Eötvös Loránd University
Faculty of Science
Institute of Mathematics

BERGISCHE UNIVERSITÄT WUPPERTAL

Summer School on Positive Operator Semigroups

September 4 – 8, 2023

Exercise Session I
Eventual Positivity

Exercise 1 (A semigroup that is (non-uniformly) eventually positive). Let $E := C([-1,1],\mathbb{C})$ be the space of continuous functions on the interval $[-1,1]$ and let $F \subset E$ denote the kernel of the functional $\varphi : f \mapsto \int_{-1}^{1} f$.
Let $S : F \to F$ denote the reflection operator, i.e., $(Sf)(x) = f(-x)$ for all $f \in E$ and all $x \in [-1,1]$.

(a) Show that $E = \operatorname{span}\{\mathbb{1}\} \oplus F$ which allows us to define the bounded operator

$$A := \mathbf{0}\big|_{\operatorname{span}\{\mathbb{1}\}} \oplus (-2I\big|_F - S)$$

on E.

(b) As A is a bounded operator, it generates a C_0-semigroup. Show that the semigroup generated by A is given by

$$T(t) = I\big|_{\operatorname{span}\{\mathbb{1}\}} \oplus e^{-2t}(\cosh(t)I\big|_F - \sinh(t)S)$$

for all $t \geq 0$.

(c) Show that $T(t) \to 0$ in the operator norm on $\ker P$ where $P := \frac{1}{2}\,\mathbb{1} \otimes \varphi$. Conclude that $(T(t))_{t\geq 0}$ is individually eventually positive.

(d) For each $n \in \mathbb{N}$, choose $f_n \in E_+$ with

$$\|f_n\|_\infty = 1, \varphi(f_n) = \frac{1}{n}, f_n(-1) = 0, \text{ and } f_n(1) = 1.$$

Use the sequence $(f_n)_{n\in\mathbb{N}}$ to justify that $(T(t))_{t\geq 0}$ is not uniformly eventually positive.