Summer School on Positive Operator Semigroups

September 4 – 8, 2023

Exercise Sessions G–H: Koopman Semigroups and Ergodic Theory

Exercise 1. Let K be a compact Hausdorff space and let $(A, D(A))$ be the generator of a positive C_0-semigroup T on $X \coloneqq C(K)$. We endow X with sup norm.
Let $f, g \in X$ and define $\xi : [0, \infty) \to X$ by $\xi(s) \coloneqq T(s)f \cdot T(s)g$ for all $s \in [0, \infty)$.

(a) Show that ξ is continuous.

(b) Show that ξ is continuously differentiable if $f, g \in D(A)$. Compute the derivative $\xi' : [0, \infty) \to X$ in this case.

(c) Now let, in addition, $t_0 > 0$ and define $\eta : [0, t_0] \to X$ by

$$\eta(s) \coloneqq T(t_0 - s)\big[T(s)f \cdot T(s)g\big]$$

for all $s \in [0, t_0]$.
Show that η is continuously differentiable if $f, g \in D(A)$. Compute the derivative $\eta' : [0, t_0] \to X$ in this case.

Exercise 2. Consider the spaces

$$X_1 \coloneqq C[0, \infty] \simeq \big\{f \in C[0, \infty) \,|\, \lim_{r \to \infty} f(r) \text{ exists}\big\} \qquad \text{and} \qquad X_2 \coloneqq C[0, 1]$$

with the supremum norm. For each $k \in \{1, 2\}$ let A_k be the differential operator $A_k f \coloneqq f'$ on X_k with its maximal domain.

(a) Show that A_1 and A_2 are derivations.

(b) Show that A_1 generates a Koopman semigroup on X_1 and determine the corresponding flow.

(c) Show that A_2 is not a generator on X_2.

(d) Use the flow from (b) to find a subspace of X_2 on which A_2 generates a Koopman semigroup.

Exercise 3. On $K \coloneqq [0, \infty]$ we take the flow $(\varphi_t)_{t \geq 0}$ that is defined as

$$\varphi_t(x) \coloneqq \begin{cases} e^t x & \text{if } x < \infty, \\ \infty & \text{if } x = \infty. \end{cases}$$

for all $t \geq 0$.

(a) Determine the generator of the corresponding Koopman semigroup and find some properties of this semigroup.

(b) Are there other "multiplicative perturbations" of the first derivative on $\mathbb{R}_+$?

Exercise 4. Take $X = C[0,\infty]$ and the operator $A_1 f := f'$ from Exercise 2. Let $m \in X$. Then the additive perturbation $B := A_1 + m$ generates a semigroup T on X that is given by

$$T(t)f(x) = \exp\Big(\int_0^t m(x+s)\mathrm{d}s\Big)f(x+t)$$

for all $f \in X$, $t \in [0,\infty)$ and $x \in [0,\infty]$.
Generalise this result to general Koopman semigroups.

Exercise 5. Let A be the generator of a Koopman semigroup T on $X = C(K)$ for a compact Hausdorff space K. Show that the spectrum $\sigma(A)$ of A is either the left half plane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \leq 0\}$ or a union of additive subgroups of $\mathrm{i}\mathbb{R}$.