Summer School on Positive Operator Semigroups

September 4 – 8, 2023

Exercise sheet 10: Numerical analysis and positive semigroups

For $d \in \mathbb{N}$, let $Q\colon \mathbb{R}^d \to \mathbb{R}^d$ be a continuous mapping such that $Q(w) \in \mathrm{R}^{d\times d}$ is a Metzler matrix for all $w \in \mathbb{R}^d$ positive vectors (that is, its off-diagonal elements are nonnegative). For $\delta > 0$, we consider the quasilinear delay equation on $\mathbb{R}^n$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Q\big(u(t-\delta)\big)u(t), & t \geq 0 \\ u(s) = \varphi(s), & s \in [-\delta, 0] \end{cases} \tag{D}$$

where $\varphi\colon [-\delta, 0] \to \mathbb{R}^d$ is given.

We choose $N \in \mathbb{N}$ and time step $\tau > 0$ such that $\delta = N\tau$. The approximate solution is denoted by $u(n\tau) \approx: u_n^{(\tau)}$ (where we will omit the τ in the upper index). Then the Magnus method has the form

$$u_{n+1/2} := \begin{cases} \varphi((n+1/2)\tau - \delta), & n = 0, \dots, N-1 \\ \mathrm{e}^{\frac{\tau}{2} Q(\varphi(n\tau - 2\delta))} u_{n-N}, & n = N, \dots, 2N-1 \\ \mathrm{e}^{\frac{\tau}{2} Q(u_{n-2N})} u_{n-N}, & n \geq 2N \end{cases} \tag{M}$$

$$u_{n+1} := \mathrm{e}^{\tau Q(u_{n+1/2})} u_n, \quad n = 0, 1, 2, \dots \quad \text{with} \quad u_0 = \varphi(0).$$

Theorem:

The Magnus method (M) preserves the positivity when applied to the quasilinear delay equation (D).

Exercise 1 (Epidemic model as quasilinear delay equation).
For $\delta > 0$ latent period, let $S, I, R\colon [-\delta, 0] \to \mathbb{R}$, $S(t), I(t), R(t)$ denote the ratios of susceptible, infected, and recovered (or immune) individuals of a population at time $t \geq 0$, respectively, and $\varphi_S, \varphi_I, \varphi_R$ their initial value on the time interval $[-\delta, 0]$. We denote by $\beta, \gamma, \nu > 0$ the infection, recovery, and vaccination rates, respectively. Then a simple epidemic model (based on the idea of Kermack and McKendrick) has the form

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} S(t) = -\beta S(t) I(t-\delta) - \nu S(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} I(t) = \beta S(t) I(t-\delta) - \gamma I(t) \\ \frac{\mathrm{d}}{\mathrm{d}t} R(t) = \nu S(t) + \gamma I(t) \end{cases} \tag{E}$$

Rewrite the epidemic model (E) into the form (D) of a quasilinear delay equation.

Hint: Find $u(t)$ and $Q(w)$.

Exercise 2 (Magnus method for the epidemic model).
Write down the Magnus method (M) when applied to the epidemic model (E). Do we need to store the values of S and/or R (besides I, of course)?

Hint: The exponential of a matrix of special structure has a special structure, too.

Exercise 3 (Positivity preservation of Magnus method for the epidemic model).
Show that the Magnus method (M) preserves the positivity when applied to the epidemic model (E).

Hint: Use the previous exercises and the Theorem above.

Supplementary exercise (Positivity preservation of Magnus method for quasilinear delay equations).

Prove the Theorem above: the Magnus method (M) preserves the positivity when applied to the quasilinear delay equation (D).

Hint: Use the method of steps, i.e., show the assertion for $n = kN, \dots, (k+1)N - 1$ for each $k = 0, 1, 2, \dots$