Summer School on Positive Operator Semigroups

September 4 – 8, 2023

Exercise Sessions E–F: Positive Semigroups on Banach Lattices

Exercise 1. Let A be the generator of a positive C_0-semigroup T on a Banach lattice E. Recall that there exist numbers $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\|_{\mathcal{L}(E)} \leq Me^{t\omega}$ for all $t \geq 0$. Hence, for all $\lambda \in (\omega, \infty)$ the resolvent $\mathcal{R}(\lambda, A) := (\lambda I - A)^{-1}$ exists and satisfies $\|\mathcal{R}(\lambda, A)\|_{\mathcal{L}(E)} \leq \frac{M}{\lambda - \omega}$.

(a) Deduce that $f = \lim_{\lambda \to \infty} \lambda \mathcal{R}(\lambda, A) f$ for all $f \in E$.

(b) Set $D(A)_+ := D(A) \cap E_+$. Show that $D(A) = D(A)_+ - D(A)_+$.

(c) Show that $\overline{D(A)_+} = E_+$.

(d) Show the *positive minimum principle*: if $0 \leq f \in D(A)$ and $0 \leq \varphi \in E'$ satisfy $\langle \varphi, f \rangle = 0$, then $\langle \varphi, Af \rangle \geq 0$.

(e) Assume that $Af \geq 0$ for all $f \in D(A)_+$. Show that $D(A) = E$.

Hints for (e): Let $f \in D(A)_+$. Show the following things:

1. For every $\mu \in \rho(A)$ one has
$$A\mathcal{R}(\mu, A)f = \mu^2 \mathcal{R}(\mu, A)f - \mu f - Af.$$
2. One has $\|Af\|_E \leq c\|f\|_E$ for an f-independent number $c \geq 0$.
3. One has $\|(\mu\mathcal{R}(\mu, A) - I)f\|_E \leq \frac{c}{\mu}\|f\|_E$ for all $\mu > \omega$.
4. One has $\|\mu\mathcal{R}(\mu, A) - I\|_{\mathcal{L}(E)} < 1$ for all sufficiently large $\mu > \omega$.
5. For μ as in Step 4., the mapping $\mu\mathcal{R}(\mu, A)$ is surjective from E to E.

Recall that a linear operator $A\colon E \supseteq D(A) \to E$ on a Banach lattice E is called *dispersive* if for all $u \in D(A)$ there exists $0 \leq \varphi \in E'$ such that $\|\varphi\| \leq 1$, $\langle \varphi, u \rangle = \|u^+\|$, and $\langle \varphi, Au \rangle \leq 0$. Also recall the following result:

Theorem 1. For a linear operator $A\colon E \supseteq D(A) \to E$ on a Banach lattice E the following assertions are equivalent:

(i) *The operator A generates of positive contractive C_0-semigroup on E.*

(ii) *The operator A is dispersive and satisfies $\overline{D(A)} = E$, and there exists a number $\lambda > 0$ such that* $\operatorname{ran}(\lambda I - A) = E$.

Exercise 2. Let $A\colon E \supseteq D(A) \to E$ be a linear operator on a Banach lattice E.

(a) Assume that A is dissipative and that one has $u^+ \in D(A)$ for each $u \in D(A)$. Assume moreover that A satisfies the *positive minimum principle* (see Exercise 1(d)). Show that A is dispersive.

(b) Let $A \in \mathcal{L}(E)$ and assume that A satisfies the positive minimum principle. Show that $e^{tA} \geq 0$ for each $t \geq 0$.
Hint: Observe that $A - \|A\|\, I$ is dissipative.

(c) Let $E = L^2(\Omega, \mu)$ for a measure space (Ω, μ). Show that A is dispersive if and only if $\big(Au\big|u^+\big) \leq 0$ for all $u \in D(A)$.

For the following exercises we need Sobolev spaces on intervals. Let $I \subseteq \mathbb{R}$ be a non-empty open interval. Let $\mathcal{D}(I) := C_c^\infty(I)$ denote the space of all infinitely differentiable scalar-valued functions on I that have compact support in I. We define

$$H^1(I) := \Big\{u \in L^2(I)\Big|\ \exists u' \in L^2(I)\ \forall \varphi \in \mathcal{D}(I) :\ \int_I u'\varphi = -\int_I u \frac{\mathrm{d}}{\mathrm{d}x}\varphi\Big\}.$$

Note that u' in this definition is uniquely determined if it exsits since $\mathcal{D}(I)$ is dense in $L^2(I)$. For instance, every function $u \in C^1(\bar{I}) \cap L^2(I)$ is in $H^1(I)$ and satisfies $\frac{\mathrm{d}}{\mathrm{d}x}u = u'$.
The space $H^1(I)$ is a Hilbert space with respect to the inner product given by

$$\big(u\big|v\big)_{H^1(I)} := \big(u\big|v\big)_{L^2(I)} + \big(u'\big|v'\big)_{L^2(I)}$$

for all $u, v \in H^1(I)$. The following theorem collects important properties of Sobolev spaces in one dimension over bounded intervals:

Theorem 2. Let $-\infty < a < b < \infty$. We use the abbreviation $H^1(a,b) := H^1\big((a,b)\big)$.

(a) *One has $H^1(a,b) \subseteq C[a,b]$.*

(b) *For all $u, v \in H^1(a,b)$ one has*

$$\int_a^b u'v = -\int_a^b uv' + u(b)v(b) - u(a)v(a).$$

(c) *For every $u \in H^1(a,b)$ one has $u^+ \in H^1(a,b)$ and*

$$(u^+)' = \mathbb{1}_{u>0}\, u'.$$

(d) *Let $H_0^1(a,b)$ denote the closure of $\mathcal{D}(a,b)$ with respect to the H^1-norm. Then $H_0^1(a,b) = \big\{u \in H^1(a,b)\big|\ u(a) = u(b) = 0\big\}$.*

Exercise 3. Let $-\infty < a < b < \infty$.

(a) Prove part (c) of Theorem 2 for the special case of a function $u \in C^1[a,b]$ that vanishes at precisely one point.

(b) Let $u \in H^1(a,b)$. Show that $u \in C^1[a,b]$ if and only if $u' \in C[a,b]$.

(c) Let $u \in H^1(a,b)$ and define

$$H^2(a,b) := \{u \in H^1(a,b) \,|\, u' \in H^1(a,b)\}.$$

Show that $H^2(a,b) \subseteq C^1[a,b]$ and that a function $u \in H^2(a,b)$ satisfies $u \in C^2[a,b]$ if and only if $u'' \in C[a,b]$.

For the rest of the exercise we let, for the sake of simplicity, $a = 0$ and $b = 1$.

(d) Let $f \in L^2(0,1)$. Show that there exists a unique $u \in H^2(0,1)$ such that

$$u - u'' = f \qquad \text{and} \qquad u'(0) = u'(1) = 0.$$

Show moreover that $u \in C^2[0,1]$ if and only if $f \in C[0,1]$.
Hint: Use the Riesz–Fréchet representation theorem in the Hilbert space $H^1(0,1)$ to find a $u \in H^1(0,1)$ that satisfies $\int_0^1 fv = \int_0^1 uv + \int_0^1 u'v'$ for all $v \in H^1(0,1)$.

(e) Define the operator A on $L^2(0,1)$ by

$$\begin{aligned} D(A) &:= \{u \in H^2(0,1) | u'(0) = u'(1) = 0\}, \\ Au &:= u''. \end{aligned}$$

The operator A is called the *Neumann Laplacian* on $L^2(0,1)$.
Show that A generates a contractive positive C_0-semigroup on $L^2(0,1)$.

Exercise 4 (The Neumann Laplacian on the space of continuous functions).
Define the operator $B\colon C[0,1] \supseteq D(B) \to C[0,1]$ by

$$\begin{aligned} D(B) &:= \{u \in C^2[0,1] \,|\, u'(0) = u'(1) = 0\}, \\ Bu &:= u''. \end{aligned}$$

(a) Show that B is dispersive.
Hint: For $u \in D(B)$ choose $x_0 \in [0,1]$ such that $u(x_0) = \|u^+\|_\infty$ and consider the point evaluation map at x_0.

(b) Show that $D(B)$ is dense in $C[0,1]$.
Hint: Use the Stone–Weierstraß approximation theorem.

(c) Show that B generates a contractive positive C_0-semigroup S on $C[0,1]$.

(d) Let A denote the operator on $L^2(0,1)$ from Exercise 3(e). Show that $\mathcal{R}(\lambda, A)C[0,1] \subseteq C[0,1]$ and $\mathcal{R}(\lambda, A)|_{C[0,1]} = \mathcal{R}(\lambda, B)$ and each $\lambda > 0$.

(e) Let T denote the semigroup on $L^2(0,1)$ generated by A. Show that $T(t)C[0,1] \subseteq C[0,1]$ and $T(t)|_{C[0,1]} = S(t)$ for each $t \geq 0$.

For the following exercise, recall that the dual operator A' of a linear operator A has the same spectrum as A and that $\mathcal{R}(\lambda, A)' = \mathcal{R}(\lambda, A')$ for all $\lambda \in \rho(A) = \rho(A')$. Also recall that a vector u in a Banach lattice E is positive if and only if $\langle \varphi, u \rangle \geq 0$ for all $0 \leq \varphi \in E'$.

Exercise 5. Let A be the generator of a positive C_0-semigroup T on a Banach lattice E and let $u \in E$. Assume that $\langle \varphi, u \rangle \geq 0$ for all $0 \leq \varphi \in D(A')$. Show that $u \geq 0$.

Exercise 6. Let A be the generator of a C_0-semigroup T on a Banach space E, let $u \in \ker A$ and $\varphi \in \ker A'$ and let $\langle \varphi, u \rangle = 1$. Define an operator $P \in \mathcal{L}(E)$ by

$$Pf := \langle \varphi, f \rangle u$$

for all $f \in E$. Show that P is a projection and that $T(t)P = PT(t) = P$ for all $t \geq 0$.

Exercise 7. Let T be a positive C_0-semigroup with generator A on a Banach lattice E. One can prove that, due to the positivity,

$$\mathcal{R}(\lambda, A)u = \lim_{T \to \infty} \int_0^T e^{-\lambda t} T(t) u \,\mathrm{d}t =: \int_0^\infty e^{-\lambda t} T(t) u \,\mathrm{d}t$$

for each $u \in E$ and each complex number λ that satisfies $\operatorname{Re} \lambda > s(A)$.

(a) Show that $\mathcal{R}(\lambda, A) \geq 0$ for each real number $\lambda > s(A)$.

(b) Show that $|\mathcal{R}(\lambda, A)u| \leq \mathcal{R}(\operatorname{Re} \lambda, A)\,|u|$ for all $u \in E$ and all λ that satisfy $\operatorname{Re} \lambda > s(A)$.

(c) Let $\lambda_0 \in \mathbb{C}$ and let (λ_n) be a sequence in the resolvent set $\rho(A)$ that converges to λ_0. Show that if $\sup_{n \in \mathbb{N}} \|\mathcal{R}(\lambda_n, A)\| < \infty$, then $\lambda_0 \in \rho(A)$.
Hint: Use the classical resolvent estimate $\operatorname{dist}(\lambda, \sigma(A)) \geq \frac{1}{\|\mathcal{R}(\lambda, A)\|}$ that holds for all $\lambda \in \rho(A)$.

(d) Show that there exists a sequence (λ_n) in $\mathbb{C}$ such that $\operatorname{Re} \lambda_n \downarrow s(A)$ and such that $\sup_{n \in \mathbb{N}} \|\mathcal{R}(\lambda_n, A)\| = \infty$.
Hint: Use the same resolvent estimate as in the hint of part (c).

(e) Assume that $s(A) > -\infty$. Show that $s(A) \in \sigma(A)$.