_k. \tag{7}$$

It satisfies $\|P_J\| \le c$ where c is the constant from (5).

Proof. (i): Let $U = \bigoplus_k^2 U_k$, $W = \bigoplus_k^2 W_k$, and $x \in U \cap W$. We expand x in the Riesz bases (U_k) of U and (W_k) of W: $x = \sum_k u_k = \sum_k w_k$ with $u_k \in U_k$, $w_k \in W_k$. As these are also expansions of x in the Riesz basis (V_k), we obtain $u_k = w_k$ and thus $u_k = 0$ and $x = 0$. The sum $U + W$ is dense in V since it contains $\sum_{k\in\mathbb{N}} V_k$.

(ii): From (5) we have the estimate

$$\Big\| \sum_{k\in J} x_k \Big\|^2 \le c \sum_{k\in J} \|x_k\|^2 \le c \sum_{k\in\mathbb{N}} \|x_k\|^2 \le c^2 \Big\| \sum_{k\in\mathbb{N}} x_k \Big\|^2.$$

This shows that P_J defined by (7) satisfies $\|P_J\| \le c$. Obviously

$$\mathcal{R}(P_J) = \bigoplus_{k\in J}^{2} V_k, \quad \ker P_J = \bigoplus_{k\in\mathbb{N}\setminus J}^{2} V_k,$$

and $V = \mathcal{R}(P_J) \oplus \ker P_J$ is topologically direct. □

3 Finitely spectral Riesz bases of subspaces

In this section we investigate operators with a Riesz basis of finite-dimensional invariant subspaces. We obtain many non-trivial compatible subspaces that are invariant under the operator. In particular, for every subset of the point spectrum there is an associated invariant compatible subspace.

We recall some concepts for a linear operator T on a Banach space V, see also [2, 16]. A point $z \in \mathbb{C}$ is called a *point of regular type* if $T - z$ is injective and the inverse $(T-z)^{-1}$ (defined on $\mathcal{R}(T-z)$) is bounded. The set of all points of regular type is denoted by $r(T)$; it is open and satisfies $\varrho(T) \subset r(T)$ and $\sigma_p(T) \cap r(T) = \varnothing$.

A subspace $W \subset V$ is called a *core* for the linear operator T if for every $x \in \mathcal{D}(T)$ there is a sequence (x_n) in W such that $x_n \to x$ and $Tx_n \to Tx$.

Finally we denote by $\mathcal{L}(\lambda)$ the space of generalised eigenvectors or root subspace of T corresponding to the eigenvalue $\lambda \in \sigma_p(T)$, i.e.

$$\mathcal{L}(\lambda) = \bigcup_{k\in\mathbb{N}} \ker(T-\lambda)^k.$$

For $\lambda \notin \sigma_p(T)$ we set $\mathcal{L}(\lambda) = \{0\}$. A sequence $x_1, \dots, x_n \in \mathcal{L}(\lambda)$ is called a *Jordan chain* if $(T-\lambda)x_k = x_{k-1}$ for $k \ge 2$ and $(T-\lambda)x_1 = 0$.

Definition 3.1 Let T be a closed operator on a separable Hilbert space V. We say that a Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$ of V is *finitely spectral* for T if

(i) each V_k is finite-dimensional, T-invariant and satisfies $V_k \subset \mathcal{D}(T)$,

(ii) the sets $\sigma(T|_{V_k})$ are pairwise disjoint, and

(iii) $\sum_{k\in\mathbb{N}} V_k$ is a core for T.

It is immediate from the definition, that T has infinitely many eigenvalues (for $\dim V = \infty$). In some situations, the conditions on the closedness and the core are automatically fulfilled:

Proposition 3.2 *Let T be a linear operator on V, $(V_k)_{k\in\mathbb{N}}$ a Riesz basis of finite-dimensional, T-invariant subspaces of V, $V_k \subset \mathcal{D}(T)$ for all k, and $\sigma(T|_{V_k})$ pairwise disjoint. Then:*

(i) $T_0 = T|_{\sum_k V_k}$ is closable and $(V_k)_{k\in\mathbb{N}}$ is finitely spectral for $\overline{T_0}$.

(ii) If $r(T) \neq \varnothing$, then T is closable and $(V_k)_{k\in\mathbb{N}}$ is finitely spectral for $\overline{T}$.

Proof. Let P_k be the projections onto the V_k corresponding to the Riesz basis. Then for $u \in \sum_k V_k$ we have $P_k T_0 u = T|_{V_k} P_k u$ for all k since u is a finite sum of elements from the T-invariant subspaces V_k.

(i): Let $x_n \in \mathcal{D}(T_0) = \sum_k V_k$ with $x_n \to 0$ and $T_0 x_n \to y$. Since the restriction $T|_{V_k}$ is bounded, we have

$$P_k y = \lim_{n\to\infty} P_k T_0 x_n = \lim_{n\to\infty} T|_{V_k} P_k x_n = T|_{V_k} P_k \lim_{n\to\infty} x_n = 0$$

for every $k \in \mathbb{N}$ and hence $y = 0$; T_0 is closable. The other assertion is now immediate.

(ii): In view of (i) it suffices to show $T \subset \overline{T_0}$: for T is closable then, and from $T_0 \subset T$ we conclude $\overline{T_0} = \overline{T}$. Let $x \in \mathcal{D}(T)$ and $z \in r(T)$. Using the Riesz basis (V_k), we have the expansion $(T - z)x = \sum_{k=0}^{\infty} y_k$ with $y_k \in V_k$. Since $T - z$ is injective and V_k is finite-dimensional and T-invariant, $T - z$ maps V_k onto V_k. We can thus set $x_k = (T - z)^{-1} y_k \in V_k$ and obtain $x = \sum_{k=0}^{\infty} x_k$ by the boundedness of $(T - z)^{-1}$. Consequently

$$\mathcal{D}(T_0) \ni \sum_{k=0}^{n} x_k \to x \qquad \text{and} \qquad (T_0 - z)\sum_{k=0}^{n} x_k = \sum_{k=0}^{n} y_k \to (T - z)x$$

as $n \to \infty$, i.e., $x \in \mathcal{D}(\overline{T_0})$ and $\overline{T_0} x = Tx$. □

The notion of a finitely spectral Riesz basis of subspaces contains many other types of bases related to eigenvectors and the spectrum as special cases:

Proposition 3.3 *Let T be a closed operator on V with $r(T) \neq \varnothing$ and $\dim \mathcal{L}(\lambda) < \infty$ for all $\lambda \in \sigma_p(T)$. Then for the assertions*

(i) T has a finitely spectral Riesz basis of subspaces,

(ii) the root subspaces $\mathcal{L}(\lambda)$, $\lambda \in \sigma_p(T)$, of T form a Riesz basis of V,

(iii) T has a Riesz basis of Jordan chains,

we have (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). Moreover *(i)* $\Leftrightarrow$ *T has a Riesz basis with parentheses of Jordan chains, and for each $\lambda \in \sigma_p(T)$ there is a parenthesis that contains all Jordan chains from the basis which correspond to λ;*

(ii) $\Leftrightarrow$ *T has a Riesz basis with parentheses of Jordan chains, and for each $\lambda \in \sigma_p(T)$ there is a parenthesis that contains all Jordan chains from the basis which correspond to λ, but no Jordan chains corresponding to other eigenvalues.*

Proof. In view of Proposition 3.2(ii), the implication (ii)$\Rightarrow$(i) is trivial. For (iii)$\Rightarrow$(ii) consider for each eigenvalue $\lambda \in \sigma_p(T)$ the subspace V_λ generated by all Jordan chains from the basis which correspond to λ. Then $(V_\lambda)_{\lambda\in\sigma_p(T)}$ is a Riesz basis of subspaces and $V_\lambda = \mathcal{L}(\lambda)$. The equivalent characterisations of (i) and (ii) are immediate from the definitions. $\square$

Remark 3.4 Example 3.7 shows that the implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are strict, even for the case that T has compact resolvent. If T has compact resolvent, then (ii) holds if and only if T is a *spectral operator* in the sense of Dunford, see [12, 36].

A closed operator T is called *Riesz-spectral* [9, 19] if all its eigenvalues are simple, T has a Riesz basis of eigenvectors, and $\overline{\sigma_p(T)}$ is totally disconnected. So if T is Riesz-spectral then (iii) holds.

Proposition 3.5 *Let T be a closed operator with a finitely spectral Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$. Then*

$$\mathcal{D}(T) = \left\{ x = \sum_{k\in\mathbb{N}} x_k \,\middle|\, x_k \in V_k,\ \sum_{k\in\mathbb{N}} \|Tx_k\|^2 < \infty \right\}, \tag{8}$$

$$Tx = \sum_{k\in\mathbb{N}} Tx_k \quad \textit{for} \quad x = \sum_{k\in\mathbb{N}} x_k \in \mathcal{D}(T),\ x_k \in V_k. \tag{9}$$

The operator T is bounded if and only if the restrictions $T|_{V_k}$ are uniformly bounded and in this case (with c from (5))

$$\|T\| \le c \sup_{k\in\mathbb{N}} \|T|_{V_k}\|.$$

Proof. (i): In order to derive (8) and (9), let first $y \in \mathcal{D}(T)$. Since $\sum_k V_k$ is a core for T, there is a sequence (y_n) in $\sum_k V_k$ with $y_n \to y$, $Ty_n \to Ty$. As in the proof of Proposition 3.2, we obtain

$$P_k T y = \lim_{n\to\infty} P_k T y_n = \lim_{n\to\infty} T|_{V_k} P_k y_n = T|_{V_k} P_k \lim_{n\to\infty} y_n = T P_k y.$$

Hence $\sum_k \|TP_k y\|^2 = \sum_k \|P_k T y\|^2 \le c\|Ty\|^2 < \infty$ and

$$y = \sum_k P_k y \in \left\{ x = \sum_k x_k \,\middle|\, x_k \in V_k,\ \sum_k \|Tx_k\|^2 < \infty \right\} \quad \text{with}$$
$$Ty = \sum_k P_k T y = \sum_k T P_k y.$$

If on the other hand $x = \sum_k x_k$ with $x_k \in V_k$, $\sum_k \|Tx_k\|^2 < \infty$, then

$$\mathcal{D}(T) \ni \sum_{k=0}^{n} x_k \to x \quad \text{and} \quad T\sum_{k=0}^{n} x_k = \sum_{k=0}^{n} Tx_k \to \sum_{k=0}^{\infty} Tx_k.$$

Hence $x \in \mathcal{D}(T)$ since T is closed.

(ii): Let $L = \sup_k \|T|_{V_k}\| < \infty$. Then for $x = \sum_k x_k \in \mathcal{D}(T)$:

$$\|Tx\|^2 = \Big\| \sum_k T|_{V_k} x_k \Big\|^2 \le c \sum_k \|T|_{V_k} x_k\|^2 \le cL^2 \sum_k \|x_k\|^2 \le c^2 L^2 \|x\|^2;$$

thus T is bounded with norm $\le c\, L$. □

For the case that the V_k are pairwise orthogonal and possibly infinite-dimensional, the spectrum of an operator defined by (8), (9) was calculated by Davies [10, Theorem 8.1.12]. We obtain:

Corollary 3.6 *Let T be a closed operator with a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$. Then*

$$\sigma_p(T) = \bigcup_{k \in \mathbb{N}} \sigma(T|_{V_k}), \tag{10}$$

$$V_k = \sum_{\lambda \in \sigma(T|_{V_k})} \mathcal{L}(\lambda), \tag{11}$$

$$\varrho(T) = r(T) = \Big\{ z \in \mathbb{C} \setminus \sigma_p(T) \,\Big|\, \sup_{k \in \mathbb{N}} \|(T|_{V_k} - z)^{-1}\| < \infty \Big\}. \tag{12}$$

Moreover, for $z \in \varrho(T)$,

$$(T - z)^{-1} \text{ compact} \quad \Leftrightarrow \quad \lim_{k \to \infty} \|(T|_{V_k} - z)^{-1}\| = 0.$$

Proof. The inclusions "$\supset$" in (10) and "$\subset$" in (11) are trivial. For the other ones let $\lambda \in \sigma_p(T)$ and $x = \sum_j x_j \in \mathcal{L}(\lambda) \setminus \{0\}$, $x_j \in V_j$. Then $(T - \lambda)^n x = 0$ for some $n \in \mathbb{N}$. Now (9) implies

$$0 = (T - \lambda)^n x = \sum_{j \in \mathbb{N}} (T|_{V_j} - \lambda)^n x_j$$

and hence $(T|_{V_j} - \lambda)^n x_j = 0$ for all j. Since $x_k \neq 0$ for some k, we obtain $\lambda \in \sigma(T|_{V_k})$. As the $\sigma(T|_{V_j})$ are disjoint, we have $\lambda \notin \sigma(T|_{V_j})$ and hence $x_j = 0$ for $j \neq k$. Therefore $x \in V_k$.

To show (12), first note that if $z \in r(T)$, then for every $k \in \mathbb{N}$, $(T|_{V_k} - z)^{-1}$ exists and is a restriction of $(T - z)^{-1}$. Thus $\sup_k \|(T|_{V_k} - z)^{-1}\| \le \|(T - z)^{-1}\| < \infty$. Furthermore, if $z \in \mathbb{C} \setminus \sigma_p(T)$ with $\sup_k \|(T|_{V_k} - z)^{-1}\| < \infty$, then

$$S : \sum_{k \in \mathbb{N}} x_k \mapsto \sum_{k \in \mathbb{N}} (T|_{V_k} - z)^{-1} x_k$$

defines a bounded operator $S : V \to V$ satisfying $(T - z)Sx = x$ for all $x \in V$. Consequently $z \in \varrho(T)$ with $(T - z)^{-1} = S$.

Now suppose that $\|(T|_{V_k} - z)^{-1}\| \to 0$ as $k \to \infty$. Then the sequence of finite-rank operators $\sum_{k=0}^n (T|_{V_k} - z)^{-1} P_k$, $n \in \mathbb{N}$, converges in norm to $(T - z)^{-1}$ since

$$\Big\| \sum_{k > n} (T|_{V_k} - z)^{-1} P_k \Big\| \le c \sup_{k > n} \|(T|_{V_k} - z)^{-1}\| \to 0 \quad \text{as} \quad n \to \infty.$$

Hence $(T-z)^{-1}$ is compact. On the other hand, let $\|(T|_{V_k}-z)^{-1}\| \not\to 0$. Then there exists a strictly increasing sequence of indices (k_l) and $x_l \in V_{k_l}$, $y_l = (T-z)^{-1}x_l$, such that $\|x_l\| = 1$ and $\inf_l \|y_l\| > 0$. From $y_l \in V_{k_l}$ it follows that every converging subsequence of (y_l) must converge to zero. So (y_l) has no converging subsequence and $(T-z)^{-1}$ is not compact. □

The following example illustrates the differences between the three Riesz basis properties from Proposition 3.3 for the case of a compact resolvent.

Example 3.7 Let $(e_k)_{k\geq 1}$ be an orthonormal basis of a separable Hilbert space V and $V_k = \operatorname{span}\{e_{2k-1}, e_{2k}\}$. Using Proposition 3.2, we can define a closed operator T on V such that $(V_k)_{k\geq 1}$ is finitely spectral for T and

$$T|_{V_k} \cong \begin{pmatrix} ik^2 & 1 \\ k & ik^2 \end{pmatrix},$$

with respect to the basis (e_{2k-1}, e_{2k}) of V_k. The eigenvalues and corresponding normalised eigenvectors of $T|_{V_k}$ are

$$\lambda_k^{\pm} = ik^2 \pm \sqrt{k}, \quad v_k^{\pm} = \frac{1}{\sqrt{1+k}}\big(e_{2k-1} \pm \sqrt{k}\, e_{2k}\big).$$

A direct computation yields $\lim_{k\to\infty} \|(T|_{V_k} - z)^{-1}\| = 0$ for $z \notin \sigma_p(T)$ and hence T has compact resolvent.

Consider now the sequences $(x_k)_{k\geq 1}$, $(x_k^+)_{k\geq 1}$ and $(x_k^-)_{k\geq 1}$ given by

$$x_k = \frac{2}{\sqrt{k}}\, e_{2k}, \qquad x_k^{\pm} = \sqrt{\frac{1+k}{k}}\, v_k^{\pm}.$$

Then $x_k = x_k^+ + x_k^-$ with $x_k^{\pm} \in \mathbb{C}v_k^{\pm}$, the sequence (x_k) converges to zero, while the sequences $(x_k^{\pm})$ do not. Consequently, the algebraically direct sum (see Proposition 2.6)

$$\bigoplus_{k\geq 1} \mathbb{C}v_k^+ \dotplus \bigoplus_{k\geq 1} \mathbb{C}v_k^-$$

is not topologically direct, and the system of root subspaces $\mathcal{L}(\lambda_k^{\pm}) = \mathbb{C}v_k^{\pm}$ of T is not a Riesz basis.

Altering the above definition, we can also consider the operator T_1 with

$$T_1|_{V_k} \cong \begin{pmatrix} ik^2 & 0 \\ k & ik^2 \end{pmatrix}.$$

Then T_1 still has a compact resolvent. The root subspace of T_1 are now $\mathcal{L}(ik^2) = V_k$ and form a Riesz basis of subspaces of V, but T_1 has no Riesz basis of Jordan chains.

Definition 3.8 Let G and S be linear operators on a Banach space. Then S is called *p-subordinate* to G with $0 \leq p < 1$ if $\mathcal{D}(G) \subset \mathcal{D}(S)$ and there exists $c \in \mathbb{R}$ such that

$$\|Sx\| \leq c\|x\|^{1-p}\|Gx\|^p \quad \text{for} \quad x \in \mathcal{D}(G).$$

If S is p-subordinate to G with $p < 1$, then S is relatively bounded to G with G-bound 0, see e.g. [18, §I.7.1] or [36, §3.2]. Recall that S is *relatively bounded* to G if $\mathcal{D}(G) \subset \mathcal{D}(S)$ and there are constants a, b such that $\|Sx\| \le a\|x\| + b\|Gx\|$ for all $x \in \mathcal{D}(G)$. The infimum of all such b is called the *G-bound* of S.

We use the notation

$$N(r, G) = \sum_{\lambda \in \sigma_p(G), |\lambda| \le r} \dim \mathcal{L}(\lambda)$$

for the sum of the algebraic multiplicities of the eigenvalues of G with $|\lambda| \le r$.

Theorem 3.9 *Let G be a normal operator with compact resolvent whose eigenvalues lie on finitely many rays from the origin. Let S be p-subordinate to G with $0 \le p < 1$. If*

$$\liminf_{r \to \infty} \frac{N(r, G)}{r^{1-p}} < \infty,$$

then $T = G + S$ has a compact resolvent and a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$.

Proof. See Theorems 4.5 and 6.1 in [37]. In particular, note that the V_k were constructed as the ranges of Riesz projections associated with disjoint parts of $\sigma(T)$, and hence the sets $\sigma(T|_{V_k})$ are disjoint. □

Now we study invariant subspaces with respect to a finitely spectral Riesz basis of subspaces.

Lemma 3.10 *Let T be a closed operator with a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$. For a compatible subspace $U = \bigoplus^2_{k \in \mathbb{N}} U_k$, $U_k \subset V_k$, the following assertions are equivalent:*

(i) U is T-invariant;

(ii) all U_k are T-invariant.

For $z \in \varrho(T)$, (i) and (ii) are equivalent to

(iii) U is $(T - z)^{-1}$-invariant.

Proof. The first claim is immediate from (9). For $z \in \varrho(T)$ note that $\dim U_k < \infty$ and $U_k \subset \mathcal{D}(T)$ imply that U_k is T-invariant if and only if U_k is $(T-z)^{-1}$-invariant. Moreover (V_k) is also finitely spectral for $(T - z)^{-1}$. □

Corollary 3.11 *Let T be closed with a finitely spectral Riesz basis of subspaces $(V_k)_{k \in \mathbb{N}}$. The subspace U is T-invariant and compatible with (V_k) if and only if*

$$U = \overline{\sum_{\lambda \in \sigma_p(T)} W_\lambda} \tag{13}$$

with T-invariant subspaces $W_\lambda \subset \mathcal{L}(\lambda)$. In this case $U = \bigoplus^2_k U_k$ with

$$U_k = \sum_{\lambda \in \sigma(T|_{V_k})} W_\lambda \quad \text{and} \quad W_\lambda = U_k \cap \mathcal{L}(\lambda),\ \lambda \in \sigma(T|_{V_k}). \tag{14}$$

In particular, for every $\sigma \subset \sigma_p(T)$ there is the T-invariant compatible subspace

$$U_\sigma = \overline{\sum_{\lambda\in\sigma} \mathcal{L}(\lambda)} \tag{15}$$

associated with σ.

Proof. Let $U = \bigoplus_k^2 U_k$ with $U_k \subset V_k$ T-invariant. Since $\dim V_k < \infty$, an argument from linear algebra implies that (14) holds, which in turn implies (13). On the other hand, if U is given by (13), and we define U_k by the first identity in (14), then U_k is T-invariant, $U_k \subset V_k$, and we obtain $U = \bigoplus_k^2 U_k$. $\square$

In the following, we will occasionally consider the subspaces $U_\pm$ associated with the point spectrum in the open right and left half-plane, respectively, i.e.

$$U_\pm = \overline{\sum_{\lambda\in\sigma_p^\pm(T)} \mathcal{L}(\lambda)} \quad \text{where} \quad \sigma_p^\pm(T) = \big\{\lambda \in \sigma_p(T) \,\big|\, \operatorname{Re}\lambda \gtrless 0\big\}.$$

Corollary 3.12 *If T is closed with a finitely spectral Riesz basis of subspaces and $\sigma_p(T) \cap i\mathbb{R} = \varnothing$, then $U_+ \dotplus U_- \subset V$ algebraically direct and dense.*

Proof. This follows from Proposition 2.6 since $U_\pm = \bigoplus_k^2 V_k^\pm$ where $V_k^\pm$ are the spectral subspaces of $T|_{V_k}$ corresponding to the right and left half-plane. $\square$

Remark 3.13 The concept of a finitely spectral Riesz basis of subspaces allows for operators that have a strip around the imaginary axis in the resolvent set, but are not *dichotomous* nonetheless:

A linear operator T is called dichotomous (see [21]) if a strip around the imaginary axis belongs to $\varrho(T)$, and there is a topologically direct decomposition $V = V_+ \oplus V_-$ such that $V_\pm$ is T-invariant and $\sigma(T|_{V_\pm})$ is contained in the open right and left half-plane, respectively. In particular $U_\pm \subset V_\pm$. Now for the operator T from Example 3.7 we have $U_\pm = \bigoplus_{k\geq 1} \mathbb{C}v_k^\pm$ and that $U_+ \dotplus U_-$ is not topologically direct. Consequently T is not dichotomous. On the other hand, T has compact resolvent and the eigenvalues $ik^2 \pm \sqrt{k}$, $k \geq 1$. Hence $\{z \,|\, |\operatorname{Re} z| < 1\} \subset \varrho(T)$.

Lemma 3.14 *Let T be a closed operator on V, $z_0 \in \varrho(T)$ and $U \subset V$ a closed $(T - z_0)^{-1}$-invariant subspace. Then U is $(T - z)^{-1}$-invariant for all z in the connected component of z_0 in $\varrho(T)$.*

Proof. It suffices to show that the set

$$A = \{z \in \varrho(T) \,|\, U \text{ is } (T - z)^{-1}\text{-invariant}\}$$

is relatively open and closed in $\varrho(T)$. Let $z \in A$. For $w \in \mathbb{C}$ sufficiently close to z a Neumann series argument shows that

$$(T - w)^{-1} = (T - z)^{-1}\big(I - (w - z)(T - z)^{-1}\big)^{-1} = \sum_{k=0}^{\infty} (w - z)^k (T - z)^{-k-1}.$$

If $x \in U$, then $(T - z)^{-k-1}x \in U$ for all $k \geq 0$. Hence also $(T - w)^{-1}x \in U$, i.e. $w \in A$; A is an open set.

Now let $w \in \varrho(T)$ with $w = \lim_{n\to\infty} z_n$, $z_n \in A$. For $x \in U$ we then have

$$U \ni (T - z_n)^{-1}x \to (T - w)^{-1}x \in U \quad \text{as} \quad n \to \infty$$

since the resolvent $(T - z)^{-1}$ is continuous in z. Hence $w \in A$, i.e., A is relatively closed. □

Proposition 3.15 *Let T be an operator with compact resolvent and a finitely spectral Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$. If U is a closed subspace that is $(T - z)^{-1}$-invariant for some $z \in \varrho(T)$, then U is T-invariant and compatible with $(V_k)_{k\in\mathbb{N}}$.*

Proof. Since T has a compact resolvent, $\sigma(T)$ consists of isolated eigenvalues only and $\varrho(T)$ is connected. The previous lemma thus implies that U is $(T - z)^{-1}$-invariant for all $z \in \varrho(T)$. Let P_k be the projections corresponding to the Riesz basis. Since $\sigma_k = \sigma(T|_{V_k})$ is an isolated part of the spectrum of T, P_k is the Riesz projection associated with σ_k:

$$P_k = \frac{i}{2\pi} \int_{\Gamma_k} (T - z)^{-1} dz \tag{16}$$

where Γ_k is a simply closed, positively oriented integration contour with σ_k in its interior and $\sigma(T) \setminus \sigma_k$ in its exterior, see e.g. [16, Theorem III.6.17]. Consequently $P_k(U) \subset U$, and U is thus compatible with (V_k). The T-invariance is now a consequence of Lemma 3.10. □

4 Hamiltonian operator matrices

The purpose of this section is to provide some basic facts about Hamiltonian operator matrices, in particular about their connection to Krein spaces. We will also derive existence results for finitely spectral Riesz bases of subspaces and for Riesz bases of generalised eigenvectors for the Hamiltonian.

We use the following definition of a Hamiltonian operator matrix, see also [4].

Definition 4.1 Let H be a Hilbert space. A *Hamiltonian operator matrix* is a block operator matrix

$$T = \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}, \qquad \mathcal{D}(T) = (\mathcal{D}(A) \cap \mathcal{D}(C)) \times (\mathcal{D}(A^*) \cap \mathcal{D}(B))$$

acting on $H \times H$ with densely defined linear operators A, B, C on H such that B and C are symmetric and T is densely defined.

If B and C are both nonnegative (positive, uniformly positive), then T is called a *nonnegative* (*positive, uniformly positive,* respectively) Hamiltonian operator matrix.[2]

Hamiltonian operator matrices are connected to two indefinite inner products on $H \times H$. We recall some corresponding notions and refer to [5, 6, 14] for more details: A vector space V together with an inner product $\langle\cdot|\cdot\rangle$ is called a *Krein*

[2]Note that the sign convention $T = \begin{pmatrix} A & -B \\ -C & -A^* \end{pmatrix}$, in particular with nonnegative B, C, is also used in the literature, e.g. in [19, 21].

space if V is also a Hilbert space with scalar product $(\cdot|\cdot)$ and there is a selfadjoint involution $J : V \to V$ such that $\langle x|y\rangle = (Jx|y)$ for all $x, y \in V$.

A subspace $U \subset V$ is called *J-neutral* if $\langle x|x\rangle = 0$ for all $x \in U$. The *J-orthogonal complement* of U is defined by

$$U^{\langle\perp\rangle} = \{x \in V \mid \langle x|y\rangle = 0 \text{ for all } y \in U\}.$$

Two subspaces $U, W \subset V$ are said to be *J-orthogonal*, $U\langle\perp\rangle W$, if $W \subset U^{\langle\perp\rangle}$. U is J-neutral if and only if $U \subset U^{\langle\perp\rangle}$. The subspace U is called *J-non-degenerate* if $U \cap U^{\langle\perp\rangle} = \{0\}$.

Let T be a densely defined linear operator on V. It is called *J-symmetric* if $\langle Tx|y\rangle = \langle x|Ty\rangle$ for all $x, y \in \mathcal{D}(T)$. The *J-adjoint* of T is defined as the maximal operator $T^{\langle *\rangle}$ such that

$$\langle Tx|y\rangle = \langle x|T^{\langle *\rangle}y\rangle \quad \text{for all} \quad x \in \mathcal{D}(T),\, y \in \mathcal{D}(T^{\langle *\rangle}).$$

The operator is called *J-selfadjoint* if $T = T^{\langle *\rangle}$, and in this case its spectrum $\sigma(T)$ is symmetric with respect to the real axis.

Consider the Krein space inner products on $H \times H$ given by

$$\langle x|y\rangle = (J_1x|y) \quad \text{with} \quad J_1 = \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}$$

and

$$[x|y] = (J_2x|y) \quad \text{with} \quad J_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Here $(\cdot|\cdot)$ denotes the usual scalar product on $H \times H$. The straightforward computation

$$\begin{aligned} \Big\langle \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \Big| \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \Big\rangle &= i(Au + Bv|\tilde{v}) - i(Cu - A^*v|\tilde{u}) \\ &= i(u|A^*\tilde{v} - C\tilde{u}) - i(v| - B\tilde{v} - A\tilde{u}) \\ &= \Big\langle \begin{pmatrix} u \\ v \end{pmatrix} \Big| \begin{pmatrix} -A & -B \\ -C & A^* \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \Big\rangle \end{aligned}$$

shows that T is *J_1-skew-symmetric*, i.e.

$$\langle Tx|y\rangle = -\langle x|Ty\rangle \quad \text{for all} \quad x, y \in \mathcal{D}(T).$$

As a consequence, T is always closable. In the following, additional assumptions on T such as in Theorem 4.6 or a lower resolvent bound 0 in Section 7 will often imply that T is already closed. From

$$\begin{aligned} \mathrm{Re}\Big[\begin{pmatrix} A & B \\ C & -A^* \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \Big| \begin{pmatrix} u \\ v \end{pmatrix}\Big] &= \mathrm{Re}\big((Au + Bv|v) + (Cu - A^*v|u)\big) \\ &= (Bv|v) + (Cu|u) \end{aligned}$$

we obtain that T is nonnegative if and only if it is *J_2-accretive*, i.e. $\mathrm{Re}[Tx|x] \geq 0$ for all $x \in \mathcal{D}(T)$.

The J_1-skew-symmetry of T yields symmetry properties of the spectrum:

Proposition 4.2 *Let T be a Hamiltonian operator matrix.*

(i) If $\lambda, \mu \in \sigma_p(T)$ with $\lambda \neq -\overline{\mu}$, then the root subspaces $\mathcal{L}(\lambda)$ and $\mathcal{L}(\mu)$ are J_1-orthogonal. In particular $\mathcal{L}(\lambda)$ is J_1-neutral for $\lambda \notin \sigma_p(T) \cap i\mathbb{R}$.

(ii) If T has a complete system of root subspaces, then $\sigma_p(T)$ is symmetric with respect to the imaginary axis, and $\mathcal{L}(\lambda) + \mathcal{L}(-\overline{\lambda})$ is J_1-non-degenerate with $\dim \mathcal{L}(\lambda) = \dim \mathcal{L}(-\overline{\lambda})$ for every $\lambda \in \sigma_p(T)$.

(iii) If there exists z such that $z, -\bar{z} \in \varrho(T)$, then T is J_1-skew-selfadjoint, i.e. $T = -T^{\langle * \rangle}$, and $\sigma(T)$ is symmetric with respect to the imaginary axis.

In particular, the point spectrum of a Hamiltonian with a finitely spectral Riesz basis of subspaces is symmetric with respect to the imaginary axis.

Proof of the proposition. (i): Since iT is J_1-symmetric, this is an immediate consequence of [6, Theorem II.3.3].

(ii): Let

$$\sigma_0 = \left\{ \lambda \in \mathbb{C} \;\middle|\; \operatorname{Re} \lambda \geq 0 \text{ and } \big(\lambda \in \sigma_p(T) \text{ or } -\overline{\lambda} \in \sigma_p(T)\big) \right\}$$

and $U_\lambda = \mathcal{L}(\lambda) + \mathcal{L}(-\overline{\lambda})$ for $\lambda \in \sigma_0$. From (i) it follows that the U_λ are pairwise J_1-orthogonal. For $x \in U_\lambda \cap U_\lambda^{\langle \perp \rangle}$ this implies that $\langle x|y \rangle = 0$ for all $y \in \sum_\mu U_\mu$. Since $\sum_\mu U_\mu \subset H \times H$ is dense by assumption, we obtain $\langle x|y \rangle = 0$ for all $y \in H \times H$ and thus $x = 0$: U_λ is J_1-non-degenerate. For $\lambda \in \sigma_0$ with $\operatorname{Re} \lambda > 0$, the subspaces $\mathcal{L}(\lambda)$ and $\mathcal{L}(-\overline{\lambda})$ are J_1-neutral and their sum is non-degenerate. This implies that $\dim \mathcal{L}(\lambda) = \dim \mathcal{L}(-\overline{\lambda})$, see [6, §I.10]. In particular $\lambda, -\overline{\lambda} \in \sigma_p(T)$ and hence the symmetry of $\sigma_p(T)$.

(iii): We have that iT is J_1-symmetric and $w, \overline{w} \in \varrho(iT)$ where $w = iz$. As in the Hilbert space situation this implies that iT is J_1-selfadjoint. Consequently, T is J_1-skew-selfadjoint. □

Remark 4.3 For any J-skew-selfadjoint operator T, $\sigma_p(T)$ is symmetric with respect to $i\mathbb{R}$ if and only if the residual spectrum $\sigma_r(T)$ is empty; this is an immediate consequence of [5, Theorem 2.1.16]. An example for a J_1-skew-selfadjoint Hamiltonian with $\sigma_r(T) \neq \varnothing$ and thus non-symmetric $\sigma_p(T)$ is constructed in [3].

The J_2-accretivity of a nonnegative Hamiltonian leads to characterisations of the spectrum on the imaginary axis:

Proposition 4.4 *Let T be a nonnegative Hamiltonian operator matrix.*

(i) We have $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ if and only if

$$\ker(A - it) \cap \ker C = \ker(A^* + it) \cap \ker B = \{0\} \quad \textit{for all} \quad t \in \mathbb{R}. \tag{17}$$

(ii) If T is uniformly positive with $B, C \geq \gamma$, then

$$\left\{ z \in \mathbb{C} \;\middle|\; |\operatorname{Re} z| < \gamma \right\} \subset r(T).$$

Proof. (i): We show that $(T - it)x = 0$ for $x = (u, v) \in \mathcal{D}(T)$ if and only if

$$u \in \ker(A - it) \cap \ker C \quad \text{and} \quad v \in \ker(A^* + it) \cap \ker B.$$

Indeed if $(T-it)x=0$, then

$$(A-it)u+Bv=0, \qquad Cu-(A^*+it)v=0, \qquad \text{and}$$
$$0=\operatorname{Re}(it[x|x])=\operatorname{Re}[Tx|x]=(Bv|v)+(Cu|u).$$

Since B and C are nonnegative, this yields $(Bv|v)=(Cu|u)=0$. Thus for all $r\in\mathbb{R}$ and $w\in\mathcal{D}(B)$,

$$0\le (B(rv+w)|rv+w)=2r\operatorname{Re}(Bv|w)+(w|w),$$

which implies $Bv=0$. Similarly $Cu=0$ and thus $(A-it)u=(A^*+it)v=0$. The other implication is immediate.

(ii): For $x=(u,v)\in\mathcal{D}(T)$ we have $\operatorname{Re}[Tx|x]=(Bv|v)+(Cu|u)\ge\gamma\|x\|^2$. Let $z\in\mathbb{C}\setminus r(T)$. Then there exists a sequence (x_n) in $\mathcal{D}(T)$ with $\|x_n\|=1$ and $(T-z)x_n\to 0$ as $n\to\infty$. For $\alpha_n=\operatorname{Re}[(T-z)x_n|x_n]$ this implies $\alpha_n\to 0$. We obtain

$$\begin{aligned}\gamma=\gamma\|x_n\|^2&\le\operatorname{Re}[Tx_n|x_n]=\alpha_n+\operatorname{Re}z\cdot[x_n|x_n]\\&\le|\alpha_n|+|\operatorname{Re}z|\,|(J_2x_n|x_n)|\le|\alpha_n|+|\operatorname{Re}z|\|x_n\|^2\to|\operatorname{Re}z|\end{aligned}$$

as $n\to\infty$, i.e. $\gamma\le|\operatorname{Re}z|$. □

We end this section with perturbation theorems for two classes of diagonally dominant Hamiltonians, which ensure the existence of finitely spectral Riesz bases of subspaces.

Definition 4.5 A Hamiltonian operator matrix is *diagonally dominant* if B is relatively bounded to A^* and C is relatively bounded to A.

Diagonally dominant block operator matrices are studied in [31, 32]. If the Hamiltonian T is diagonally dominant, then in particular

$$\mathcal{D}(A)\subset\mathcal{D}(C), \qquad \mathcal{D}(A^*)\subset\mathcal{D}(B). \tag{18}$$

If A is closed, these inclusions are even sufficient for diagonally dominance since B and C are closable; see [32, Remark 2.2.2].

Theorem 4.6 *Let T be a Hamiltonian operator matrix where A is normal with compact resolvent and B, C are p-subordinate to A with $0\le p<1$. If $\sigma(A)$ lies on finitely many rays from the origin and*

$$\liminf_{r\to\infty}\frac{N(r,A)}{r^{1-p}}<\infty, \tag{19}$$

then T has compact resolvent, is J_1-skew-selfadjoint, and there exists a finitely spectral Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$ for T.

Proof. This is an application of Theorem 3.9 to the decomposition

$$T=G+S \quad\text{with}\quad G=\begin{pmatrix}A&0\\0&-A^*\end{pmatrix}, \quad S=\begin{pmatrix}0&B\\C&0\end{pmatrix},$$

see [37, Theorem 7.2] for details. The skew-selfadjointness then follows by Proposition 4.2. □

Theorem 4.7 *Let T be a uniformly positive Hamiltonian such that A is skew-selfadjoint with compact resolvent, B, C are bounded and satisfy $B, C \geq \gamma$. Let ir_k be the eigenvalues of A where $(r_k)_{k\in\Lambda}$ is increasing and $\Lambda \in \{\mathbb{Z}_+, \mathbb{Z}_-, \mathbb{Z}\}$. Suppose that almost all eigenvalues ir_k are simple and that for some $l > b = \max\{\|B\|, \|C\|\}$ we have*

$$r_{k+1} - r_k \geq 2l \quad \text{for almost all } k \in \Lambda.$$

Then T has compact resolvent, almost all of its eigenvalues are simple,

$$\sigma(T) \subset \{z \in \mathbb{C} \,|\, \gamma \leq |\operatorname{Re} z| \leq b\},$$

and T admits a Riesz basis of eigenvectors and finitely many Jordan chains.

Proof. See [37, Theorem 7.3]. □

Remark 4.8 Due to [37, Remark 6.7], Theorem 4.6 continues to hold if A is an operator with compact resolvent and a Riesz basis of Jordan chains, B is p-subordinate to A^*, C is p-subordinate to A, $0 \leq p < 1$, almost all eigenvalues of A lie inside sets $\{e^{i\theta_j}(x+iy) \,|\, x > 0,\, |y| \leq \alpha x^p\}$ with $\alpha \geq 0$, $-\pi \leq \theta_j < \pi$, $j = 1, \dots, n$, and (19) is satisfied. Theorem 4.7 also holds if A has a compact resolvent, a Riesz basis of eigenvectors and finitely many Jordan chains, and almost all eigenvalues of A are simple, contained in a strip around the imaginary axis and their imaginary parts are uniformly separated. The constant b has to be adjusted then.

5 Invariant subspaces of Hamiltonians

In this section we construct invariant compatible subspaces of the Hamiltonian that are hypermaximal J_1-neutral. For a nonnegative Hamiltonian, the subspaces associated with the point spectrum in the open right and left half-plane are J_2-nonnegative and -nonpositive, respectively.

Let V be a Krein space. Recall that a subspace $U \subset V$ is J-neutral if and only if $U \subset U^{\langle\perp\rangle}$. It is called *hypermaximal J-neutral* if $U = U^{\langle\perp\rangle}$, see [5, 6]. It is not hard to see that if U, W are J-neutral subspaces with $V = U \oplus W$, then U and W are hypermaximal J-neutral. For $\dim V < \infty$, this is even an equivalence:

Lemma 5.1 *Let V be a finite-dimensional Krein space. If $U \subset V$ is hypermaximal J-neutral, then there exists a J-neutral subspace W such that $V = U \oplus W$.*

Proof. By induction on $n = \dim U$ we show that there exist systems $(e_1, \dots, e_n)$ in U and $(f_1, \dots, f_n)$ in V that form a *dual pair*, i.e. $\langle e_j | f_k\rangle = \delta_{jk}$, and are such that $W = \operatorname{span}\{f_1, \dots, f_n\}$ is neutral. Indeed, if $\dim U = n+1$ and $e \in U \setminus \operatorname{span}\{e_1, \dots, e_n\}$, we can set

$$e_{n+1} = e - \sum_{j=1}^{n} \langle e | f_j\rangle e_j.$$

Since V is non-degenerate, there exists $f \in V$ with $\langle e_{n+1} | f\rangle = 1$. Then

$$\widetilde{f} = f - \sum_{j=1}^{n} \langle f | e_j\rangle f_j - \sum_{j=1}^{n} \langle f | f_j\rangle e_j \quad \text{and} \quad f_{n+1} = \widetilde{f} - \frac{\langle \widetilde{f} | \widetilde{f}\rangle}{2} e_{n+1}$$

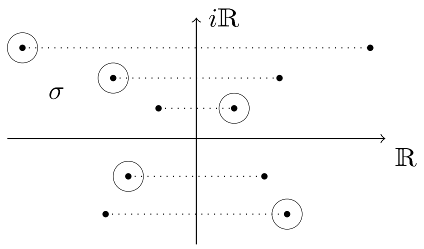

Figure 1: A point spectrum symmetric with respect to $i\mathbb{R}$ and a skew-conjugate subset σ.

yields the desired properties.

If $\sum_{j=1}^n \alpha_j e_j + \beta_j f_j = 0$, then we can take the inner product of this equation with the elements e_j, f_j and find $\alpha_j = \beta_j = 0$ for all j: $(e_1, \dots, e_n, f_1, \dots, f_n)$ is linearly independent. In particular $(e_1, \dots, e_n)$ is a basis of U and $U \cap W = \{0\}$. To show $V = U \oplus W$, let $x \in V$ and set $u = x - w$ where $w = \sum_{j=1}^n \langle x|e_j\rangle f_j \in W$. Then $\langle u|e_j\rangle = 0$ for all j, i.e. $u \in U^{\langle\perp\rangle} = U$. □

Definition 5.2 For a linear operator whose point spectrum $\sigma_p(T)$ is symmetric with respect to the imaginary axis, we say that a subset $\sigma \subset \sigma_p(T) \setminus i\mathbb{R}$ is *skew-conjugate* if

(i) $\lambda \in \sigma \Rightarrow -\overline{\lambda} \notin \sigma$ and

(ii) $\lambda \in \sigma_p(T) \setminus i\mathbb{R} \Rightarrow \lambda \in \sigma$ or $-\overline{\lambda} \in \sigma$.

In other words, σ contains one eigenvalue from each skew-conjugate pair $(\lambda, -\overline{\lambda})$ in $\sigma_p(T) \setminus i\mathbb{R}$, compare Figure 1.

Theorem 5.3 *Let T be a closed Hamiltonian operator matrix with a finitely spectral Riesz basis of subspaces. Then T possesses a hypermaximal J_1-neutral, T-invariant, compatible subspace if and only if for all $it \in \sigma_p(T) \cap i\mathbb{R}$ we have*

$$\mathcal{L}(it) = M_{it} \oplus N_{it} \quad \text{with} \quad M_{it}, N_{it}\ J_1\text{-neutral and } M_{it}\ T\text{-invariant.} \tag{20}$$

In this case, for every skew-conjugate subset $\sigma \subset \sigma_p(T) \setminus i\mathbb{R}$ the T-invariant compatible subspace

$$U = \overline{\sum_{\lambda\in\sigma} \mathcal{L}(\lambda) + \sum_{it\in\sigma_p(T)\cap i\mathbb{R}} M_{it}} \tag{21}$$

is hypermaximal J_1-neutral.

Proof. Let $(V_k)_{k\in\mathbb{N}}$ be a finitely spectral Riesz basis of subspaces for T and write $\sigma_k = \sigma(T|_{V_k})$. Suppose first that U is hypermaximal J_1-neutral, T-invariant, and compatible with (V_k). So U is of the form

$$U = \bigoplus_{k\in\mathbb{N}}^{2} U_k = \overline{\sum_{\lambda\in\sigma_p(T)} M_\lambda}$$

where the subspaces $U_k \subset V_k$ and $M_\lambda \subset \mathcal{L}(\lambda)$ are all T-invariant, compare Corollary 3.11. By Proposition 4.2, each $\mathcal{L}(it)$, $it \in \sigma_p(T) \cap i\mathbb{R}$, is J_1-non-degenerate and thus itself a Krein space. In view of the previous lemma it suffices to show that M_{it} is hypermaximal neutral with respect to $\mathcal{L}(it)$, i.e., $M_{it}^{\langle\perp\rangle} \cap \mathcal{L}(it) = M_{it}$.

Since $M_{it} \subset U$ we have that M_{it} is neutral and hence $M_{it} \subset M_{it}^{\langle\perp\rangle} \cap \mathcal{L}(it)$. Let $x \in M_{it}^{\langle\perp\rangle} \cap \mathcal{L}(it)$. Since $\mathcal{L}(it)$ is J_1-orthogonal to $\mathcal{L}(\lambda)$ for every $\lambda \neq it$, we see that $x \langle\perp\rangle M_\lambda$ for all λ and hence $x \in U^{\langle\perp\rangle} = U$. On the other hand $x \in \mathcal{L}(it) \subset V_{k_0}$ with k_0 such that $it \in \sigma_{k_0}$. Consequently $x \in U \cap V_{k_0} = U_{k_0}$. Now the decomposition

$$U_{k_0} = \bigoplus_{\lambda \in \sigma_{k_0}} M_\lambda$$

implies that $x \in U_{k_0} \cap \mathcal{L}(it) = M_{it}$.

For the other implication, suppose now that for every $it \in \sigma_p(T) \cap i\mathbb{R}$ there is a decomposition $\mathcal{L}(it) = M_{it} \oplus N_{it}$ into neutral subspaces where M_{it} is T-invariant, let $\sigma \subset \sigma_p(T) \setminus i\mathbb{R}$ be skew-conjugate, and let U be given by (21). Since U is the closure of the sum of neutral, pairwise orthogonal subspaces, U is neutral. Moreover, U is T-invariant and compatible with (V_k) with decomposition

$$U = \bigoplus_{k \in \mathbb{N}}^{2} U_k, \qquad U_k = \sum_{\lambda \in \sigma_k \cap \sigma} \mathcal{L}(\lambda) + \sum_{it \in \sigma_k \cap i\mathbb{R}} M_{it}.$$

It remains to show that $U^{\langle\perp\rangle} \subset U$. We have $V_k = U_k \oplus W_k$ with

$$W_k = \sum_{\lambda \in \tau_k} \mathcal{L}(\lambda) + \sum_{it \in \sigma_k \cap i\mathbb{R}} N_{it}, \qquad \tau_k = \sigma_k \setminus (\sigma \cup i\mathbb{R}).$$

Let $x \in U^{\langle\perp\rangle}$. We expand x in the Riesz basis (V_k) as $x = \sum_k (u_k + w_k)$ with $u_k \in U_k$, $w_k \in W_k$. To show that all w_k are zero, we consider now the subspaces

$$\widetilde{U}_k = \sum_{\lambda \in \tau_k} \mathcal{L}(-\overline{\lambda}) + \sum_{it \in \sigma_k \cap i\mathbb{R}} M_{it}.$$

The fact that σ is skew-conjugate yields $\lambda \in \tau_k \Rightarrow -\overline{\lambda} \in \sigma$, and therefore $\widetilde{U}_k \subset U$. Moreover $\widetilde{U}_k$ is J_1-orthogonal to W_j for $j \neq k$, and W_k is neutral. For $\tilde{u} \in \widetilde{U}_k$, $\tilde{w} \in W_k$ we thus compute

$$0 = \langle x|\tilde{u}\rangle = \sum_{j \in \mathbb{N}} \langle u_j + w_j|\tilde{u}\rangle = \langle w_k|\tilde{u}\rangle = \langle w_k|\tilde{u} + \tilde{w}\rangle.$$

In view of Proposition 4.2, $\widetilde{U}_k + W_k$ is non-degenerate since it is the orthogonal sum of subspaces $\mathcal{L}(\lambda) + \mathcal{L}(-\overline{\lambda})$, $\lambda \in \tau_k \cup (\sigma_k \cap i\mathbb{R})$. Consequently $w_k = 0$ for all k and hence $x = \sum_k u_k \in U$. □

Remark 5.4 Since all root subspaces of T are finite-dimensional, results about the Jordan structure of J-symmetric matrices (e.g. [20, Theorem 2.3.2]) may be used to reformulate condition (20): It turns out that (20) holds if and only if $\mathcal{L}(it) = M'_{it} \oplus N'_{it}$ with neutral subspaces M'_{it}, N'_{it}.

Now we consider the subspaces associated with $\sigma_p^\pm(T)$, the point spectrum of T in the open right and left half-plane, respectively.

Lemma 5.5 *Let T be a linear operator on a Banach space with $\sigma_p(T) \cap i\mathbb{R} = \varnothing$. Consider the algebraically direct decomposition*

$$\sum_{\lambda \in \sigma_p(T)} \mathcal{L}(\lambda) = W_+ \dotplus W_-, \qquad W_\pm = \sum_{\lambda \in \sigma_p^\pm(T)} \mathcal{L}(\lambda),$$

and the associated algebraic projections $P_\pm$ onto $W_\pm$. Then

$$\frac{1}{i\pi} \int_{i\mathbb{R}}' (T-z)^{-1} x \, dz = P_+ x - P_- x \qquad \text{for all} \quad x \in \sum_{\lambda \in \sigma_p(T)} \mathcal{L}(\lambda), \tag{22}$$

where the prime denotes the Cauchy principal value at infinity, that is $\int_{i\mathbb{R}}' f \, dz = \lim_{r\to\infty} \int_{-ir}^{ir} f \, dz$.

Note that the integrand in (22) is well-defined since $(T-z)^{-1}$ acts, for each x, on a finite sum of finite-dimensional subspaces generated by Jordan chains; $(T-z)^{-1}x$ is thus continuous in z.

Proof of the lemma. By linearity it suffices to consider $x \in \mathcal{L}(\lambda)$ and the Jordan chain generated by x. With respect to this Jordan chain, T is represented by the matrix

$$E_\lambda = \begin{pmatrix} \lambda & 1 & \\ & \ddots & \ddots \\ & & \lambda \end{pmatrix}, \tag{23}$$

and it suffices to show that

$$\int_{i\mathbb{R}}' (E_\lambda - z)^{-1} dz = \pm i\pi I$$

for $\operatorname{Re}\lambda \gtrless 0$. This is a straightforward calculation. □

Lemma 5.6 *Let T be an operator with a Riesz basis $(x_k)_{k\in\mathbb{N}}$ consisting of Jordan chains. If $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ and $\sigma_p(T)$ is contained in a strip around the imaginary axis, then*

$$\int_{-\infty}^{\infty} \|(T-it)^{-1}x\|^2 \, dt \;\geq\; c_0 \|x\|^2 \qquad \text{for} \quad x \in \operatorname{span}\{x_k \mid k \in \mathbb{N}\}$$

with some constant $c_0 > 0$.

Proof. Let $x \in \operatorname{span}\{x_k \mid k \in \mathbb{N}\}$. Then there is a finite system $F = (y_1, \dots, y_n) \subset (x_k)_{k\in\mathbb{N}}$ consisting of Jordan chains such that $x = \alpha_1 y_1 + \ldots + \alpha_n y_n$. $\operatorname{span} F$ is a T-invariant subspace with basis F. With respect to F, $(T-it)^{-1}$ is represented by a block diagonal matrix D with blocks of the form $(E_\lambda - it)^{-1}$, E_λ as in (23). Hence

$$(T-it)^{-1}x = \sum_{k=1}^n \alpha_k (T-it)^{-1} y_k = \sum_{j,k=1}^n \alpha_k D_{jk} y_j.$$

Let $m, M > 0$ be the constants from (3) for the Riesz basis (x_k). Putting $\xi = (\alpha_1, \dots, \alpha_n)$ and using the Euclidean norm on $\mathbb{C}^n$, we find

$$\|(T-it)^{-1}x\|^2 \geq m \sum_{j=1}^n \Big| \sum_{k=1}^n \alpha_k D_{jk} \Big|^2 = m \|D\xi\|^2.$$

Now $\|D\xi\|^2$ is the sum of terms of the form $\|(E_\lambda - it)^{-1}\nu\|^2$, one for each Jordan chain in F with ν the part of ξ corresponding to that Jordan chain. From

$$\|E_\lambda - it\| \le |\lambda - it| + \left\| \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0 \end{pmatrix} \right\| \le |\lambda - it| + 1$$

it follows that

$$\|(E_\lambda - it)^{-1}\nu\|^2 \ge \frac{1}{(|\lambda - it| + 1)^2}\|\nu\|^2.$$

With $u = \operatorname{Re}\lambda$, $v = \operatorname{Im}\lambda$, we calculate

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dt}{(|\lambda - it| + 1)^2} &\ge \int_{-\infty}^{\infty} \frac{dt}{2(|\lambda - it|^2 + 1)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dt}{1 + u^2 + (t - v)^2} \\ &= \frac{1}{2\sqrt{1+u^2}} \arctan\left(\frac{t - v}{\sqrt{1+u^2}}\right)\Big|_{t=-\infty}^{\infty} = \frac{\pi}{2\sqrt{1+u^2}}. \end{aligned}$$

Choosing $a > 0$ such that $|\operatorname{Re}\lambda| \le a$ for all $\lambda \in \sigma_p(T)$, we obtain

$$\int_{-\infty}^{\infty} \|(T - it)^{-1}x\|^2\, dt \ge m\frac{\pi}{2\sqrt{1+a^2}}\|\xi\|^2 \ge \frac{m\pi}{2M\sqrt{1+a^2}}\|x\|^2.$$

□

A subspace $U \subset V$ of a Krein space is called *J-nonnegative*, *J-positive* and *uniformly J-positive* if $\langle x|x\rangle \ge 0$, > 0 and $\ge \alpha\|x\|^2$, respectively, for all $x \in U\setminus\{0\}$, with some constant $\alpha > 0$. Nonpositive, negative and uniformly negative subspaces are defined accordingly.

A variant of the following result for dichotomous operators was obtained in [22, Theorem 1.4]. In [5, Corollary 2.2.22], the first assertion is proved for so-called W-dissipative operators.

Proposition 5.7 *Let T be a nonnegative Hamiltonian operator matrix and consider the subspaces*

$$U_\pm = \overline{\sum_{\lambda \in \sigma_p^\pm(T)} \mathcal{L}(\lambda)}.$$

(i) U_+ is J_2-nonnegative and U_- is J_2-nonpositive.

(ii) If in addition T is uniformly positive, has a Riesz basis of Jordan chains, and $\sigma_p(T)$ is contained in a strip around the imaginary axis, then $U_\pm$ is uniformly J_2-positive/-negative.

Proof. (i): To show that U_+ is J_2-nonnegative, it suffices to show that $[x|x] \ge 0$ for every $x = x_1 + \cdots + x_n$, $x_j \in \mathcal{L}(\lambda_j)$, $\lambda_j \in \sigma_p^+(T)$. Let

$$W_0 = \operatorname{span}\big\{(T - \lambda_j)^k x_j \,\big|\, j = 1, \ldots, n,\ k \in \mathbb{N}\big\}.$$

Then W_0 is finite-dimensional, T-invariant, $x \in W_0$, and the restriction $T_0 = T|_{W_0}$ satisfies $\sigma(T_0) \subset \{\lambda_1, \ldots, \lambda_n\}$. Lemma 5.5 and the J_2-accretivity of T imply

$$\begin{aligned} [x|x] = \operatorname{Re}[x|x] &= \frac{1}{\pi}\int_{\mathbb{R}}' \operatorname{Re}[(T_0 - it)^{-1}x|x]\, dt \\ &= \frac{1}{\pi}\int_{\mathbb{R}}' \operatorname{Re}[T_0(T_0 - it)^{-1}x|(T_0 - it)^{-1}x]\, dt \ge 0. \end{aligned}$$

A similar calculation shows that U_- is J_2-nonpositive.

(ii): We have $\mathrm{Re}[Tx|x] \ge \gamma\|x\|^2$ with some $\gamma > 0$ and $i\mathbb{R} \subset r(T)$. Let W_+ be the span of the Jordan chains from the Riesz basis which correspond to $\sigma_p^+(T)$. As a consequence of $0 \in r(T)$ we have $\overline{W_+} = U_+$. For $x \in W_+$, Lemmas 5.5 and 5.6 now yield

$$\begin{aligned}[x|x] &= \frac{1}{\pi}\int_{\mathbb{R}}' \mathrm{Re}[T(T-it)^{-1}x|(T-it)^{-1}x]\,dt \\ &\ge \frac{\gamma}{\pi}\int_{\mathbb{R}} \|(T-it)^{-1}x\|^2\,dt \ge \frac{\gamma c_0}{\pi}\|x\|^2.\end{aligned}$$

Consequently U_+ is uniformly J_2-positive. Again, a similar reasoning applies to U_-. □

Remark 5.8 Proposition 4.2 and Theorem 5.3 also hold for arbitrary (skew-) symmetric operators on Krein spaces since in the proofs the particular structure of the Hamiltonian as a block operator matrix was not used. Similarly, Proposition 5.7 holds for arbitrary (uniformly) accretive operators.

Lemma 5.9 *Let T be a nonnegative Hamiltonian operator matrix with*

$$C > 0 \quad \text{and} \quad \ker(A^* - \lambda) \cap \ker B = \{0\} \quad \text{for all} \quad \lambda \in \mathbb{C}. \tag{24}$$

Then the root subspaces $\mathcal{L}(\lambda)$ of T are J_2-positive for $\mathrm{Re}\,\lambda > 0$ and J_2-negative for $\mathrm{Re}\,\lambda < 0$.

Proof. Suppose that $\mathrm{Re}\,\lambda > 0$; the proof for $\mathrm{Re}\,\lambda < 0$ is analogous. From Proposition 5.7 we know that $\mathcal{L}(\lambda)$ is J_2-nonnegative. Let $x = (u,v) \in \mathcal{L}(\lambda) \setminus \{0\}$ and $n \in \mathbb{N}$ minimal such that $(T-\lambda)^n x = 0$. We use induction on n to show that $[x|x] \ne 0$ and thus $[x|x] > 0$.

For $n = 1$ we have

$$\mathrm{Re}\,\lambda \cdot [x|x] = \mathrm{Re}[Tx|x] = (Bv|v) + (Cu|u).$$

If $[x|x] = 0$, then $u = 0$ since B is nonnegative and C positive. Hence

$$Tx = \begin{pmatrix} Bv \\ -A^*v \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ v \end{pmatrix},$$

and (24) yields $v = 0$, a contradiction.

For $n > 1$ we set $y = (T-\lambda)x$; so $y \ne 0$ and thus $[y|y] > 0$ by the induction hypothesis. If $[x|x] = 0$, then

$$0 = \mathrm{Re}\,\lambda \cdot [x|x] = \mathrm{Re}[Tx|x] - \mathrm{Re}[y|x],$$

i.e.,

$$\mathrm{Re}[y|x] = \mathrm{Re}[Tx|x] = (Bv|v) + (Cu|u).$$

For $r \in \mathbb{R}$ let $w = rx + y \in \mathcal{L}(\lambda)$. Then $0 \le [w|w] = 2r\,\mathrm{Re}[y|x] + [y|y]$. Since r is arbitrary, this implies $\mathrm{Re}[y|x] = (Bv|v) + (Cu|u) = 0$. So again $u = 0$ and also $Bv = 0$, see the proof of Proposition 4.4. Consequently, the first component of y is zero and hence $[y|y] = 0$, again a contradiction. □

6 Solutions of the Riccati equation

In this section we consider diagonally dominant Hamiltonian operator matrices. We derive conditions that ensure the existence of (generally unbounded) solutions of the corresponding Riccati equation.

For a linear operator X on the Hilbert space H we consider the graph subspace

$$\Gamma(X) = \left\{ \begin{pmatrix} u \\ Xu \end{pmatrix} \,\middle|\, u \in \mathcal{D}(X) \right\}.$$

It is well known that invariant graph subspaces of block operator matrices are connected to solutions of Riccati equations. In particular, the notions of strong as well as weak solutions are considered. In [19] it was shown that these two notions are equivalent for bounded solutions corresponding to Hamiltonian operator matrices with bounded B, C. The equivalence for unbounded solutions corresponding to bounded selfadjoint block operator matrices was proved in [17]. We consider the situation where all involved operators are unbounded, see also Remark 7.7 and [36, §4.3].

Proposition 6.1 *Let T be a diagonally dominant Hamiltonian and X a linear operator on H.*

(i) $\Gamma(X)$ is T-invariant if and only if X satisfies the Riccati equation

$$X(Au + BXu) = Cu - A^*Xu \quad \text{for all} \quad u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*). \tag{25}$$

(In particular $Au + BXu \in \mathcal{D}(X)$ for $u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$.)

(ii) If T has a finitely spectral Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$ and $\Gamma(X)$ is compatible with $(V_k)_{k\in\mathbb{N}}$, then $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for X.

(iii) If X is selfadjoint and $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for X, then (25) holds if and only if the weak Riccati equation

$$(Xu|Av) + (Au|Xv) + (BXu|Xv) - (Cu|v) = 0 \tag{26}$$

holds for all $u, v \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$.

Proof. (i): $\Gamma(X)$ is T-invariant if and only if for all $u \in \mathcal{D}(A) \cap \mathcal{D}(X)$ with $Xu \in \mathcal{D}(A^*)$ there exists $v \in \mathcal{D}(X)$ such that

$$T\begin{pmatrix} u \\ Xu \end{pmatrix} = \begin{pmatrix} Au + BXu \\ Cu - A^*Xu \end{pmatrix} = \begin{pmatrix} v \\ Xv \end{pmatrix},$$

and this is obviously equivalent to (25).

(ii): By assumption, we have $\Gamma(X) = \bigoplus_k^2 U_k$ with $U_k \subset V_k \subset \mathcal{D}(T)$. Then $\sum_k U_k$ is dense in $\Gamma(X)$, and hence the subspace $D \subset H$ obtained by projecting $\sum_k U_k$ onto the first component is a core for X. Moreover $D \subset \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ since $\sum_k U_k \subset \mathcal{D}(T)$; hence $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for X.

(iii): Taking the scalar product of (25) with $v \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$, we immediately get (26). On the other hand, (26) can be rewritten as

$$(Au + BXu|Xv) = (Cu - A^*Xu|v).$$

Since $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for X, this equation holds for all $v \in \mathcal{D}(X)$. Consequently $Au + BXu \in \mathcal{D}(X^*) = \mathcal{D}(X)$ and (25) follows. $\square$

Graph subspaces are also naturally connected to the Krein space inner products considered in Section 4, see also [11].

Lemma 6.2 *Consider a linear operator X on the Hilbert space H.*

(i) X is Hermitian, i.e. $(Xu|v) = (u|Xv)$ for all $u, v \in \mathcal{D}(X)$, if and only if $\Gamma(X)$ is J_1-neutral.

(ii) X is selfadjoint if and only if $\Gamma(X)$ is hypermaximal J_1-neutral.

If X is Hermitian, then

(iii) X is nonnegative and nonpositive if and only if $\Gamma(X)$ is J_2-nonnegative and J_2-nonpositive, respectively;

(iv) X is bounded and uniformly positive (negative) if and only if $\Gamma(X)$ is uniformly J_2-negative (positive).

Proof. The assertions (i) and (iii) are immediate. For (ii) suppose $\Gamma(X)$ is hypermaximal J_1-neutral. If $w \in \mathcal{D}(X)^\perp$ then

$$\left\langle \begin{pmatrix} u \\ Xu \end{pmatrix} \,\middle|\, \begin{pmatrix} 0 \\ w \end{pmatrix} \right\rangle = i(u|w) = 0 \quad \text{for all } u \in \mathcal{D}(X).$$

Hence $(0, w) \in \Gamma(X)^{\langle\perp\rangle} = \Gamma(X)$ and so $w = 0$; X is densely defined. Since X is also Hermitian, it is thus symmetric, $X \subset X^*$. If now $v \in \mathcal{D}(X^*)$, then

$$\left\langle \begin{pmatrix} u \\ Xu \end{pmatrix} \,\middle|\, \begin{pmatrix} v \\ X^*v \end{pmatrix} \right\rangle = i(u|X^*v) - i(Xu|v) = 0 \quad \text{for all } u \in \mathcal{D}(X),$$

which implies $(v, X^*v) \in \Gamma(X)$ and so $v \in \mathcal{D}(X)$ and $X^*v = Xv$. X is thus selfadjoint. The converse implication in (ii) is proved similarly.

(iv): Let X be Hermitian and $\Gamma(X)$ uniformly J_2-positive. Then

$$2\|Xu\|\|u\| \geq 2(Xu|u) = \left[\begin{pmatrix} u \\ Xu \end{pmatrix} \,\middle|\, \begin{pmatrix} u \\ Xu \end{pmatrix} \right] \geq \alpha \left\| \begin{pmatrix} u \\ Xu \end{pmatrix} \right\|^2 = \alpha\|u\|^2 + \alpha\|Xu\|^2,$$

implies that $(Xu|u) \geq \frac{\alpha}{2}\|u\|^2$ and $\|Xu\| \leq \frac{2}{\alpha}\|u\|$. The proof of the other assertions is similar. □

Now we formulate our main existence result for solutions of the Riccati equation. Similar existence results have been obtained for bounded B and C: for Riesz-spectral Hamiltonians in [19] and for dichotomous ones in [21]. Here we allow B and C to be unbounded and consider finitely spectral Riesz bases of subspaces. This concept is more general than a Riesz-spectral operator and it also allows for non-dichotomous operators, see Remarks 3.4 and 3.13. Moreover, in [19] the conditions for the existence of solutions were formulated in terms of the eigenvectors of T while our conditions are solely on the operators A, B, C.

Theorem 6.3 *Let T be a diagonally dominant, nonnegative Hamiltonian operator matrix with $\varrho(T) \cap i\mathbb{R} \neq \varnothing$ and a finitely spectral Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$. Suppose that*

(a) B is positive, or

(b) there is a connected component M of $\varrho(A)$ such that $M \cap \varrho(T) \cap i\mathbb{R} \neq \varnothing$ and

$$\operatorname{span}\big\{(A-z)^{-1}B^*u \,\big|\, z \in M,\, u \in \mathcal{D}(B^*)\big\} \subset H \quad \text{is dense.} \tag{27}$$

Then the following holds:

(i) Every hypermaximal J_1-neutral, T-invariant, compatible subspace U is the graph $U = \Gamma(X)$ of a selfadjoint operator X satisfying the Riccati equation

$$X(Au + BXu) = Cu - A^*Xu, \qquad u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*), \tag{28}$$

and $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*)$ is a core for X.

(ii) If

$$\ker(A - it) \cap \ker C = \{0\} \quad \text{for all} \quad t \in \mathbb{R},$$

then $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ and for every skew-conjugate set $\sigma \subset \sigma_p(T)$ the associated invariant compatible subspace U_σ is hypermaximal J_1-neutral; thus $U_\sigma = \Gamma(X_\sigma)$ with a selfadjoint solution X_σ of (28). The solutions $X_\pm$ corresponding to $\sigma = \sigma_p^\pm(T)$ are nonnegative/nonpositive.

(iii) If C is even positive, then every X_σ is injective. In addition, $X_\pm$ is the uniquely determined nonnegative/nonpositive selfadjoint solution of (28) whose graph is compatible with $(V_k)_{k\in\mathbb{N}}$.

Before proving the theorem, we give some remarks on its conditions and assertions.

Remark 6.4 If $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ (and $\dim H = \infty$), then T has infinitely many skew-conjugate pairs of eigenvalues since $\sigma_p(T)$ is symmetric to $i\mathbb{R}$ and infinite. Hence there are infinitely many skew-conjugate sets $\sigma \subset \sigma_p(T)$ and infinitely many corresponding solutions X_σ.

Remark 6.5 There is the following symmetric version of the assertions in Theorem 6.3; it is immediate from the consideration of the transformed Hamiltonian

$$\widetilde{T} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & -A^* \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} -A^* & C \\ B & A \end{pmatrix} : \tag{29}$$

Suppose that C is positive or that there is a connected component M of $\varrho(A)$ such that $M \cap \varrho(T) \cap i\mathbb{R} \neq \varnothing$ and

$$\operatorname{span}\big\{(A^* - \bar{z})^{-1}C^*v \,\big|\, z \in M,\, v \in \mathcal{D}(C^*)\big\} \subset H \quad \text{is dense.} \tag{30}$$

Then a hypermaximal J_1-neutral, T-invariant, compatible subspace U is the "inverse" graph

$$U = \Gamma_{\mathrm{inv}}(Y) = \left\{ \begin{pmatrix} Yv \\ v \end{pmatrix} \,\middle|\, v \in \mathcal{D}(Y) \right\}$$

of a selfadjoint operator Y such that

$$Y(CYv - A^*v) = AYv + Bv, \qquad v \in \mathcal{D}(A^*) \cap Y^{-1}\mathcal{D}(A), \tag{31}$$

and $\mathcal{D}(A^*) \cap Y^{-1}\mathcal{D}(A)$ is a core for Y. If in addition $\ker(A^* - it) \cap \ker B = \{0\}$ for all $t \in \mathbb{R}$, then $\sigma_p(T) \cap i\mathbb{R} = \varnothing$ and every skew-conjugate set $\sigma \subset \sigma_p(T)$ yields a selfadjoint solution Y_σ of (31). The solutions $Y_\pm$ corresponding to $\sigma_p^\pm(T)$ are nonnegative/nonpositive. If B is positive, then every Y_σ is injective and the $Y_\pm$ are uniquely determined.

For bounded B, C, conditions analogous to (27) and (30) have been used in [21]. In that setting, they are equivalent to the approximate controllability of the pair (A, B) and the approximate observability of (A, C), respectively. Here we use the following relation:

Proposition 6.6 *Let A, B be densely defined operators on a Hilbert space H and $M \subset \varrho(A)$. Then for the assertions*

(i) $\operatorname{span}\bigl\{(A-z)^{-1}B^*v \bigm| z \in M, v \in \mathcal{D}(B^*)\bigr\} \subset H$ *dense,*

(ii) $\ker(A^* - \lambda) \cap \ker B = \{0\}$ *for all* $\lambda \in \mathbb{C}$,

we have the implication (i) $\Rightarrow$ (ii). If A is normal with compact resolvent, B is closable, $\mathcal{D}(A) \subset \mathcal{D}(B)$, and M has an accumulation point in $\varrho(A)$, then (i) $\Leftrightarrow$ (ii).

Proof. For (i)$\Rightarrow$(ii) consider $A^*u = \lambda u$, $Bu = 0$. Then

$$((A-z)^{-1}B^*v|u) = (v|B(A^* - \bar{z})^{-1}u) = (v|(\lambda - \bar{z})^{-1}Bu) = 0$$

for every $z \in M$, $v \in \mathcal{D}(B^*)$ and (i) implies $u = 0$.

Now let A be normal with compact resolvent. Let $(\lambda_k)_{k\in\mathbb{N}}$ be the eigenvalues of A and P_k the corresponding orthogonal projections onto the eigenspaces. To prove (i), let $u \in H$ be such that $((A-z)^{-1}B^*v|u) = 0$ for all $z \in M$, $v \in \mathcal{D}(B^*)$; we aim to show $u = 0$. The function

$$f(z) = \bigl((A-z)^{-1}B^*v\big|u\bigr) = \sum_{k=0}^{\infty} \frac{1}{\lambda_k - z}(P_kB^*v|u)$$

is holomorphic on $\varrho(A)$ and vanishes on M; hence $f = 0$ by the identity theorem. If we integrate the series along a circle in $\varrho(A)$ enclosing exactly one λ_k, we obtain

$$0 = (P_kB^*v|u) = (B^*v|P_ku) \quad \text{for all} \quad v \in \mathcal{D}(B^*),$$

i.e. $P_ku \in \mathcal{R}(B^*)^\perp = \ker \overline{B}$. Since $P_ku \in \mathcal{D}(A) \subset \mathcal{D}(B)$, we have in fact $P_ku \in \ker B$. Since the eigenspaces of A and A^* coincide, (ii) now implies $P_ku = 0$ for all $k \in \mathbb{N}$ and thus $u = 0$. □

Proof of Theorem 6.3. (i): In view of Proposition 6.1 and Lemma 6.2, we only need to show that U is a graph subspace. For this it is sufficient that $(0, w) \in U$ implies $w = 0$. Suppose that (a) holds, let $it \in \varrho(T)$, $t \in \mathbb{R}$, and set $(u, v) = (T - it)^{-1}(0, w)$. Then

$$(A - it)u + Bv = 0, \quad Cu - (A^* + it)v = w.$$

Since U is J_1-neutral and invariant under $(T - it)^{-1}$, this implies

$$0 = \left\langle \begin{pmatrix} 0 \\ w \end{pmatrix} \middle| \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = -i(w|u)$$

and thus

$$0 = (w|u) = (Cu|u) - (v|(A - it)u) = (Cu|u) + (Bv|v).$$

Since B is positive and C nonnegative, we obtain $v = 0$ and, with the reasoning from the proof of Proposition 4.4, $Cu = 0$. Hence $w = 0$.

In the case of (b), for $it \in M \cap \varrho(T) \cap i\mathbb{R}$ we consider u, v as above and obtain now $Cu = Bv = 0$. Since $it \in \varrho(A)$, we have $-it \in \varrho(A^*)$. For $\tilde{u} \in \mathcal{D}(B^*)$ we get

$$\big((A^* + it)^{-1}w \big| B^*\tilde{u}\big) = -(v|B^*\tilde{u}) = -(Bv|\tilde{u}) = 0.$$

Consequently, the function $f(z) = ((A^* - \bar{z})^{-1}w|B^*\tilde{u})$, which is holomorphic on M, vanishes on $M \cap \varrho(T) \cap i\mathbb{R}$. From the identity theorem we thus obtain

$$0 = \big((A^* - \bar{z})^{-1}w \big| B^*\tilde{u}\big) = \big(w \,\big|\, (A - z)^{-1}B^*\tilde{u}\big) \quad \text{for all } z \in M,$$

and (27) now implies $w = 0$.

(ii): The assertion that T has no eigenvalues on $i\mathbb{R}$ is immediate from Proposition 4.4 and 6.6. Condition (20) in Theorem 5.3 is now trivially satisfied and hence U_σ is hypermaximal J_1-neutral. The subspace $U_\pm$ associated with $\sigma_p^\pm(T)$ is J_2-nonnegative/-nonpositive by Proposition 5.7 and hence $X_\pm$ is nonnegative/nonpositive by Lemma 6.2.

(iii): If $C > 0$, then $U_\sigma = \Gamma_{\mathrm{inv}}(Y_\sigma)$ by Remark 6.5. Hence X_σ is injective with $X_\sigma^{-1} = Y_\sigma$. Now let X be nonnegative selfadjoint and $\Gamma(X) = \bigoplus_k^2 U_k$ with $U_k \subset V_k$ T-invariant. Then each U_k is J_2-nonnegative and the span of certain root vectors of T. By Proposition 6.6, Lemma 5.9 can be applied and yields that U_k is the span of root vectors corresponding to eigenvalues in the right half-plane. Therefore $U_k \subset U_+$ and hence $\Gamma(X) \subset U_+$. Consequently $X \subset X_+$ and thus $X = X_+$ since both operators are selfadjoint. The proof of the uniqueness of X_- is analogous. □

Under the additional assumptions of Theorem 4.6, which yields a finitely spectral Riesz basis of subspaces for T, condition (27) for the existence of solutions of the Riccati equation simplifies:

Theorem 6.7 *Let T be a nonnegative Hamiltonian operator matrix such that A is normal with compact resolvent and B, C are p-subordinate to A with $0 \le p < 1$. Suppose that $\sigma(A)$ lies on finitely many rays from the origin, that*

$$\liminf_{r\to\infty} \frac{N(r, A)}{r^{1-p}} < \infty,$$

and that

$$\begin{aligned} \ker(A - \lambda) \cap \ker B &= \{0\} \quad \textit{for all} \quad \lambda \in \mathbb{C}, \\ \ker(A - it) \cap \ker C &= \{0\} \quad \textit{for all} \quad t \in \mathbb{R}. \end{aligned}$$

Then T has compact resolvent, a finitely spectral Riesz basis of subspaces, and $\sigma(T) \cap i\mathbb{R} = \varnothing$. For every skew-conjugate set $\sigma \subset \sigma(T)$ the associated T-invariant compatible subspace U_σ is the graph $U_\sigma = \Gamma(X_\sigma)$ of a selfadjoint solution X_σ of the Riccati equation

$$X_\sigma(Au + BX_\sigma u) = Cu - A^*X_\sigma u, \qquad u \in \mathcal{D}(A) \cap X_\sigma^{-1}\mathcal{D}(A^*), \tag{32}$$

and $\mathcal{D}(A) \cap X_\sigma^{-1}\mathcal{D}(A^*)$ is a core for X_σ. The solutions $X_\pm$ corresponding to $\sigma = \sigma_p^\pm(T)$ are nonnegative/nonpositive.

Proof. This follows from Theorem 4.6, Theorem 6.3 and Proposition 6.6. Note that $\ker(A - \lambda) = \ker(A^* - \lambda)$ since A is normal. □

7 Bounded solutions of the Riccati equation

In this section we prove the existence of bounded solutions of the Riccati equation and also obtain relations between those solutions. We make use of the following quantity associated with relatively bounded perturbations:

Definition 7.1 Let G, S be linear operators on a Banach or Hilbert space such that S is relatively bounded to G. We call

$$c_0 = \inf_{z \in \varrho(G)} \|S(G-z)^{-1}\|$$

the *lower resolvent bound* of S with respect to G.

Note that if $c_0 < 1$, then there exists $z \in \varrho(G)$ with $\|S(G-z)^{-1}\| < 1$, and a Neumann series argument implies that $z \in \varrho(G+S)$ and

$$(G+S-z)^{-1} = (G-z)^{-1}\left(I + S(G-z)^{-1}\right)^{-1}.$$

Recall that a linear operator G on a Hilbert space is called *accretive* if $\operatorname{Re}(Gx|x) \geq 0$ for all $x \in \mathcal{D}(G)$ and *m-accretive* if in addition $z \in \varrho(G)$ for one (and hence for all) z with $\operatorname{Re} z < 0$, see [16, §V.3.10].

Lemma 7.2 *Let S be relatively bounded to G with G-bound b_0 and lower resolvent bound c_0. Then:*

(i) $b_0 \leq c_0$.

(ii) If G is selfadjoint or m-accretive on a Hilbert space, then $b_0 = c_0$.

(iii) If there is a sequence (z_k) in $\varrho(G)$ and a constant $M \in \mathbb{R}$ such that

$$\lim_{k\to\infty} |z_k| = \infty \qquad \text{and} \qquad \|(G-z_k)^{-1}\| \leq \frac{M}{|z_k|},$$

then $b_0 = 0 \Leftrightarrow c_0 = 0$.

Proof. (i) follows from the estimate

$$\|Sx\| \leq \|S(G-z)^{-1}\|\|(G-z)x\| \leq \|S(G-z)^{-1}\|\bigl(\|Gx\| + |z|\|x\|\bigr).$$

(ii): If G is selfadjoint, then $\|(G-ir)^{-1}\| \leq |r|^{-1}$ and $\|G(G-ir)^{-1}\| \leq 1$ for $r \in \mathbb{R}\setminus\{0\}$. Hence for every $b > b_0$ there exists $a \in \mathbb{R}$ such that

$$\|S(G-ir)^{-1}\| \leq a\|(G-ir)^{-1}\| + b\|G(G-ir)^{-1}\| \leq \frac{a}{|r|} + b$$

for every $r \in \mathbb{R}\setminus\{0\}$; consequently $c_0 \leq b_0$. If G is m-accretive, then $\|(G+r)^{-1}\| \leq r^{-1}$ and $\|G(G+r)^{-1}\| \leq 1$ for $r > 0$. Hence $\|S(G+r)^{-1}\| \leq ar^{-1} + b$.

(iii): Let $b_0 = 0$. The assumptions yield

$$\|G(G-z_k)^{-1}\| = \|I + z_k(G-z_k)^{-1}\| \leq 1 + M$$

and

$$\|S(G-z_k)^{-1}\| \leq a\|(G-z_k)^{-1}\| + b\|G(G-z_k)^{-1}\| \leq \frac{aM}{|z_k|} + b(1+M),$$

where $b > 0$ can be chosen arbitrarily small. Hence $c_0 = 0$. □

Remark 7.3 For selfadjoint G, the above proof shows that in fact

$$\lim_{r\to\pm\infty} \|S(G-ir)^{-1}\| = b_0.$$

This result is already contained in the book of Weidmann [34, Satz 9.1]. If G is m-accretive, then

$$\lim_{r\to\infty} \|S(G+r)^{-1}\| = b_0.$$

For general G, the inequality $b_0 \le c_0$ may be strict: Consider for example some $b_0 > 0$ and normal operators G, S with a common orthonormal basis of eigenvectors (e_k), $Ge_k = \lambda_k e_k$, $\sigma_p(G) = \mathbb{Z} \cup i\mathbb{Z}$, $Se_k = b_0 e_k$ for $\lambda_k = 0$ and $Se_k = b_0\lambda_k e_k$ for $\lambda_k \neq 0$. Then straightforward computations show that b_0 is the G-bound of S and $c_0 \ge b_0\sqrt{8/5}$.

Definition 7.4 For a diagonally dominant Hamiltonian T we consider the decomposition

$$T = G + S, \quad G = \begin{pmatrix} A & 0 \\ 0 & -A^* \end{pmatrix}, \quad S = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}. \tag{33}$$

We define the *lower resolvent bound* of T to be the lower resolvent bound of S with respect to G.

As seen above, if T has lower resolvent bound < 1, then $\varrho(T) \neq \varnothing$. Moreover, the Hamiltonians from Theorem 4.6 and 4.7 have lower resolvent bound 0, which is a consequence of Lemma 7.2(iii).

Proposition 7.5 *Let T be a diagonally dominant Hamiltonian with lower resolvent bound 0. Let $X : H \to H$ be bounded such that $\Gamma(X)$ is T- and $(T-z)^{-1}$-invariant for all $z \in \varrho(T)$. Then $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and*

$$A^*Xu + XAu + XBXu - Cu = 0, \qquad u \in \mathcal{D}(A). \tag{34}$$

Moreover

$$\sigma(A+BX) = \sigma(T|_{\Gamma(X)}), \quad \sigma_p(A+BX) = \sigma_p(T|_{\Gamma(X)}),$$

and for every $\lambda \in \sigma_p(A+BX)$ the root subspace of $A+BX$ corresponding to λ is the projection onto the first component of the root subspace of $T|_{\Gamma(X)}$ corresponding to λ.

Proof. We consider the isomorphism φ and the projection pr_1 given by

$$\begin{aligned} \varphi : H &\to \Gamma(X), \\ u &\mapsto (u, Xu), \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathrm{pr}_1 : H \times H &\to H, \\ (u,v) &\mapsto u. \end{aligned}$$

Hence $\varphi^{-1} = \mathrm{pr}_1|_{\Gamma(X)}$. Using the decomposition (33) and writing $E = \varphi^{-1}T|_{\Gamma(X)}\varphi$ and $F = \mathrm{pr}_1 S\varphi$, we have

$$E - F = \mathrm{pr}_1 T\varphi - \mathrm{pr}_1 S\varphi = \mathrm{pr}_1 G\varphi = A|_{\mathcal{D}(A)\cap X^{-1}\mathcal{D}(A^*)}.$$

The $(T-z)^{-1}$-invariance of $\Gamma(X)$ implies that $\varphi^{-1}(T-z)^{-1}\varphi = (E-z)^{-1}$ for $z \in \varrho(T)$. Since T has lower resolvent bound 0, we can now use a Neumann series argument to find $z \in \varrho(G) \cap \varrho(T)$ such that

$$\begin{aligned} F(E-z)^{-1} &= \mathrm{pr}_1 S\varphi \circ \varphi^{-1}(T-z)^{-1}\varphi = \mathrm{pr}_1 S(T-z)^{-1}\varphi \\ &= \mathrm{pr}_1 S(G-z)^{-1}\big(I + S(G-z)^{-1}\big)^{-1}\varphi \end{aligned}$$

and $\|F(E-z)^{-1}\| < 1$. Consequently $z \in \varrho(E-F) = \varrho(A|_{\mathcal{D}(A)\cap X^{-1}\mathcal{D}(A^*)})$. Since also $z \in \varrho(A)$, we obtain $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A^*) = \mathcal{D}(A)$, i.e. $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$. The Riccati equation (34) then follows from (25). Moreover, we have

$$\varphi^{-1}T|_{\Gamma(X)}\varphi = A + BX,$$

which immediately implies the equality of the spectra and point spectra of $T|_{\Gamma(X)}$ and $A + BX$, and that φ maps the root subspaces of $A + BX$ bijectively onto the corresponding ones of $T|_{\Gamma(X)}$. □

In [19, Theorem 5.6], a correspondence between bounded solutions of the Riccati equation and eigenvectors of Riesz-spectral Hamiltonians is derived. We obtain:

Theorem 7.6 *Let T be a diagonally dominant Hamiltonian with lower resolvent bound* 0, *compact resolvent and a finitely spectral Riesz basis of subspaces $(V_k)_{k\in\mathbb{N}}$. Let $X : H \to H$ be bounded. Then $\Gamma(X)$ is T-invariant and compatible with $(V_k)_{k\in\mathbb{N}}$ if and only if $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and X is a solution of the Riccati equation*

$$A^*Xu + XAu + XBXu - Cu = 0, \qquad u \in \mathcal{D}(A). \tag{35}$$

Proof. If $\Gamma(X)$ is invariant and compatible, then the assertion follows from Proposition 7.5. So suppose that $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and that (35) holds. In view of Proposition 3.15 it suffices to find $z \in \varrho(T)$ such that $\Gamma(X)$ is $(T-z)^{-1}$-invariant. Let φ and pr_1 be as above. Let $z \in \varrho(G)$, in particular $z \in \varrho(A)$. Since $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$, we have

$$A - z = \mathrm{pr}_1(G-z)\varphi.$$

Set $W = (G-z)\varphi(\mathcal{D}(A))$. Then pr_1 maps W bijectively onto H and we have

$$(A-z)^{-1} = \varphi^{-1}(G-z)^{-1}(\mathrm{pr}_1|_W)^{-1}.$$

We want to show that W is closed. Let $x_n \in W$ with $x_n \to x$ as $n \to \infty$ and set $y_n = (G-z)^{-1}x_n$. Then $y_n \to (G-z)^{-1}x$ as well as

$$y_n = \varphi(A-z)^{-1}\mathrm{pr}_1 x_n \to \varphi(A-z)^{-1}\mathrm{pr}_1 x.$$

Consequently $(G-z)^{-1}x = \varphi(A-z)^{-1}\mathrm{pr}_1 x$ and hence $x \in W$. The open mapping theorem now implies that $(\mathrm{pr}_1|_W)^{-1}$ is bounded. Since

$$BX(A-z)^{-1} = \mathrm{pr}_1 S\varphi \circ \varphi^{-1}(G-z)^{-1}(\mathrm{pr}_1|_W)^{-1} = \mathrm{pr}_1 S(G-z)^{-1}(\mathrm{pr}_1|_W)^{-1}$$

and since T has lower resolvent bound 0, we can find $z \in \varrho(G) \cap \varrho(T)$ such that $\|BX(A-z)^{-1}\| < 1$, which in turn yields $z \in \varrho(A+BX)$. Since (35) holds, $\Gamma(X)$ is T-invariant and $\varphi^{-1}T|_{\Gamma(X)}\varphi = A+BX$; in particular $\varrho(T|_{\Gamma(X)}) = \varrho(A+BX)$. We end up with $z \in \varrho(T) \cap \varrho(T|_{\Gamma(X)})$, which implies that $\Gamma(X)$ is $(T-z)^{-1}$-invariant. □

Remark 7.7 Let X be bounded and selfadjoint. Then $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and

$$A^*Xu + XAu + XBXu - Cu = 0, \qquad u \in \mathcal{D}(A),$$

if and only if $X\mathcal{D}(A) \subset \mathcal{D}(B)$ and

$$(Xu|Av) + (Au|Xv) + (BXu|Xv) - (Cu|v) = 0, \quad u, v \in \mathcal{D}(A).$$

Indeed, the second equation implies that $(Xu|Av)$ is bounded in v; hence $Xu \in \mathcal{D}(A^*)$ and the first equation follows.

Lemma 7.8 *Let X_+, X_- be bounded selfadjoint operators on a Hilbert space H with X_+ uniformly positive and X_- nonpositive. If X is a Hermitian operator on H satisfying $\mathcal{D}(X) = D_+ \dot{+} D_-$, $X|_{D_\pm} = X_\pm|_{D_\pm}$, then X is bounded.*

Proof. First consider $u \in D_+$, $v \in D_-$ with $\|u\| = \|v\| = 1$. Then

$$\begin{aligned}\operatorname{Re}(u - v|X_+u + X_-v) &= \operatorname{Re}\big((u|X_+u) - (v|Xu) + (u|Xv) - (v|X_-v)\big)\\ &= (u|X_+u) - (v|X_-v) \geq \gamma\end{aligned}$$

where $X_+ \geq \gamma > 0$. Hence

$$\gamma \leq |(u - v|X_+u + X_-v)| \leq \|u - v\| \cdot \big(\|X_+\| + \|X_-\|\big).$$

This implies

$$1 - \operatorname{Re}(u|v) = \frac{1}{2}\|u - v\|^2 \geq \delta \quad \text{with} \quad \delta = \frac{1}{2}\left(\frac{\gamma}{\|X_+\| + \|X_-\|}\right)^2 > 0.$$

Consequently

$$|(u|v)| \leq 1 - \delta \quad \text{for all} \quad u \in D_+, v \in D_- \text{ with } \|u\| = \|v\| = 1.$$

Now for arbitrary $u \in D_+$, $v \in D_-$ we have the estimates

$$\begin{aligned}&\|X(u+v)\| = \|X_+u + X_-v\| \leq \max\{\|X_+\|, \|X_-\|\}\big(\|u\| + \|v\|\big),\\ &\big(\|u\| + \|v\|\big)^2 \leq 2\big(\|u\|^2 + \|v\|^2\big),\\ &\|u+v\|^2 \geq \|u\|^2 + \|v\|^2 - 2|(u|v)| \geq \|u\|^2 + \|v\|^2 - 2(1-\delta)\|u\|\|v\|\\ &\qquad\quad \geq \|u\|^2 + \|v\|^2 - (1-\delta)\big(\|u\|^2 + \|v\|^2\big) = \delta\big(\|u\|^2 + \|v\|^2\big).\end{aligned}$$

Therefore

$$\|X(u+v)\| \leq \sqrt{\frac{2}{\delta}}\max\big\{\|X_+\|, \|X_-\|\big\}\|u+v\|,$$

X is bounded. □

For uniformly positive Hamiltonians we can now prove the existence of bounded solutions of the Riccati equation and also obtain relations between them. The existence of bounded solutions was shown in [21] for the case that $\sigma(A)$ lies in a sector in the open left half-plane; here $\sigma(A)$ may also have points in the closed right half-plane. The relations (37) and (39) were derived in [8, 26] under the assumption of the existence of X_-. On the other hand, no uniform positivity was needed there.

Recall from Proposition 4.4 and (12) that a closed uniformly positive Hamiltonian with a finitely spectral Riesz basis of subspaces satisfies $\{z \in \mathbb{C} \,|\, |\operatorname{Re} z| < \gamma\} \subset \varrho(T)$ for some $\gamma > 0$.

Theorem 7.9 *Let T be a uniformly positive, diagonally dominant Hamiltonian with lower resolvent bound 0 and a Riesz basis of Jordan chains. Suppose that all eigenvalues of T have finite multiplicity and that they are contained in a strip around $i\mathbb{R}$.*

(i) If U is a hypermaximal J_1-neutral, T-invariant, compatible subspace, then $U = \Gamma(X)$ where X is bounded, selfadjoint, boundedly invertible, $X\mathcal{D}(A) = \mathcal{D}(A^*)$, and X is a solution of the Riccati equation

$$A^*Xu + XAu + XBXu - Cu = 0, \qquad u \in \mathcal{D}(A). \tag{36}$$

Moreover, the solutions $X_\pm$ corresponding to the compatible subspaces $U_\pm$ associated with $\sigma_p^\pm(T)$ are uniformly positive/negative and

$$X_- \le X \le X_+, \qquad X_-^{-1} \le X^{-1} \le X_+^{-1}. \tag{37}$$

(ii) If X is a closed symmetric operator satisfying $\mathcal{D}(A) \subset X^{-1}\mathcal{D}(B)$, and

$$(Xu|Av) + (Au|Xv) + (BXu|Xv) - (Cu|v) = 0, \quad u, v \in \mathcal{D}(A), \tag{38}$$

then X is bounded, $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and (36) and the first inequality in (37) hold. If in addition T has a compact resolvent, then $\Gamma(X)$ is hypermaximal J_1-neutral, T-invariant and compatible, and hence all conclusions of (i) hold.

(iii) If X is bounded and $\Gamma(X)$ is T-invariant and compatible, then there exists a projection P such that

$$X = X_+P + X_-(I - P). \tag{39}$$

Proof. (i): Let $(\lambda_k)_{k\in\mathbb{N}}$ be the eigenvalues of T. In view of Proposition 3.3, the root subspaces $\mathcal{L}(\lambda_k)$ of T form a finitely spectral Riesz basis. Theorem 6.3 and Remark 6.5 thus yield that U is a graph $U = \Gamma(X)$ with X selfadjoint and injective. In particular $U_\pm = \Gamma(X_\pm)$ where $X_\pm$ is also bounded and uniformly positive/negative by Proposition 5.7 and Lemma 6.2. We have $\Gamma(X) = \bigoplus^2_{k\in\mathbb{N}} U_k$ with T-invariant subspaces $U_k \subset \mathcal{L}(\lambda_k)$. Hence

$$\Gamma(X) = W_+ \oplus W_- \quad \text{with} \quad W_+ = \bigoplus^2_{\operatorname{Re}\lambda_k>0} U_k, \quad W_- = \bigoplus^2_{\operatorname{Re}\lambda_k<0} U_k, \tag{40}$$

and $W_\pm \subset \Gamma(X_\pm)$. Setting $D_\pm = \mathrm{pr}_1(W_\pm)$ where pr_1 is the projection onto the first component, we get $\mathcal{D}(X) = D_+ \dot{+} D_-$, $X|_{D_\pm} = X_\pm|_{D_\pm}$, and Lemma 7.8 implies that X is bounded. From Proposition 7.5 we thus obtain $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and (36). Then (38) holds too, and the first inequality in (37) will be a consequence of (ii). As $\Gamma(X_{(\pm)}) = \Gamma_{\mathrm{inv}}(X_{(\pm)}^{-1})$, the above reasoning applied to the Hamiltonian $\widetilde{T}$ from (29) yields the boundedness of X^{-1}, $X^{-1}\mathcal{D}(A^*) \subset \mathcal{D}(A)$ (hence $X\mathcal{D}(A) = \mathcal{D}(A^*)$), and the second inequality in (37).

(ii): Since equation (38) holds for X_+, we have

$$\begin{aligned} 0 &= (Au|(X_+ - X)u) + ((X_+ - X)u|Au) + (BX_+u|X_+u) - (BXu|Xu) \\ &= ((A + BX_+)u|(X_+ - X)u) + ((X_+ - X)u|(A + BX_+)u) \\ &\qquad\qquad\qquad\qquad - (B(X_+ - X)u|(X_+ - X)u) \end{aligned}$$

for $u \in \mathcal{D}(A)$. With $\Delta = X_+ - X$ and $t \in \mathbb{R}$ we obtain

$$2\operatorname{Re}\big((A + BX_+ - it)u\big|\Delta u\big) = (B\Delta u|\Delta u) \ge 0.$$

Proposition 7.5 implies that $i\mathbb{R} \subset \varrho(A + BX_+)$, that $\sigma_p(A + BX_+)$ is contained in the right half-plane, and that the system of root subspaces (L_λ) of $A + BX_+$ is complete in H. Then

$$\operatorname{Re}\bigl(v\big|\Delta(A + BX_+ - it)^{-1}v\bigr) \geq 0 \quad \text{for} \quad v \in H,$$

and Lemma 5.5 yields

$$(\Delta v|v) = \frac{1}{\pi}\int_{\mathbb{R}}' \operatorname{Re}\bigl(\Delta v\big|(A + BX_+ - it)^{-1}v\bigr)\,dt \geq 0 \quad \text{for} \quad v \in \sum_{\lambda \in \sigma_p(A+BX_+)} L_\lambda.$$

Hence $X \leq X_+$ on $\sum_\lambda L_\lambda$. Analogously we find $X_- \leq X$ on $\sum_\lambda L_\lambda$. Since X_+ and X_- are bounded, this implies that X is bounded on $\sum_\lambda L_\lambda$ and hence on H since X is closed. Consequently $X_- \leq X \leq X_+$ holds on H, and $X\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and (36) follow by Remark 7.7.

Let now T have a compact resolvent. Theorem 7.6 implies that $\Gamma(X)$ is a compatible subspace. It is also hypermaximal J_1-neutral since X is selfadjoint.

(iii): We have again the decomposition (40). In particular, (U_k) is a Riesz basis of $\Gamma(X)$. Let $D_k = \operatorname{pr}_1(U_k)$. Then (D_k) is complete in H. Moreover, if c is the constant from (4) for the basis (U_k) and $u_k \in D_k$, then

$$\begin{aligned}
c^{-1}\sum_{k=0}^n \|u_k\|^2 \leq c^{-1}\sum_{k=0}^n \left\|\begin{pmatrix} u_k \\ Xu_k \end{pmatrix}\right\|^2 \leq \left\|\sum_{k=0}^n \begin{pmatrix} u_k \\ Xu_k \end{pmatrix}\right\|^2 \leq (1 + \|X\|^2)\left\|\sum_{k=0}^n u_k\right\|^2,\\
\left\|\sum_{k=0}^n u_k\right\|^2 \leq \left\|\sum_{k=0}^n \begin{pmatrix} u_k \\ Xu_k \end{pmatrix}\right\|^2 \leq c\sum_{k=0}^n \left\|\begin{pmatrix} u_k \\ Xu_k \end{pmatrix}\right\|^2 \leq c(1 + \|X\|^2)\sum_{k=0}^n \|u_k\|^2.
\end{aligned}$$

So $(D_k)_{k\in\mathbb{N}}$ is a Riesz basis of subspaces of H. Consequently, we have the decomposition

$$H = \bigoplus_{\operatorname{Re}\lambda_k > 0}^2 D_k \oplus \bigoplus_{\operatorname{Re}\lambda_k < 0}^2 D_k.$$

Let $P : H \to H$ be the corresponding projection onto $\bigoplus^2_{\operatorname{Re}\lambda_k > 0} D_k$. Since $X|_{D_k} = X_\pm|_{D_k}$ for $\operatorname{Re}\lambda_k \gtrless 0$, we obtain $X = X_+P + X_-(I - P)$. □

8 Examples

We start with an explicit example for the existence of infinitely many unbounded solutions of the Riccati equation. The Hamiltonian here is neither Riesz-spectral nor dichotomous, and so the results from [19] and [21] can not be applied.

Example 8.1 Let T be a nonnegative Hamiltonian with normal A, $B = I$ and selfadjoint C such that A and C admit an orthonormal basis $(f_k)_{k\geq 1}$ of common eigenvectors, $Af_k = ik^2 f_k$ and $Cf_k = kf_k$ for $k \geq 1$. Then C is $1/2$-subordinate to A and Theorem 6.7 can be applied. In fact, T is the operator from Example 3.7, its eigenvalues and corresponding normalised eigenvectors are

$$\lambda_k^\pm = ik^2 \pm \sqrt{k}, \quad v_k^\pm = \frac{1}{\sqrt{1+k}}\begin{pmatrix} f_k \\ \pm\sqrt{k}\,f_k \end{pmatrix},$$

and $V_k = \mathbb{C}f_k \times \mathbb{C}f_k$ yields a finitely spectral orthogonal basis of subspaces. In particular, T is neither Riesz-spectral nor dichotomous, see also Remark 3.13. The hypermaximal J_1-neutral compatible subspace corresponding to a skew-conjugate set $\sigma \subset \sigma(T)$ is given by

$$U_\sigma = \bigoplus_{k\geq 1}^{2} U_k \quad \text{with} \quad U_k = \begin{cases} \mathbb{C}v_k^+ & \text{if } \lambda_k^+ \in \sigma, \\ \mathbb{C}v_k^- & \text{if } \lambda_k^- \in \sigma. \end{cases}$$

It is the graph $U_\sigma = \Gamma(X_\sigma)$ of a selfadjoint solution X_σ of the Riccati equation (32),

$$X_\sigma f_k = \begin{cases} \sqrt{k}\, f_k & \text{if } \lambda_k^+ \in \sigma, \\ -\sqrt{k}\, f_k & \text{if } \lambda_k^- \in \sigma. \end{cases}$$

In particular, X_σ is unbounded and boundedly invertible. Moreover, every closed densely defined solution X such that $\Gamma(X)$ is compatible with (V_k) is of the above form.

By choosing different eigenvalues for the operators A and C, it is easy to construct solutions X_σ with different properties, for example solutions that are unbounded and not boundedly invertible, see also [36, §5.1].

We now prove the existence of infinitely many solutions of a Riccati equation involving differential operators. Since B and C are unbounded, this example is again not covered by [19, 21].

Example 8.2 Let $H = L^2([a,b])$ and consider the operators A, B, C on H given by

$$\begin{aligned} &Au = u''', \quad Bu = -(g_1 u')' + h_1 u, \quad Cu = -(g_2 u')' + h_2 u, \\ &\mathcal{D}(A) = \{u \in W^{3,2}([a,b]) \,|\, u(a) = u(b) = 0,\, u'(a) = u'(b)\}, \\ &\mathcal{D}(B) = \mathcal{D}(C) = \{u \in C^2([a,b]) \,|\, u(a) = u(b) = 0\} \end{aligned}$$

where $g_1, g_2 \in C^1([a,b])$, $h_1, h_2 \in L^2([a,b])$, $g_1, g_2, h_1, h_2 \geq 0$, and $W^{k,2}([a,b])$ denotes the Sobolev space of k times weakly differentiable, square integrable functions. Then A is skew-selfadjoint with compact resolvent, $0 \in \varrho(A)$, and $\sigma(A)$ consists of at most two sequences of eigenvalues

$$\lambda_{jk} = c_{jk} k^3, \quad k \geq k_{j0},\ j = 1, 2,$$

with converging sequences (c_{jk}), see [25]. Since the multiplicity of every eigenvalue is at most three, this implies that

$$\sup_{r\geq 1} \frac{N(r,A)}{r^{1/3}} < \infty.$$

The operators B and C are symmetric and nonnegative. Using Sobolev and interpolation inequalities, see [1], we can find constants $b_1, b_2, b_3 \geq 0$ such that

$$\begin{aligned} \|Bu\|_{L^2} &\leq \|g_1\|_\infty \|u''\|_{L^2} + \|g_1'\|_{L^2} \|u'\|_\infty + \|h_1\|_{L^2} \|u\|_\infty \leq b_1 \|u\|_{W^{2,2}} \\ &\leq b_2 \|u\|_{L^2}^{1/3} \|u\|_{W^{3,2}}^{2/3} \leq b_3 \|u\|_{L^2}^{1/3} \big(\|u\|_{L^2} + \|u'''\|_{L^2}\big)^{2/3} \\ &\leq b_3 \big(\|A^{-1}\| + 1\big)^{2/3} \|u\|_{L^2}^{1/3} \|Au\|_{L^2}^{2/3} \end{aligned}$$

for $u \in \mathcal{D}(A)$. Hence B, and similarly C, are 2/3-subordinate to A. By Theorem 4.6, the Hamiltonian corresponding to A, B, C thus has a finitely spectral Riesz basis of subspaces. If $g_1 > 0$ or $h_1 > 0$, and if $g_2 > 0$ or $h_2 > 0$, then both B and C are positive, and Theorem 6.3 yields an injective selfadjoint solution of the Riccati equation (28) for every skew-conjugate set $\sigma \subset \sigma(T)$.

This example immediately generalises to normal differential operators A on $[a, b]$ of order n and nonnegative symmetric differential operators B, C of order at most $n - 1$.

In our final example we obtain the existence of bounded solutions. In contrast to [8], the explicit assumption of the existence of X_- is not needed here. We also construct a solution whose graph is not a compatible subspace.

Example 8.3 Let $H = L^2([-1, 1])$ and consider the operators

$$\begin{aligned} &Au = u', \quad \mathcal{D}(A) = \big\{u \in W^{1,2}([-1,1]) \,\big|\, u(-1) = u(1)\big\}, \\ &Bu = bu, \quad Cu = cu, \quad \mathcal{D}(B) = \mathcal{D}(C) = H \end{aligned}$$

with $b, c \in L^\infty([-1, 1])$ and $b(t), c(t) \geq \gamma > 0$ for almost all $t \in [-1, 1]$. A is skew-selfadjoint with compact resolvent and simple eigenvalues $\lambda_k = i\pi k$. B and C are bounded and uniformly positive. If now $\|b\|_\infty, \|c\|_\infty < \pi/2$, then we can apply Theorems 4.7 and 7.9 and obtain bounded, selfadjoint, boundedly invertible solutions of the Riccati equation (36) as well as the relations

$$X_- \leq X \leq X_+ \qquad \text{and} \qquad X = X_+P + X_-(I - P).$$

Consider now the special case that $c = \chi^2 b$ with

$$\chi(t) = \begin{cases} 1, & t < 0, \\ \alpha, & t \geq 0, \end{cases} \qquad \alpha \in \mathbb{R} \setminus \{0, 1\}.$$

Let $X \in L(H)$ be the operator of multiplication with χ. It is not hard to see that

$$\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A) = \big\{u \in W^{1,2}([-1,1]) \,\big|\, u(-1) = u(0) = u(1) = 0\big\}$$

and $AXu = \chi u'$ for $u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A)$. Hence

$$-AXu + XAu + XBXu - Cu = -\chi u' + \chi u' + \chi^2 bu - cu = 0.$$

Consequently, X is a solution of the Riccati equation

$$-AXu + XAu + XBXu - Cu = 0, \qquad u \in \mathcal{D}(A) \cap X^{-1}\mathcal{D}(A),$$

and $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A) \subset H$ is dense. In particular, $\Gamma(X)$ is a T-invariant subspace. On the other hand, since $\mathcal{D}(A) \cap X^{-1}\mathcal{D}(A) \neq \mathcal{D}(A)$ we have $X\mathcal{D}(A) \not\subset \mathcal{D}(A)$, and with Theorem 7.6 we conclude that $\Gamma(X)$ is not a compatible subspace.

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References

[1] R. A. Adams. *Sobolev spaces.* Academic Press, New York-London, 1975.

[2] N. I. Akhiezer and I. M. Glazman. *Theory of Linear Operators in Hilbert Space.* Dover Publications Inc., New York, 1993.

[3] Alatancang, J. Huang, and H. Wu. Descriptions of spectra of infinite dimensional Hamiltonian operators and their applications. *Math. Nachr.*, 283(8):1144–1154, 2010.

[4] T. Y. Azizov, A. Dijksma, and I. V. Gridneva. On the boundedness of Hamiltonian operators. *Proc. Amer. Math. Soc.*, 131(2):563–576, 2003.

[5] T. Y. Azizov and I. S. Iokhvidov. *Linear Operators in Spaces With an Indefinite Metric.* John Wiley & Sons, Chichester, 1989.

[6] J. Bognar. *Indefinite Inner Product Spaces.* Springer, Berlin, 1974.

[7] P. Bubák, C. V. M. van der Mee, and A. C. M. Ran. Approximation of solutions of Riccati equations. *SIAM J. Control Optim.*, 44(4):1419–1435, 2005.

[8] R. F. Curtain, O. V. Iftime, and H. J. Zwart. A representation of all solutions of the control algebraic Riccati equation for infinite-dimensional systems. *Internat. J. Control*, 78(7):505–520, 2005.

[9] R. F. Curtain and H. J. Zwart. *An Introduction to Infinite Dimensional Linear Systems Theory.* Springer, New York, 1995.

[10] E. B. Davies. *Linear Operators and Their Spectra.* Cambridge University Press, 2007.

[11] A. Dijksma and H. S. V. de Snoo. Symmetric and selfadjoint relations in Kreĭn spaces. I. In *Operators in indefinite metric spaces, scattering theory and other topics (Bucharest, 1985)*, volume 24 of *Oper. Theory Adv. Appl.*, pages 145–166. Birkhäuser, Basel, 1987.

[12] N. Dunford and J. T. Schwartz. *Linear Operators, Part III.* Wiley-Interscience, New York, 1971.

[13] I. C. Gohberg and M. G. Krein. *Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space.* American Mathematical Society, Providence, 1969.

[14] I. S. Iohvidov and M. G. Kreĭn. Spectral theory of operators in spaces with an indefinite metric. I. *Trudy Moskov. Mat. Obšč.*, 5:367–432, 1956. English transl. in Amer. Math. Soc. Transl. (2), **13** (1960), 105–175.

[15] B. Jacob, C. Trunk, and M. Winklmeier. Analyticity and Riesz basis property of semigroups associated to damped vibrations. *J. Evol. Equ.*, 8(2):263–281, 2008.

[16] T. Kato. *Perturbation Theory for Linear Operators.* Springer, Berlin, 1980.

[17] V. Kostrykin, K. A. Makarov, and A. K. Motovilov. Existence and uniqueness of solutions to the operator Riccati equation. A geometric approach. In *Advances in differential equations and mathematical physics (Birmingham, AL, 2002)*, volume 327 of *Contemp. Math.*, pages 181–198. Amer. Math. Soc., Providence, 2003.

[18] S. G. Krein. *Linear Differential Equations in Banach Space.* Amer. Math. Soc., Providence, 1971.

[19] C. R. Kuiper and H. J. Zwart. Connections between the algebraic Riccati equation and the Hamiltonian for Riesz-spectral systems. *J. Math. Systems Estim. Control*, 6(4):1–48, 1996.

[20] P. Lancaster and L. Rodman. *Algebraic Riccati Equations.* Oxford University Press, Oxford, 1995.

[21] H. Langer, A. C. M. Ran, and B. A. van de Rotten. Invariant subspaces of infinite dimensional Hamiltonians and solutions of the corresponding Riccati equations. In *Linear Operators and Matrices*, volume 130 of *Oper. Theory Adv. Appl.*, pages 235–254. Birkhäuser, Basel, 2002.

[22] H. Langer and C. Tretter. Diagonalization of certain block operator matrices and applications to Dirac operators. In *Operator theory and analysis (Amsterdam, 1997)*, volume 122 of *Oper. Theory Adv. Appl.*, pages 331–358. Birkhäuser, Basel, 2001.

[23] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theorems. I: Abstract Parabolic Systems.* Cambridge University Press, 2000.

[24] K. Mårtensson. On the matrix Riccati equation. *Inform. Sci.*, 3:17–49, 1971.

[25] M. A. Naimark. *Linear Differential Operators. Part 1.* Frederick Ungar Publishing Co., New York, 1967.

[26] M. R. Opmeer and O. V. Iftime. A representation of all bounded selfadjoint solutions of the algebraic Riccati equation for systems with an unbounded observation operator. In *Proceedings of the 43rd IEEE Conference on Decision and Control*, 2004. 6 pages.

[27] J. E. Potter. Matrix quadratic solutions. *SIAM J. Appl. Math.*, 14(3):496–501, 1966.

[28] A. J. Pritchard and D. Salamon. The linear quadratic control problem for infinite-dimensional systems with unbounded input and output operators. *SIAM J. Control Optim.*, 25(1):121–144, 1987.

[29] A. C. M. Ran and C. van der Mee. Perturbation results for exponentially dichotomous operators on general Banach spaces. *J. Funct. Anal.*, 210(1):193–213, 2004.

[30] I. Singer. *Bases in Banach Spaces II.* Springer, Berlin, 1981.

[31] C. Tretter. Spectral issues for block operator matrices. In *Differential equations and mathematical physics (Birmingham, AL, 1999)*, volume 16 of *AMS/IP Stud. Adv. Math.*, pages 407–423. Amer. Math. Soc., Providence, 2000.

[32] C. Tretter. *Spectral Theory of Block Operator Matrices and Applications.* Imperial College Press, London, 2008.

[33] V. N. Vizitei and A. S. Markus. On convergence of multiple expansions in the eigenvectors and associated vectors of an operator pencil. *Mat. Sb.*, 66(108):287–320, 1965. English transl. in Amer. Math. Soc. Transl., (2) **87** (1970), 187–227.

[34] J. Weidmann. *Lineare Operatoren in Hilberträumen. Teil 1.* Mathematische Leitfäden. [Mathematical Textbooks]. B. G. Teubner, Stuttgart, 2000. Grundlagen. [Foundations].

[35] M. Weiss and G. Weiss. Optimal control of stable weakly regular linear systems. *Math. Control Signals Systems*, 10(4):287–330, 1997.

[36] C. Wyss. *Perturbation theory for Hamiltonian operator matrices and Riccati equations.* PhD thesis, University of Bern, 2008.

[37] C. Wyss. Riesz bases for p-subordinate perturbations of normal operators. *J. Funct. Anal.*, 258(1):208–240, 2010.

[38] G. Q. Xu and S. P. Yung. The expansion of a semigroup and a Riesz basis criterion. *J. Differential Equations*, 210(1):1–24, 2005.

[39] H. Zwart. Riesz basis for strongly continuous groups. Eprint, `arXiv:0808.3447 [math.FA]`, 2008.