Online seminar on port-Hamiltonian systems (pHOne)
6th March 2024, online

The Partitioned Finite Element Method for port-Hamiltonian systems: a structure-preserving discretization for boundary-controlled wave and heat PDEs.

Denis Matignon[1]

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Outline

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Goal of the presentation and a few references

- **Goal**: the underlying structure of physical systems must be preserved by numerical methods at the discrete level, i.e. from infinite dimension to finite dimension (but still in continuous time).
- **Applications**: the 2D wave PDE, and the 2D heat PDE with collocated boundary control and observation.
- **Proof of concept**: the heat-wave coupling and refined asymptotics can be recovered at the discrete level.

Twenty years of distributed port-Hamiltonian systems: a literature review
Rashad, R., Califano, F., van der Schaft, A.J. and Stramigioli, S.
IMA J. Mathematics of Control and Information,
vol.37 (4), pp. 1400–1422 (2020)

$\Longrightarrow$ More than 170 up-to-date references on:

- Theoretical Framework
- Modeling
- Analysis and Control
- Discretization

and more specifically

- Numerical Methods for Distributed Parameter Port-Hamiltonian Systems **Kotyczka P.** *TUM University Press, Munich (2019)*,
- Structure preserving approximation of dissipative evolution problems **Egger H.** *Numerische Mathematik vol. 143(1), pp. 85–106 (2019)*
- A Partitioned Finite-Element Method for power-preserving discretization of open systems of conservation laws, **Cardoso-Ribeiro F.L., Matignon D., Lefèvre L.** *IMA J. Mathematics of Control and Information, vol. 38(2), pp. 493–533 (2021)*
- Long-time behavior of a coupled heat-wave system using a structure-preserving finite element method, **Haine G., Matignon D., Monteghetti F.** *Mathematical Reports, vol. 22(1–2), pp. 187–215 (2022)*.

PFEM 4 pHs? the team!

- Origin of the method:

  F. L. Cardoso-Ribeiro, D. Matignon, and L. Lefèvre. A structure-preserving Partitioned Finite Element Method for the 2D wave equation. In 6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control (LHMNLC), Valparaíso, Chile, 2018. IFAC-PapersOnLine, Vol. 51, Issue 3, 2018, pp. 119–124

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- Collaborators on the **PFEM 4 pHs** project since then:
  - Anass SERHANI
  - Andrea BRUGNOLI
  - Ghislain HAINE
  - Valérie POMMIER - BUDINGER
  - Daniel ALAZARD
  - Michel SALAÜN
  - Xavier VASSEUR
  - Florian MONTEGHETTI
  - Michel FOURNIÉ
  - Giuseppe FERRARO
  - Antoine BENDIMERAD-HOHL

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Simulate complex open physical systems by ensuring the *conservation of the* ***power balance*** for a chosen functional: the **Hamiltonian**.

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- *Partitioned Finite Element Method (PFEM):*
  → It approximates the Stokes-Dirac structure into a **Dirac structure**.
  → The **discrete Hamiltonian** satisfies a "discrete" power balance.

A Partitioned Finite-Element Method for power-preserving discretization of open systems of conservation laws
Cardoso-Ribeiro F.L., Matignon D., Lefèvre L.
IMA J. Mathematics of Control and Information, vol.38(2) , pp. 493–533 (2021)

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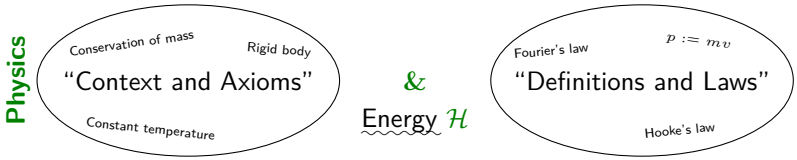

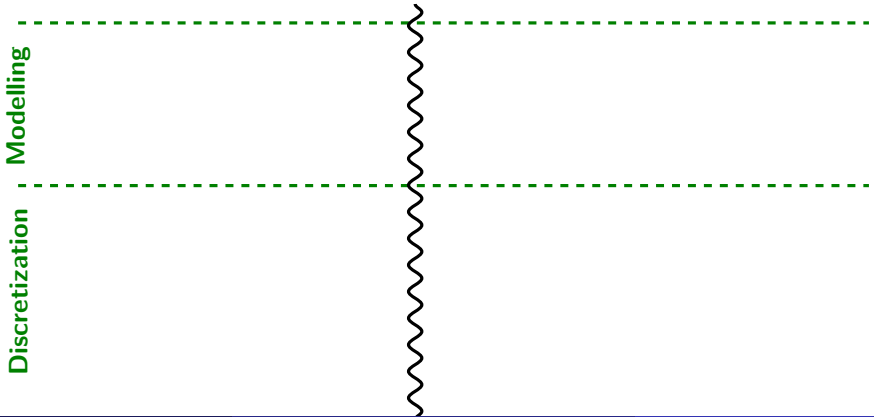

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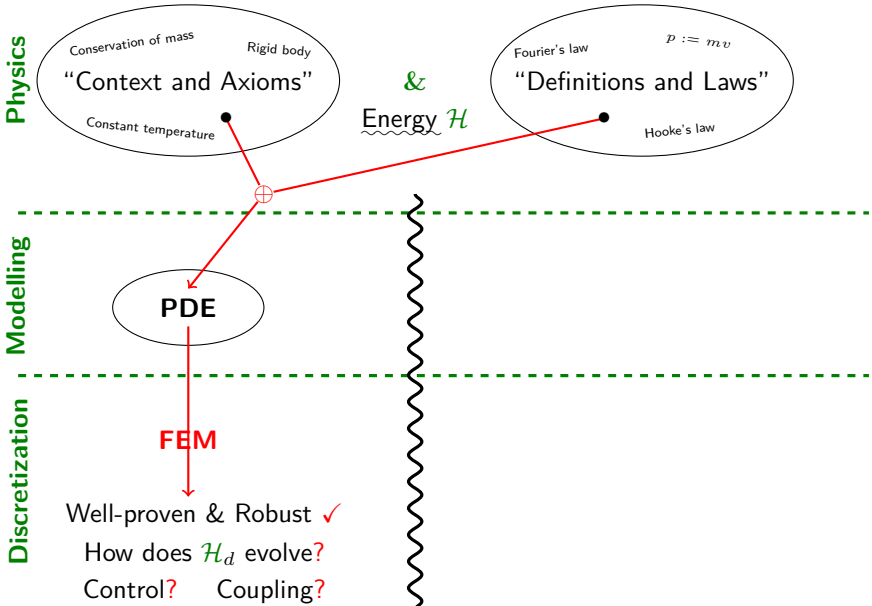

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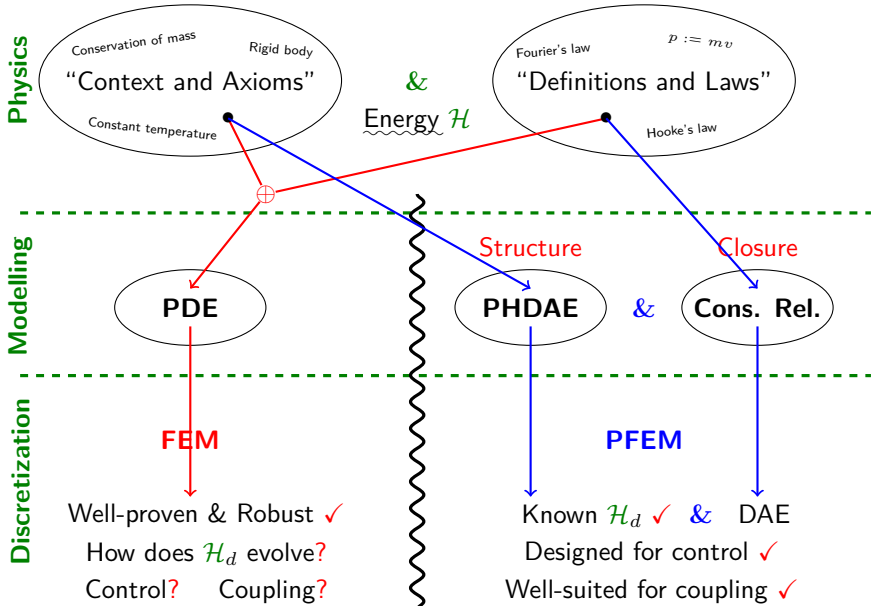

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Lossy Power Balance

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}(t)) = -\left\langle R\overrightarrow{\boldsymbol{e}}_{\overrightarrow{\boldsymbol{\alpha}}}(t), \overrightarrow{\boldsymbol{e}}_{\overrightarrow{\boldsymbol{\alpha}}}(t)\right\rangle_J + \langle \boldsymbol{u}(t), \boldsymbol{y}(t)\rangle_B \;\leq\; \langle \boldsymbol{u}(t), \boldsymbol{y}(t)\rangle_B .$$

Although **the underlying geometry** is well-determined with the above equality, **constitutive relations** between $\overrightarrow{\boldsymbol{\alpha}}$ and $\overrightarrow{\boldsymbol{e}}_{\overrightarrow{\boldsymbol{\alpha}}}$ are also needed to solve the system!

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Conservative System: Wave as PH-DAE

Deflection w of a 2D-membrane, *boundary deflection velocity* as control.

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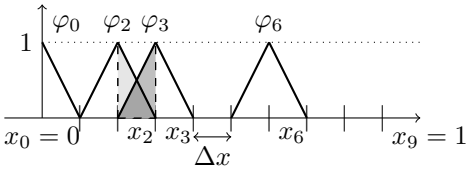
Example of a $\mathbb{P}^1$-Lagrange mass matrix

Discretization: $\alpha^{ap}(x) := \sum_{k=0}^{N-1} \varphi_k(x)\,\alpha^k = \boldsymbol{\phi}^\top \underline{\alpha}$, with N unknown coeff. $\underline{\alpha}$, N shape functions:



Mass matrix:

$$M_{i,j} = \int_0^1 \varphi_j(x)\varphi_i(x)\,\mathrm{d}x = \sum_{k=0}^{9} \underbrace{\int_{x_k}^{x_{k+1}} \varphi_j(x)\varphi_i(x)\,\mathrm{d}x}_{\neq 0 \Rightarrow i,j=k,k+1}. \quad \Rightarrow \quad \text{sparsity}$$

Elementary matrix:

$$(M_k^{elt})_{i,j} := \int_{x_k}^{x_{k+1}} \varphi_j(x)\varphi_i(x)\,\mathrm{d}x \in \mathbb{R}^{2\times 2}, \qquad i,j = k, k+1.$$

Assembly algorithm (pseudo-code): for k in $(0, 9)$; do $M[k : k+1, k : k+1] = M_k^{elt}$;

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For all test functions $\overrightarrow{\boldsymbol{v}}_q$, v_p and v_∂ (smooth enough):

$$\begin{cases} \left\langle \partial_t \overrightarrow{\boldsymbol{\alpha}}_q, \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2} = \left\langle \overrightarrow{\mathbf{grad}}\left(e_p\right), \overrightarrow{\boldsymbol{v}}_q \right\rangle_{\mathbf{L}^2}, \\ \left\langle \partial_t \alpha_p, v_p \right\rangle_{L^2} = \left\langle \operatorname{div}\left(\overrightarrow{\boldsymbol{e}}_q\right), v_p \right\rangle_{L^2}, \\ \left\langle \boldsymbol{y}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} = \left\langle \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}, v_\partial \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}. \end{cases}$$

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Applying Green's formula on the 1st line and using the definition of $\boldsymbol{u}$:

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Green's formula applied on the 2nd line would lead to normal stress control $\boldsymbol{u} = \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}}$. The energy variables are **partitioned** accordingly.

Conservative System: FEM Application

The energy, co-energy, boundary and test functions of the *same* index are discretized by using the *same* bases, either scalar- or **vector**-valued:

$$\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) := \sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}}) \alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t),$$

with $\overrightarrow{\boldsymbol{\Phi}}_q$ an $N_q \times 2$ matrix,

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$$\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) := \sum_{\ell=1}^{N_q} \overrightarrow{\boldsymbol{\phi}}_q^{\ell}(\overrightarrow{\boldsymbol{x}})\alpha_q^{\ell}(t) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{\alpha}_q(t), \quad \overrightarrow{\boldsymbol{e}}_q^{ap}(t,\overrightarrow{\boldsymbol{x}}) = \overrightarrow{\boldsymbol{\Phi}}_q^{\top} \cdot \underline{e}_q(t),$$

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The discretized system (giving the structure) then reads:

$$\begin{cases}
\overrightarrow{\boldsymbol{M}}_q \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) = D\cdot\underline{e}_p(t) + B\cdot\underline{\boldsymbol{u}}(t),\\
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$$D := -\int_\Omega \operatorname{div}\left(\overrightarrow{\boldsymbol{\Phi}}_q\right)\cdot\boldsymbol{\phi}_p^{\top}, \qquad B := \int_{\partial\Omega}\left(\overrightarrow{\boldsymbol{\Phi}}_q\cdot\overrightarrow{\boldsymbol{n}}\right)\cdot\boldsymbol{\Psi}^{\top}.$$

Conservative System: Power Balance

Finite-Dimensional extended structure operator

$$\mathcal{J}_d := \begin{pmatrix} 0 & D & B \\ -D^\top & 0 & 0 \\ -B^\top & 0 & 0 \end{pmatrix}.$$

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$$\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) := \mathcal{H}\left(\overrightarrow{\boldsymbol{\alpha}}_q^{ap}, \alpha_p^{ap}\right) = \tfrac{1}{2}\left(\underline{\alpha}_q^\top \cdot \underline{\overrightarrow{\boldsymbol{M}}}_{\overline{\overline{T}}} \cdot \underline{\alpha}_q + \underline{\alpha}_p^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p\right),$$

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Constitutive relations: $\overrightarrow{M}_q \cdot \underline{e}_q = \overrightarrow{\overline{\overline{\boldsymbol{T}}}} \cdot \underline{\alpha}_q \quad \& \quad M_p \cdot \underline{e}_p = M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p$ ✓✓

Conservative System: Power Balance

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Denote $\underline{f} := \left(\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q, \quad \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p, \quad -\underline{\boldsymbol{y}}\right)^\top$ and $\underline{e} := \left(\underline{e}_q, \quad \underline{e}_p, \quad \underline{\boldsymbol{u}}\right)^\top$, then:

Discrete Lossless **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) = \underline{\boldsymbol{u}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

A Proof of the Discrete Power Balance

Since by definition the discrete Hamiltonian reads:

$$\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) = \frac{1}{2}\left(\underline{\alpha}_q^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{T}}} \cdot \underline{\alpha}_q + \underline{\alpha}_p^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p\right),$$

we can compute its time derivative along the trajectories:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) &= (\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q)^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{T}}} \cdot \underline{\alpha}_q + (\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p)^\top \cdot M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p, \\
&= (\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q)^\top \cdot \overrightarrow{\boldsymbol{M}}_q \cdot \underline{e}_q + (\frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p)^\top \cdot M_p \cdot \underline{e}_p, \\
&= (\overrightarrow{\boldsymbol{M}}_q \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q)^\top \cdot \underline{e}_q + (M_p \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p)^\top \cdot \underline{e}_p, \\
&= (D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t))^\top \cdot \underline{e}_q + (-D^\top \cdot \underline{e}_q(t))^\top \cdot \underline{e}_p, \\
&= \underline{\boldsymbol{u}}(t)^\top \cdot B^\top \cdot \underline{e}_q \\
&= \underline{\boldsymbol{u}}(t)^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}(t). \quad \square
\end{aligned}$$

PFEM for pHs gives rise to PH-DAEs

Summarizing the main steps: discretization of the structure and of the constitutive relations are made separately.

The discretized system is a PH-DAE:

$$\begin{cases} \overrightarrow{M}_q \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) = D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t), \\ M_p \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) = -D^\top \cdot \underline{e}_q(t), \\ M_\partial \cdot \underline{\boldsymbol{y}}(t) = B^\top \cdot \underline{e}_q(t), \end{cases}$$

together with

$$\begin{cases} \overrightarrow{M}_q \cdot \underline{e}_q(t) = \overrightarrow{M}_{\overline{\overline{T}}} \cdot \underline{\alpha}_q(t), \\ M_p \cdot \underline{e}_p(t) = M_{\frac{1}{\rho}} \cdot \underline{\alpha}_p(t) \end{cases}$$

$\Longrightarrow$ in general, PFEM for pHs gives rise to finite-dimensional PH-DAEs, for which efficient numerical methods can be used.

Remark: in the linear case only, an alternative implementation of PFEM gives rise to PH-ODEs in terms of the co-energy variables only.

Outline

Impedance Boundary Condition (IBC)

The Impedance Boundary Condition, with $Z \geq 0$ on $\partial\Omega$, and $\boldsymbol{\nu}$ as new control, is considered:
$\boldsymbol{\nu} = e_p + Z \overrightarrow{\boldsymbol{e}}_q \cdot \overrightarrow{\boldsymbol{n}} \Leftrightarrow \boldsymbol{\nu} = \partial_t w + Z \left(\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\mathbf{grad}}\,(w) \right) \cdot \overrightarrow{\boldsymbol{n}}$.

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It can be seen as an *output feedback law* $\boldsymbol{u} = -Z\boldsymbol{y} + \boldsymbol{\nu}$ in the previous case.

Lossy **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(\overrightarrow{\boldsymbol{\alpha}}_q, \alpha_p) = -\left\langle \boldsymbol{y}, Z\boldsymbol{y} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} + \left\langle \boldsymbol{y}, \boldsymbol{\nu} \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} .$$

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Add **impedance ports** (f_i, e_i) and **dissipative constitutive relation** $e_i = Zf_i$,
and approximate f_i and e_i in the boundary FEM basis $\boldsymbol{\Psi}$:

$$\begin{pmatrix} \overline{\boldsymbol{M}}_q & 0 & 0 & 0 & 0 \\ 0 & M_p & 0 & 0 & 0 \\ 0 & 0 & M_p & 0 & 0 \\ 0 & 0 & 0 & M_\partial & 0 \\ 0 & 0 & 0 & 0 & M_\partial \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_q(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\alpha}_p(t) \\ \underline{f}_i(t) \\ -\underline{\boldsymbol{y}}(t) \end{pmatrix} = \begin{pmatrix} 0 & D & -B & B \\ -D^\top & 0 & 0 & 0 \\ B^\top & 0 & 0 & 0 \\ -B^\top & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{e}_q(t) \\ \underline{e}_p(t) \\ \underline{e}_i(t) \\ \underline{\boldsymbol{\nu}}(t) \end{pmatrix}$$

and $M_\partial \cdot \underline{e}_i = \langle Z \rangle \cdot \underline{f}_i$, with $\langle Z \rangle := \int_{\partial\Omega} Z\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^\top \geq 0$.

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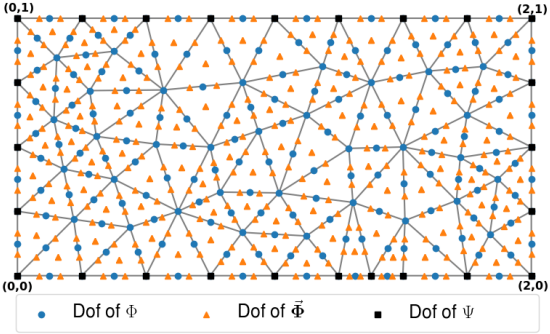
and $M_\partial \cdot \underline{e}_i = \langle Z \rangle \cdot \underline{f}_i$, with $\langle Z \rangle := \int_{\partial\Omega} Z\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}^\top \geq 0$.

Discrete Lossy **Power Balance**

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right) = -\underline{\boldsymbol{y}}^\top \cdot \langle Z \rangle \cdot \underline{\boldsymbol{y}} + \underline{\boldsymbol{\nu}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

Boundary Dissipation: Simulations

- Heteregenous ($\rho \not\equiv constant$);
- Anisotropic (tensor $\overline{\overline{\boldsymbol{T}}} \not\equiv constant$);
- $\epsilon \equiv 0$;
- $Z \neq 0$ for $t \geq 2$;
- Raviart-Thomas FEM for q-variables;
- Lagrange FEM for p-variables;
- Lagrange FEM for ∂-variables;



Outline

Mixed boundary control: principle

1. The basic idea is: $\partial\Omega = \Gamma_D \cup \Gamma_N$ and $\int_{\partial\Omega} = \int_{\Gamma_D} + \int_{\Gamma_N}$.
2. Where the control is not known, a Lagrange multiplier λ is introduced instead + a constraint is added to the system, an extended skew-symmetric J_e matrix is obtained.
3. $\Longrightarrow$ a PH-DAE is readily obtained, with a Lagrange multiplier of very small dimension.
4. This method is detailed in one early reference,
   Brugnoli, A., Cardoso-Ribeiro, F.L., Haine, G., and Kotyczka, P. **Partitioned finite element method for structured discretization with mixed boundary conditions** In *IFAC-PapersOnLine*, volume 53(2), 7557–7562. (2020),
   but several other possibilities have been explored since then.
5. a PH-ODE can be obtained, taking advantage of the Hellinger-Reissner principle, see e.g.
   Brugnoli, A., Haine, G., and Matignon, D. **Explicit structure-preserving discretization of port-Hamiltonian systems with mixed boundary control** In *IFAC-PapersOnLine*, volume 55(30), 418–423. (2022).

Outline

Convergence rate: theory

Theorem (Haine, Matignon & Serhani, 2023)

Let $\kappa \geq 1$ be an integer, and $T > 0$.

Let $\begin{pmatrix} \overrightarrow{\boldsymbol{\alpha}}_{q_0} \\ \alpha_{p_0} \end{pmatrix} \in \mathcal{Z}_\kappa := \begin{pmatrix} \overline{\overline{T}} & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{H}^{\kappa+1}_{\mathrm{div}}(\Omega) \\ H^{\kappa+1}(\Omega) \end{bmatrix}$, $\boldsymbol{u} \in C^2([0,\infty); H^{\kappa+1}(\partial\Omega))$.

Let $\begin{pmatrix} \overrightarrow{\boldsymbol{\alpha}}^{ap}_q(0) \\ \alpha^{ap}_p(0) \end{pmatrix}$, $\boldsymbol{u}^{ap}$ be their interpolations with $\left(\mathbb{P}^k\right)^N \times \mathbb{P}^\ell \times \mathbb{P}^m$.

Let $\mathbf{E}(t) := \left\| (\overrightarrow{\boldsymbol{\alpha}}_q - \overrightarrow{\boldsymbol{\alpha}}^{ap}_q)(t), \quad (\alpha_p - \alpha^{ap}_p)(t) \right)^\top \Big\|_{\mathbf{L}^2 \times L^2}$,

$\exists C_T > 0$, independent of $\begin{pmatrix} \overrightarrow{\boldsymbol{\alpha}}_{q_0} \\ \alpha_{p_0} \end{pmatrix}$, and $\boldsymbol{u}$: for all h and all $t \in [0,T]$

$$\mathbf{E}(t) \leq C_T h^{\min\{\ell\,;\,k\,;\,m\}} \left(\left\| \begin{pmatrix} \overrightarrow{\boldsymbol{\alpha}}_q \\ \alpha_p \end{pmatrix} \right\|_{L^\infty([0,T];\mathcal{Z}_\kappa)} + \|\boldsymbol{u}\|_{L^\infty([0,T];H^{\kappa+1}(\partial\Omega))} \right).$$

The **optimal** order is κ, when $k = \kappa$, $\ell = \kappa$ and $m = \kappa$.

Convergence rate: theory

Theorem (Haine, Matignon & Serhani, 2023)

Let $\kappa \geq 1$ be an integer, and $T > 0$.

Let $\begin{pmatrix} \overrightarrow{\boldsymbol{\alpha}}_{q_0} \\ \alpha_{p_0} \end{pmatrix} \in \mathcal{Z}_\kappa := \begin{pmatrix} \overline{\overline{\boldsymbol{T}}} & 0 \\ 0 & \frac{1}{\rho} \end{pmatrix}^{-1} \begin{bmatrix} \mathbf{H}^{\kappa+1}_{\mathrm{div}}(\Omega) \\ H^{\kappa+1}(\Omega) \end{bmatrix}$, $\boldsymbol{u} \in C^2([0,\infty); H^{\kappa+1}(\partial\Omega))$.

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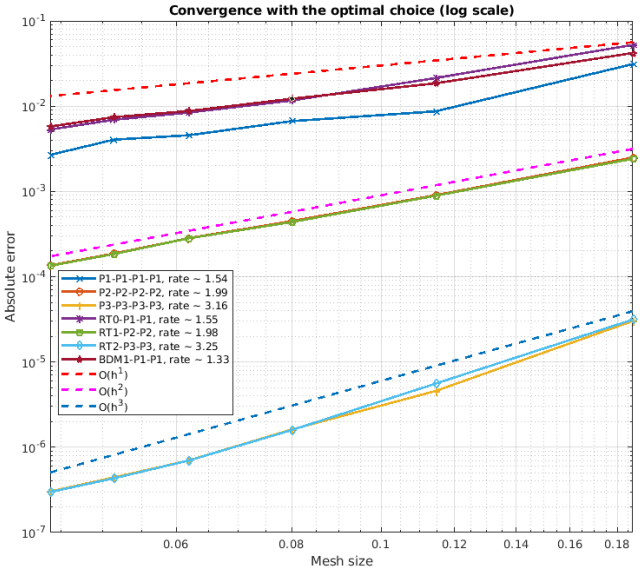
The **optimal** order is κ, when $k = \kappa$, $\ell = \kappa$ and $m = \kappa$.

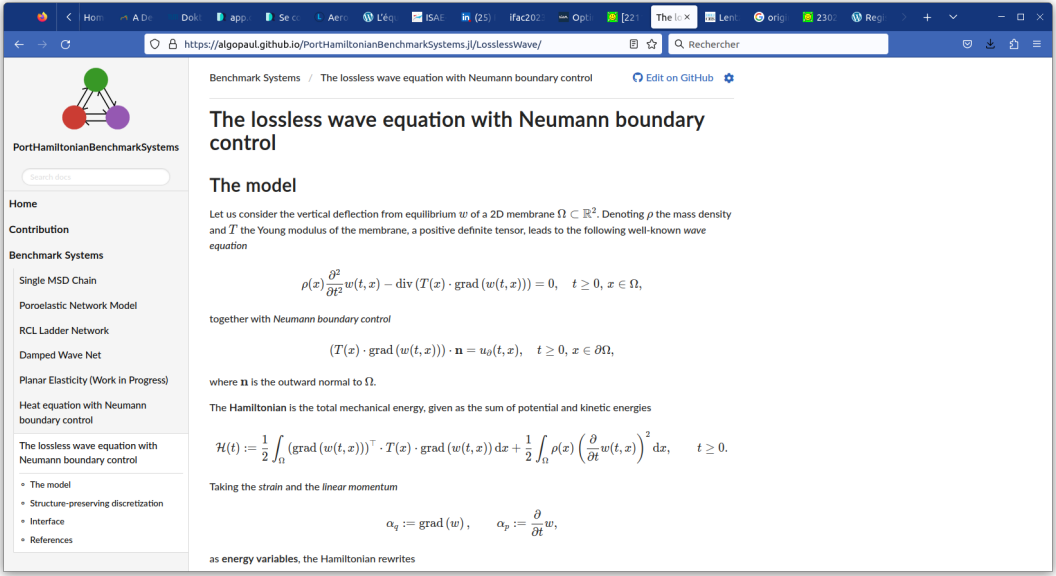
$RT_{\kappa-1} \times \mathbb{P}^\kappa \times \mathbb{P}^\kappa$ $\qquad$ $BDM_\kappa \times \mathbb{P}^\kappa \times \mathbb{P}^\kappa$ $\qquad$ $BDFM_\kappa \times \mathbb{P}^\kappa \times \mathbb{P}^\kappa$

Convergence rate: numerics



Periodic Table of the Finite Elements

Make your own 2D membrane simulations using the pHs benchmark!



Outline

- **Space domain and physical parameters:**
  - $\Omega \subset \mathbb{R}^{n \geq 1}$ is a bounded open connected set;
  - $\overrightarrow{\boldsymbol{n}}$ is the outward unit normal on the boundary $\partial\Omega$;
  - $\rho(\overrightarrow{\boldsymbol{x}})$ is the mass density;
  - $\overline{\overline{\boldsymbol{\lambda}}}(\overrightarrow{\boldsymbol{x}})$ is the conductivity tensor (symmetric, positive definite)

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- **Notations:**
  - T is the local temperature;
  - $\beta := \frac{1}{T}$ is the reciprocal temperature;
  - u is the internal energy density;
  - s is the entropy density;
  - $\overrightarrow{\boldsymbol{J}}_Q$ is the heat flux;
  - $\overrightarrow{\boldsymbol{J}}_S := \beta \overrightarrow{\boldsymbol{J}}_Q$ is the entropy flux;
  - $C_V := \left(\frac{\mathrm{d}u}{\mathrm{d}T} \right)_V$ is the isochoric heat capacity.

- **"Context & Axioms":**
  - **Medium:** rigid body without chemical reaction;
  - **1st law of thermodynamics:**

$$\rho(\overrightarrow{\boldsymbol{x}})\partial_t u(t,\overrightarrow{\boldsymbol{x}}) = -\mathrm{div}\left(\overrightarrow{\boldsymbol{J}}_Q(t,\overrightarrow{\boldsymbol{x}})\right);$$

  - **Gibbs' relation:**

$$\mathrm{d}U = T\ \mathrm{d}S, \quad \Longrightarrow \quad \partial_t u(t,\overrightarrow{\boldsymbol{x}}) = T(t,\overrightarrow{\boldsymbol{x}})\partial_t s(t,\overrightarrow{\boldsymbol{x}});$$

  - **Entropy evolution:**

$$\rho(\overrightarrow{\boldsymbol{x}})\partial_t s(t,\overrightarrow{\boldsymbol{x}}) = -\mathrm{div}\left(\overrightarrow{\boldsymbol{J}}_S(t,\overrightarrow{\boldsymbol{x}})\right) + \sigma(t,\overrightarrow{\boldsymbol{x}}),$$

  with $\sigma := \overrightarrow{\mathbf{grad}}\,(\beta)\cdot\overrightarrow{\boldsymbol{J}}_Q$ is the *irreversible entropy production*, with $\sigma \geq 0$.

Thermodynamics

- **"Context & Axioms":**
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- **"Laws":**
  - **Fourier's law:**

$$\overrightarrow{\boldsymbol{J}}_Q(t,\overrightarrow{\boldsymbol{x}}) = -\overline{\overline{\boldsymbol{\lambda}}}(t,\overrightarrow{\boldsymbol{x}})\cdot\overrightarrow{\mathbf{grad}}\left(T(t,\overrightarrow{\boldsymbol{x}})\right);$$

  - **Dulong-Petit's law:**

$$u(t,\overrightarrow{\boldsymbol{x}}) = C_V(\overrightarrow{\boldsymbol{x}})T(t,\overrightarrow{\boldsymbol{x}}).$$

Three useful Hamiltonian functionals

- **Lyapunov functional:** $\boldsymbol{\nu} := T$ and $\boldsymbol{y} := \overrightarrow{\boldsymbol{J}}_Q \cdot \overrightarrow{\boldsymbol{n}}$ (or the other way),
  $\mathcal{H}(T(t, \overrightarrow{\boldsymbol{x}})) := \int_\Omega \rho(\overrightarrow{\boldsymbol{x}}) C_V(\overrightarrow{\boldsymbol{x}}) \left(T(t, \overrightarrow{\boldsymbol{x}})\right)^2 \, \mathrm{d}\overrightarrow{\boldsymbol{x}}$,

Lossy Power Balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} := -\int_\Omega \overrightarrow{\boldsymbol{f}}_Q \cdot \overline{\overline{\boldsymbol{\lambda}}} \cdot \overrightarrow{\boldsymbol{f}}_Q + \langle \boldsymbol{y}, \boldsymbol{\nu} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} := -\underline{f}_Q \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{\lambda}}}} \cdot \underline{f}_Q + \underline{\boldsymbol{\nu}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

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- **Internal energy:** $\boldsymbol{\nu} := T$, and $\boldsymbol{y} := \overrightarrow{\boldsymbol{J}}_S \cdot \overrightarrow{\boldsymbol{n}}$, $\mathcal{U}(s(t, \overrightarrow{\boldsymbol{x}})) := \int_\Omega \rho(\overrightarrow{\boldsymbol{x}}) u(s(t, \overrightarrow{\boldsymbol{x}})) \, \mathrm{d}\overrightarrow{\boldsymbol{x}}$,

Lossless Power Balance (first law of thermodynamics)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(s) = \langle \boldsymbol{y}, \boldsymbol{\nu} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}_d(\overline{s}) = \underline{\boldsymbol{\nu}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}.$$

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- **Entropy:** $\boldsymbol{\nu} := \overrightarrow{\boldsymbol{J}}_Q \cdot \overrightarrow{\boldsymbol{n}}$, $\boldsymbol{y} := \frac{1}{T}$, $\mathcal{S}(u(t, \overrightarrow{\boldsymbol{x}})) := \int_\Omega \rho(\overrightarrow{\boldsymbol{x}}) s(u(t, \overrightarrow{\boldsymbol{x}})) \, \mathrm{d}\overrightarrow{\boldsymbol{x}}$,

Accretive Power Balance (second law of thermodynamics)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}(u) = \int_\Omega \sigma + \langle \boldsymbol{\nu}, \boldsymbol{y} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S}_d(\overline{u}) = \underline{\sigma}^\top \cdot M \cdot \underline{\mathbf{1}} + \underline{\boldsymbol{y}}^\top \cdot M_\partial \cdot \underline{\boldsymbol{\nu}}.$$

Outline

Diffusion: Lyapunov Functional

Quadratic Hamiltonian: Lyapunov Functional

$$\mathcal{H}(\rho(\overrightarrow{\boldsymbol{x}})u(t,\overrightarrow{\boldsymbol{x}})) := \frac{1}{2}\int_\Omega \rho(\overrightarrow{\boldsymbol{x}})\frac{\left(u(t,\overrightarrow{\boldsymbol{x}})\right)^2}{C_V(\overrightarrow{\boldsymbol{x}})}\,\mathrm{d}\overrightarrow{\boldsymbol{x}}$$
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Energy variable : $\alpha_u := \rho u$, **co-energy variable** : $e_u := \delta_{\alpha_u}\mathcal{H} = T$.

This is the *usual* functional used in the mathematics community...

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Defining $f_u := \partial_t\alpha_u = \rho\partial_t u$, $e_u = T$, $\overrightarrow{\boldsymbol{f}}_Q := -\overrightarrow{\mathbf{grad}}\,(T)$, and $\overrightarrow{\boldsymbol{e}}_Q := \overrightarrow{\boldsymbol{J}}_Q$, we get

$$\begin{pmatrix} f_u \\ \overrightarrow{\boldsymbol{f}}_Q \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{div} \\ -\overrightarrow{\mathbf{grad}} & 0 \end{pmatrix}\begin{pmatrix} e_u \\ \overrightarrow{\boldsymbol{e}}_Q \end{pmatrix}, \qquad \text{with constitutive relation: } \overrightarrow{\boldsymbol{e}}_Q = \overline{\overline{\boldsymbol{\lambda}}}\,\overrightarrow{\boldsymbol{f}}_Q.$$

Diffusion: Lyapunov Functional

At least two choices for boundary control: e_u or $\overrightarrow{e}_Q \cdot \overrightarrow{n}$.
With *inward* heat flux $\nu = -\overrightarrow{e}_Q \cdot \overrightarrow{n}$, the output is $y = e_u$, *i.e.* the boundary temperature, and the discretized system is:

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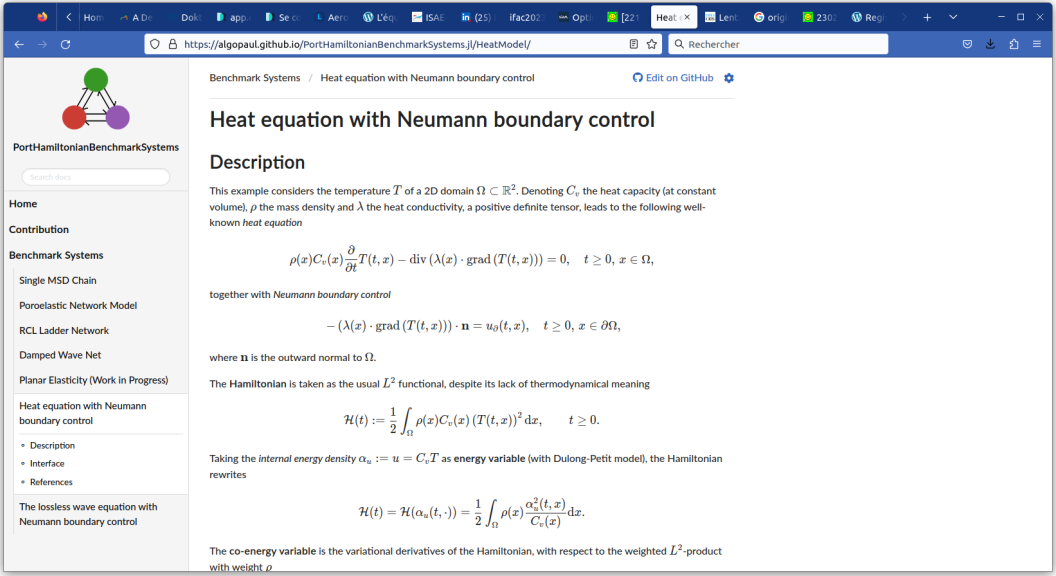
Discrete Lossy **Power Balance**

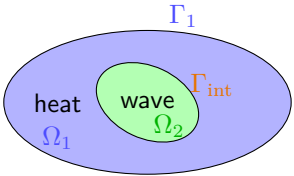
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_d = -\underline{f}_Q \cdot \overrightarrow{M}_{\overline{\overline{\lambda}}} \cdot \underline{f}_Q + \underline{\nu}^\top \cdot M_\partial \cdot \underline{y} \;\leq\; \underline{\nu}^\top \cdot M_\partial \cdot \underline{y}.$$

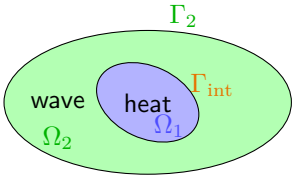
Make your own 2D heat simulations using the pHs benchmark!

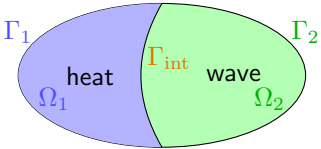


Outline

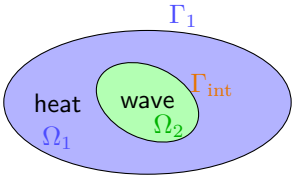
System and configurations

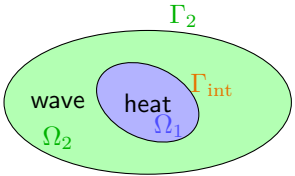


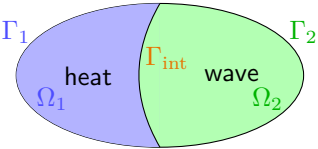




System and configurations



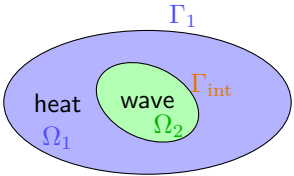


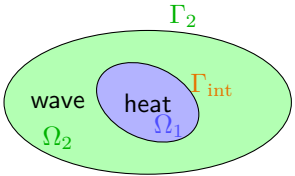


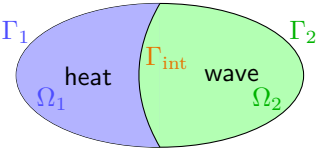
Heat:

$$\left\{ \begin{array}{rcll} \partial_t T(t,\overrightarrow{\boldsymbol{x}}) - \Delta T(t,\overrightarrow{\boldsymbol{x}}) & = & 0, & \overrightarrow{\boldsymbol{x}} \in \Omega_1, \\ T(t,\overrightarrow{\boldsymbol{x}}) & = & 0, & \overrightarrow{\boldsymbol{x}} \in \Gamma_1, \end{array} \right.$$

System and configurations







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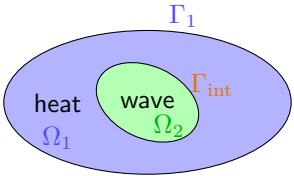
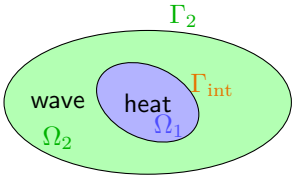
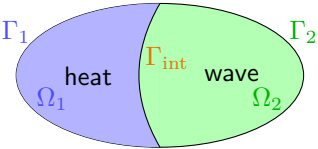
System and configurations







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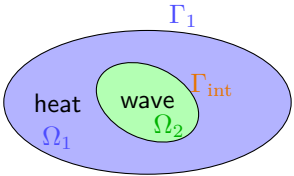

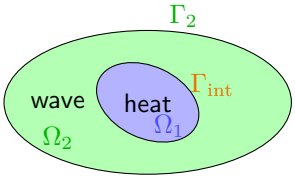

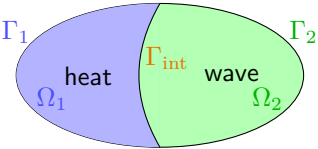

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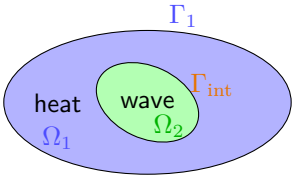

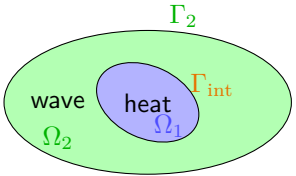

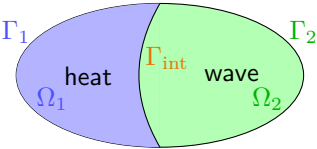

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Geometric Control Condition (GCC):

All characteristics of the wave equation must encounter Ω_1 in finite time.

Known decays

Long-time behavior of a coupled heat-wave system arising in fluid-structure interaction

Zhang X. and Enrique Z.

Archive for Rational Mechanics and Analysis, 184(1):49–120, (2007)

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Remark

In our numerical simulations, the initial data are such that the constant solution is the null solution.

Coupled model with transmission condition

Gyrator interconnection on Γ_{int}:

$$\boldsymbol{u}_1(t, \overrightarrow{\boldsymbol{x}}) = -\boldsymbol{y}_2(t, \overrightarrow{\boldsymbol{x}}), \qquad \boldsymbol{u}_2(t, \overrightarrow{\boldsymbol{x}}) = \boldsymbol{y}_1(t, \overrightarrow{\boldsymbol{x}}), \quad \forall t > 0, \ \overrightarrow{\boldsymbol{x}} \in \Gamma_{\text{int}}.$$

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The **total Hamiltonian** of the coupled Heat-Wave system is given by:

$$\mathcal{H}(T,\partial_t w, \overrightarrow{\mathbf{grad}} w) := \underbrace{\frac{1}{2}\int_{\Omega_1} T(t,\overrightarrow{\boldsymbol{x}})^2 \,\mathrm{d}\overrightarrow{\boldsymbol{x}}}_{\mathcal{H}_T} + \underbrace{\frac{1}{2}\int_{\Omega_2} \partial_t w(t,\overrightarrow{\boldsymbol{x}})^2 + \left\|\overrightarrow{\mathbf{grad}} w(t,\overrightarrow{\boldsymbol{x}})\right\|^2 \,\mathrm{d}\overrightarrow{\boldsymbol{x}}}_{\mathcal{H}_w}.$$

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Dissipative Power Balance

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H} &= \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_T + \tfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_w \\
&= -\textstyle\int_{\Omega_1} \|\overrightarrow{\boldsymbol{J}}_Q\|^2 + \langle \boldsymbol{u}_1, \boldsymbol{y}_1\rangle_{H^{-\frac{1}{2}}(\Gamma_{\text{int}}),H^{\frac{1}{2}}(\Gamma_{\text{int}})} + \langle \boldsymbol{y}_2, \boldsymbol{u}_2\rangle_{H^{-\frac{1}{2}}(\Gamma_{\text{int}}),H^{\frac{1}{2}}(\Gamma_{\text{int}})} \\
&= -\textstyle\int_{\Omega_1} \|\overrightarrow{\boldsymbol{J}}_Q\|^2 - \langle \boldsymbol{y}_2, \boldsymbol{y}_1\rangle_{H^{-\frac{1}{2}}(\Gamma_{\text{int}}),H^{\frac{1}{2}}(\Gamma_{\text{int}})} + \langle \boldsymbol{y}_2, \boldsymbol{y}_1\rangle_{H^{-\frac{1}{2}}(\Gamma_{\text{int}}),H^{\frac{1}{2}}(\Gamma_{\text{int}})} \\
&= -\textstyle\int_{\Omega_1} \|\overrightarrow{\boldsymbol{J}}_Q\|^2.
\end{aligned}$$

Outline

Coupled model at the discrete level

Let $C := B_{1,\mathrm{int}} M_{\mathrm{int}}^{-1} B_{2,\mathrm{int}}^\top$ the **gyrator interconnection** matrix.

$$\mathrm{Diag}\begin{pmatrix} M_1 \\ \overrightarrow{M}_1 \\ M_2 \\ \overrightarrow{M}_2 \\ M_{bnd,1} \\ M_{bnd,2} \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\underline{T} \\ \underline{J_Q} \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\partial_t w} \\ \frac{\mathrm{d}}{\mathrm{d}t}\underline{\overrightarrow{\mathbf{grad}} w} \\ \underline{0} \\ -\underline{y_w} \end{pmatrix} = \begin{pmatrix} 0 & D_1 & 0 & -C & B_1 & 0 \\ -D_1^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D_2 & 0 & 0 \\ C^\top & 0 & -D_2^\top & 0 & 0 & B_2 \\ -B_1^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_2^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{T} \\ \underline{J_Q} \\ \underline{\partial_t w} \\ \underline{\overrightarrow{\mathbf{grad}} w} \\ \underline{y_T} \\ \underline{0} \end{pmatrix}$$

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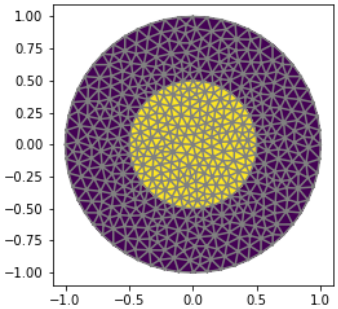
Discrete Lossy Power Balance

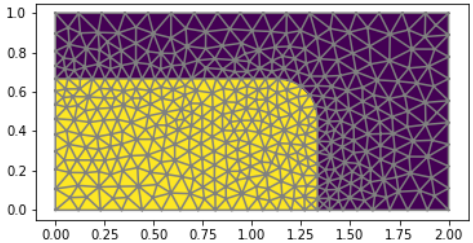
The discrete Hamiltonian $\mathcal{H}^d$ is defined as the continuous Hamiltonian $\mathcal{H}$ evaluated in the *approximated solution*.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}^d(\underline{T}, \underline{\partial_t w}, \underline{\overrightarrow{\mathbf{grad}} w}) = -\underline{J_Q}^\top \overrightarrow{M}_1 \underline{J_Q}.$$

Outline

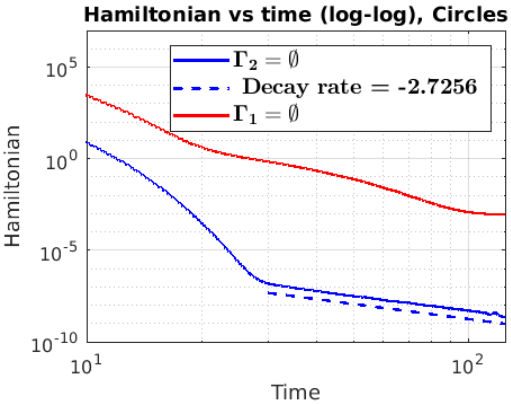
Numerical simulations: configurations

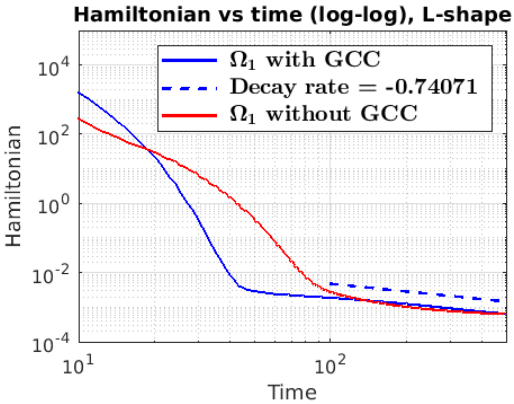




Switching Ω_1 and Ω_2, **four cases** are covered.

Hamiltonian decays





GCC holds, $\Gamma_1 \neq \emptyset$ and $\Gamma_2 = \emptyset$

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Outline

PFEM applies to many more models (1/2)

As soon as J is formally skew-symmetric...

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- **the full von-Kármán plate** $J := \begin{bmatrix} \mathbf{0} & \mathrm{Div} & \mathbf{0} & \mathbf{0} \\ \mathrm{Grad} & \mathbf{0} & -\mathcal{C}(w)^* & \mathbf{0} \\ 0 & \mathcal{C}(w) & 0 & -\mathrm{div}\,\mathrm{Div} \\ \mathbf{0} & \mathbf{0} & \mathrm{Grad}\,\overrightarrow{\mathbf{grad}} & \mathbf{0} \end{bmatrix}$ where

$\mathcal{C}(w)(\boldsymbol{T}) = \mathrm{div}(\boldsymbol{T}\overrightarrow{\mathbf{grad}}w)$

PFEM applies to many more models (2/2)

- **2D Incompressible Navier Stokes equation:** with vorticity ω and stream function ψ.
  $J := \mathsf{curl}_{\mathsf{2D}}(G(\omega)\,\overrightarrow{\mathbf{grad}}^{\perp})$, where $G(\omega) = \omega \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

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Constitutive relations are postponed!
Dissipation **is not** a drawback!

https://g-haine.github.io/scrimp/

scrimp github haine

SCRIMP

0.5.0

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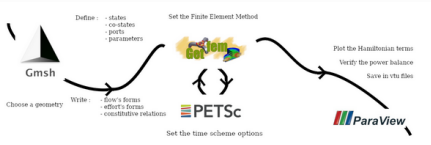
Simulation and ContRol of Interactions in Multi-Physics



What is SCRIMP?

SCRIMP (Simulation and ContRol of Interactions in Multi-Physics) is a python collection, *namely* a package, of *methods* and *classes* for the structure-preserving discretization and simulation of multi-physics models, using the formalism of port-Hamiltonian systems (van der Schaft and Maschke (2002)).

SCRIMP aims at speeding the coding process of the **Partitioned Finite Element Method** on a wide range of (multi-)physical systems (Brugnoli *et al.* (2021)), and scrimp and save time!

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The End

THANK YOU FOR YOUR ATTENTION!

Outline

Co-energy variables and PH-ODEs

In order to transform the PH-DAEs into PH-ODEs, in the linear case, the constitutive relations can be first inverted, second discretized.

$$\overrightarrow{\boldsymbol{\alpha}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) = \overline{\overline{\boldsymbol{T}}}^{-1} \cdot \overrightarrow{\boldsymbol{e}}_q^{ap}(t, \overrightarrow{\boldsymbol{x}}) \quad \text{and} \quad \alpha_p^{ap}(t, \overrightarrow{\boldsymbol{x}}) = \rho\, e_p^{ap}(t, \overrightarrow{\boldsymbol{x}}).$$

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The discretization in the same bases as previously gives:

$$\overrightarrow{\boldsymbol{M}}_q \cdot \underline{\alpha}_q = \overrightarrow{\boldsymbol{M}}_{\overrightarrow{\overline{\overline{\boldsymbol{T}}}}^{-1}} \cdot \underline{e}_q \quad \text{and} \quad M_p \cdot \underline{\alpha}_p = M_\rho \cdot \underline{e}_p,$$

where new mass matrices, or spatial averages, have been defined: $\overrightarrow{\boldsymbol{M}}_{\overrightarrow{\overline{\overline{\boldsymbol{T}}}}^{-1}} := \int_\Omega \overrightarrow{\boldsymbol{\Phi}}_q \cdot \overrightarrow{\overline{\overline{\boldsymbol{T}}}}^{-1} \cdot \overrightarrow{\boldsymbol{\Phi}}_q^\top$ & $M_\rho := \int_\Omega \rho\, \boldsymbol{\phi}_p \cdot \boldsymbol{\phi}_p^\top$.

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The discretized system now is a PH-ODE:

$$\begin{cases} \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}^{-1}}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_q(t) = D\cdot\underline{e}_p(t) + B\cdot\underline{\boldsymbol{u}}(t), \\ M_\rho\cdot\frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_p(t) = -D^\top\cdot\underline{e}_q(t), \\ M_\partial\cdot\underline{\boldsymbol{y}}(t) = B^\top\cdot\underline{e}_q(t), \end{cases}$$

and enjoys the same conservative power balance at the discrete level.

With the same definition the discrete Hamiltonian:

$$\tilde{\mathcal{H}}_d\left(\underline{e}_q, \underline{e}_p\right) := \mathcal{H}\left(\overrightarrow{\boldsymbol{\alpha}}_q^{ap}, \alpha_p^{ap}\right) = \frac{1}{2}\left(\underline{e}_q^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}^{-1}} \cdot \underline{e}_q + \underline{e}_p^\top \cdot M_\rho \cdot \underline{e}_p\right),$$

we can easily compute its time derivative along the trajectories:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{H}}_d\left(\underline{e}_q, \underline{e}_p\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_q\right)^\top \cdot \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}^{-1}} \cdot \underline{e}_q + \left(\frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_p\right)^\top \cdot M_\rho \cdot \underline{e}_p, \\
&= \left(\overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}^{-1}} \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_q\right)^\top \cdot \underline{e}_q + \left(M_\rho \cdot \frac{\mathrm{d}}{\mathrm{d}t}\underline{e}_p\right)^\top \cdot \underline{e}_p, \\
&= \left(D \cdot \underline{e}_p(t) + B \cdot \underline{\boldsymbol{u}}(t)\right)^\top \cdot \underline{e}_q + \left(-D^\top \cdot \underline{e}_q(t)\right)^\top \cdot \underline{e}_p, \\
&= \underline{\boldsymbol{u}}(t)^\top \cdot B^\top \cdot \underline{e}_q \\
&= \underline{\boldsymbol{u}}(t)^\top \cdot M_\partial \cdot \underline{\boldsymbol{y}}(t). \quad \square
\end{aligned}$$

Remark: both definitions do coincide, i.e. $\tilde{\mathcal{H}}_d\left(\underline{e}_q, \underline{e}_p\right) = \mathcal{H}_d\left(\underline{\alpha}_q, \underline{\alpha}_p\right)$, since the discretization of the constitutive relations now provides: $\overrightarrow{\boldsymbol{M}}_q \cdot \underline{\alpha}_q = \overrightarrow{\boldsymbol{M}}_{\overline{\overline{\boldsymbol{T}}}^{-1}} \cdot \underline{e}_q$ and $M_p \cdot \underline{\alpha}_p = M_\rho \cdot \underline{e}_p$ (exercise).

Outline

Accretion: Entropy

Hamiltonian: Entropy

$$\mathcal{S}(\rho(\overrightarrow{\boldsymbol{x}})u(t,\overrightarrow{\boldsymbol{x}})) := \int_\Omega \rho(\overrightarrow{\boldsymbol{x}})s(\rho(\overrightarrow{\boldsymbol{x}})u(t,\overrightarrow{\boldsymbol{x}}))\ \mathrm{d}\overrightarrow{\boldsymbol{x}},$$

Energy variable : $\alpha_u := \rho u$, **co-energy variable** : $e_u := \delta_{\alpha_u}\mathcal{S} = \beta$.

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Power Balance (second law of thermodynamics)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S} = \int_\Omega \sigma - \left\langle \overrightarrow{\boldsymbol{J}}_Q \cdot \overrightarrow{\boldsymbol{n}}, \beta \right\rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}.$$

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Defining $f_u := \partial_t \alpha_u = \rho \partial_t u$, $e_u = \beta$, $\overrightarrow{\boldsymbol{f}}_Q := -\overrightarrow{\mathbf{grad}}\,(\beta)$, and $\overrightarrow{\boldsymbol{e}}_Q := \overrightarrow{\boldsymbol{J}}_Q$

$$\begin{pmatrix} f_u \\ \overrightarrow{\boldsymbol{f}}_Q \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{div} \\ -\overrightarrow{\mathbf{grad}} & 0 \end{pmatrix} \begin{pmatrix} e_u \\ \overrightarrow{\boldsymbol{e}}_Q \end{pmatrix}.$$

Accretion: Entropy

At least two choices for boundary control: e_u or $\overrightarrow{e}_Q \cdot \overrightarrow{n}$.

With *inward* heat flux $\nu = -\overrightarrow{e}_Q \cdot \overrightarrow{n}$, the output is $y = e_u$, *i.e.* the boundary reciprocal temperature, and the discretized system is:

$$\begin{pmatrix} M & 0 & 0 \\ 0 & \overrightarrow{M} & 0 \\ 0 & 0 & M_\partial \end{pmatrix} \begin{pmatrix} \underline{f}_u \\ \underline{f}_Q \\ -\underline{y} \end{pmatrix} = \begin{pmatrix} 0 & \widetilde{D} & \widetilde{B} \\ -\widetilde{D}^\top & 0 & 0 \\ -\widetilde{B}^\top & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{e}_u \\ \underline{e}_Q \\ \underline{\nu} \end{pmatrix}.$$

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***Non-linear* constitutive relations** from: $\rho C_V = \alpha_u e_u$ & $e_u^2 \overrightarrow{\boldsymbol{e}}_Q = -\overline{\overline{\boldsymbol{T}}} \cdot \overrightarrow{\boldsymbol{f}}_Q$.

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Accretive Power Balance (second law of thermodynamics)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{S} = -\int_\Omega \overrightarrow{\boldsymbol{f}}_Q \cdot \overrightarrow{\boldsymbol{e}}_Q + \langle \boldsymbol{\nu}, \boldsymbol{y} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}} \;\geq\; \langle \boldsymbol{\nu}, \boldsymbol{y} \rangle_{H^{-\frac{1}{2}}, H^{\frac{1}{2}}}.$$

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Accretive Power Balance (second law of thermodynamics)

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Outline

The irrotational shallow water equations

* Energy variables: α_h the fluid height, $\boldsymbol{\alpha}_v$ the linear momentum,

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* Non-quadratic and non-separable Hamiltonian functional:

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* Dynamical system:

$$\frac{\partial}{\partial t} \begin{pmatrix} \alpha_h \\ \boldsymbol{\alpha}_v \end{pmatrix} = \begin{bmatrix} 0 & -\mathrm{div} \\ -\overrightarrow{\mathbf{grad}} & \mathbf{0} \end{bmatrix} \begin{pmatrix} e_h \\ \mathbf{e}_v \end{pmatrix}, \qquad (x, y) \in \Omega = \{x^2 + y^2 \leq R\},$$

$$\begin{pmatrix} e_h \\ \mathbf{e}_v \end{pmatrix} = \begin{pmatrix} \delta_{\alpha_h} H \\ \delta_{\boldsymbol{\alpha}_v} H \end{pmatrix} = \begin{pmatrix} \frac{1}{2\rho} \left\| \boldsymbol{\alpha}_v \right\|^2 + \rho g \alpha_h \\ \frac{1}{\rho} \alpha_h \boldsymbol{\alpha}_v \end{pmatrix},$$

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* Consider a uniform Neumann boundary control

$$u_\partial = -\mathbf{e}_v \cdot \mathbf{n}|_{\partial\Omega} = -\frac{1}{\rho} \alpha_h \boldsymbol{\alpha}_v \cdot \mathbf{n}|_{\partial\Omega}, \qquad \text{Volumetric inflow rate.}$$

The corresponding output reads

$$y_\partial = e_h|_{\partial\Omega} = (\rho g \alpha_h + \frac{1}{2\rho} \left\| \boldsymbol{\alpha}_v \right\|^2)|_{\partial\Omega}.$$

Outline

Numerics: PFEM in the polynomial case

The difficulty lies in the non-linear nature of both constitutive relations. However, since they remain polynomial, off-line Finite Element computations can be performed, and makes possible the online computation of the discrete constitutive relations at each time step.

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A general result

$$M_h \cdot \underline{e_h} := \nabla_{\underline{\alpha_h}} \mathcal{H}_d(\underline{\alpha_h}, \underline{\alpha_v}), \text{ and } M_v \cdot \underline{e_v} := \nabla_{\underline{\alpha_v}} \mathcal{H}_d(\underline{\alpha_h}, \underline{\alpha_v}).$$

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Here quadratic quantities have to be computed in the integrals, namely $\underline{q_h} := \int_\Omega \boldsymbol{\phi}_h \frac{1}{2\rho} \underline{\alpha_v}^\top \cdot \overrightarrow{\boldsymbol{\Phi}}_v \cdot \overrightarrow{\boldsymbol{\Phi}}_v^\top \cdot \underline{\alpha_v}$ and $\underline{q_v} := \int_\Omega \overrightarrow{\boldsymbol{\Phi}}_v \frac{1}{\rho} \underline{\alpha_h}^\top \cdot \boldsymbol{\phi}_h \cdot \overrightarrow{\boldsymbol{\Phi}}_v^\top \cdot \underline{\alpha_v}$.

$$\Rightarrow 1 \leq i \leq N_h, \quad q_h^i(t) = \underline{\alpha_v}(t)^\top \cdot \left(\int_\Omega \phi_h^i \frac{1}{2\rho} \overrightarrow{\boldsymbol{\Phi}}_v \cdot \overrightarrow{\boldsymbol{\Phi}}_v^\top \right) \cdot \underline{\alpha_v}(t),$$

$$\Rightarrow 1 \leq k \leq N_v, \quad q_v^k(t) = \underline{\alpha_h}(t)^\top \cdot \left(\int_\Omega \boldsymbol{\phi}_h \frac{1}{\rho} \overrightarrow{\boldsymbol{\phi}}_v^k{}^\top \cdot \overrightarrow{\boldsymbol{\Phi}}_v^\top \right) \cdot \underline{\alpha_v}(t).$$

Remark: the sizes of the vectors and matrices do match as well (exercise).
$\Longrightarrow$ Off-line computation proves possible!

Outline

Boundary stabilization of the 2D SWE

A simple proportional control stabilizes the system around the desired point h^{des}

$$u_\partial = -k(y_\partial - y_\partial^{\text{des}}), \qquad y_\partial^{\text{des}} = \rho g h^{\text{des}}, \quad k > 0.$$

This control law ensures that the Lyapunov functional

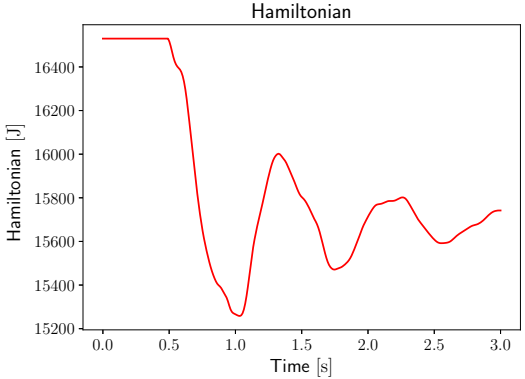
$$V = \frac{1}{2}\int_\Omega \left\{ \frac{1}{2}\rho g(\alpha_h - \alpha_h^{\text{des}})^2 + \frac{1}{2\rho}\alpha_h \left\|\boldsymbol{\alpha}_v\right\|^2 \right\} \, \mathrm{d}\Omega \geq 0,$$

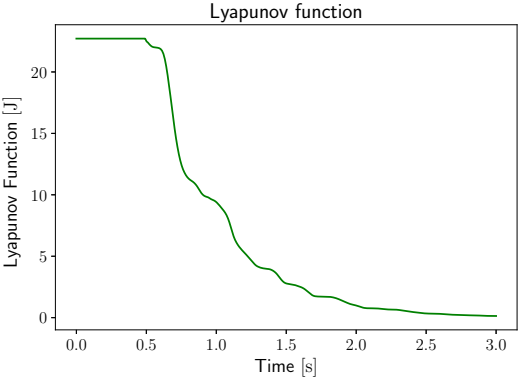
where $\alpha_h^{\text{des}} = h^{\text{des}}$, has negative semi-definite time derivative

$$\dot{V} = -k \int_{\partial\Omega} \left(y_\partial - y_\partial^{\text{des}}\right)^2 \, \mathrm{d}\Gamma \leq 0.$$

Simulation Results for the 2D SWE

Simulation Results for the 2D SWE





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