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BERGISCHE UNIVERSITÄT WUPPERTAL

Summer School on Positive Operator Semigroups

September 4 – 8, 2023

Exercise Sessions A–D: The Finite-Dimensional Case and Introduction to Infinite-Dimensional Banach Lattices

Exercise 1. Let A be a $n \times n$ diagonalisable matrix with m distinct eigenvalues $\lambda_1, \ldots, \lambda_m$. Prove that in this case its spectral projections are of the form

$$P_i = \prod_{j \neq i} \frac{A - \lambda_j}{\lambda_i - \lambda_j}, \quad i = 1, \ldots, m.$$

Exercise 2. Prove that if $|A| \leq B$, then the following inequalities hold

$$\|A\| \leq \|B\| \quad \text{and} \quad \mathrm{r}(A) \leq \mathrm{r}(|A|) \leq \mathrm{r}(B).$$

Exercise 3. Show that if there exists an operator norm $\|\cdot\|$ on $M_n(\mathbb{C})$ such that $\|T\| < 1$, then the sequence (T^k) is stable.

Exercise 4. Describe the asymptotic behavior of sequence (T^k) for the following special classes of matrices $T \in M_n(\mathbb{R})$.

(a) T is idempotent (or involutary), i.e., $T^2 = I$.

(b) T is nilpotent, i.e., $T^q = 0$ for some $q \in \mathbb{N}$.

(c) T is unipotent, i.e., $T - I$ is nilpotent.

(d) T is orthogonal, i.e., $T^\top T = TT^\top = I$.

Exercise 5. For each of the following matrices determine whether its powers are convergent or Cesàro summable. Evaluate the limit of each convergent matrix and the Cesàro limit of each summable matrix.

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A_3 = \frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 2 & 0 & -1 \\ 2 & -2 & 1 \end{pmatrix}.$$

Exercise 6. Let $T \geq 0$. Prove that, for $\mu \in \rho(T)$,

$$R(\mu, T) \geq 0 \quad \text{implies} \quad \mu > \mathrm{r}(T).$$

Exercise 7. Show that, if $a_1, \ldots, a_n \in \mathbb{C}$ are all non-zero, then the following matrix is irreducible:

$$\begin{pmatrix} 0 & a_1 & 0 & \dots & 0 \\ 0 & 0 & a_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & 0 & a_{n-1} \\ a_n & 0 & \dots & 0 & 0 \end{pmatrix}.$$

Exercise 8. Show that $T \geq 0$ is irreducible if and only if the eigenspaces of T and of $T^\top$ belonging to $\mathrm{r}(T) = \mathrm{r}(T^\top)$ are one dimensional and spanned by a strictly positive vector.

Exercise 9. Let T be a positive irreducible matrix. Prove that, if the trace $\mathrm{tr}\, T > 0$, then T is primitive.

Exercise 10. Verify irreducibility and imprimitivity of the matrices T_i, $i = 1, 2$, below and discuss the asymptotic behavior of the sequence $\left((T_i/\mathrm{r}(T_i))^k\right)$.

$$T_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Exercise 11 (Matrix exponential function).

(a) Characterize those matrices $A \in M_n(\mathbb{C})$ for which e^{tA} is positive for all $t \in \mathbb{R}$.

(b) Find all positive periodic matrix semigroups, and all positive, periodic, irreducible matrix semigroups.

Exercise 12. Consider the Competitive Markets Model given by

$$p(t) = p^0 + e^{tKA}c, \quad t \geq 0, \quad \text{where} \quad c = p(0) - p^0,$$

where p^0 are equilibrium prices, $p(0)$ initial prices, $K = \mathrm{diag}(k_1, \ldots, k_n)$ a diagonal matrix of positive adjustment speeds while for the coefficients of the matrix $A = (a_{ij})$ we have

$$a_{ij} \geq 0 \text{ for } i \neq j \text{ and } a_{ii} < 0.$$

List the conditions for the matrix A under which the prices will behave periodically.

Exercise 13. Assume that we have a group of individuals who are arranged in the vertices of a graph and the disease can spread along the edges according to the differential equation

$$\dot{y}(t) = (\eta G - \mu I)y(t),$$

where G is the (weighted) adjacency matrix of the graph. Each individual can recover from he illness with a rate of $\mu = 1/4$. Discuss the role of the infection rate η, if he graph is

(a) a complete graph with 4 vertices,

(b) a cycle of length 5 (regular pentagon),

(c) a cube (8 vertices),

(d) the graph in Figure 1.

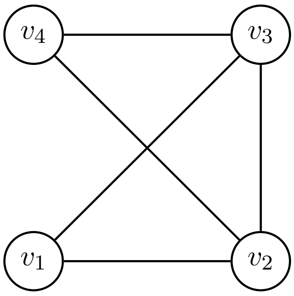

Figure 1: The graph in Exercise 13.

Exercise 14 (Properties of vector lattices). Let E be a vector lattice and x, y, $z \in E$. Prove

(a) $x \vee y = \frac{1}{2}(x + y + |x - y|)$, and $x \wedge y = \frac{1}{2}(x + y - |x - y|)$.

(b) $|x| \vee |y| = \frac{1}{2}(|x + y| + |x - y|)$ and deduce that

$$|x| \wedge |y| = \frac{1}{2}(|x + y| - |x - y|).$$

(c) Deduce that $x \perp y$ is equivalent to $|x - y| = |x + y|$.

(d) The triangle inequality: $||x| - |y|| \leq |x + y| \leq |x| + |y|$.

(e) Deduce that $x \perp y$ is equivalent to $|x| \vee |y| = |x| + |y|$ and in this case $||x| - |y|| = |x + y| = |x| + |y|$.

(f) Birkhoff's inequalities: $|x \vee z - y \vee z| \leq |x - y|$ and $|x \wedge z - y \wedge z| \leq |x - y|$.

Exercise 15. Let us consider the Banach space $E := C^1[0,1]$ of continuously differentiable functions on $[0,1]$ with the norm

$$\|f\| = \max_{s\in[0,1]} |f(s)| + \max_{s\in[0,1]} |f'(s)|$$

and the natural order $f \geq 0$ if $f(s) \geq 0$ for all $s \in [0,1]$. Prove that E is not a vector lattice.

Exercise 16. Consider $C^1[0,1]$ equipped with the norm

$$\|f\| = \max_{s\in[0,1]} |f'(s)| + |f(0)|$$

and the order $f \geq 0$ whenever $f(0) \geq 0$ and $f' \geq 0$. Show that $E := (C^1[0,1], \geq, \|\cdot\|)$ is a Banach lattice.

Exercise 17. Let E be a Banach lattice. Then,

(a) the lattice operations are continuous,

(b) the positive cone E_+ is closed, and

(c) order intervals are closed and bounded.

Exercise 18 (Properties of ideals). Prove that a subspace I of a Banach lattice is an ideal if and only if

$$[\, x \in I,\ |y| \leq |x| \,] \Longrightarrow y \in I.$$

Exercise 19. Show that an operator is positive, i.e., $TE_+ \subset F_+$, if and only if $|Tx| \leq T|x|$ holds for all $x \in X$.

Exercise 20. Consider the Banach lattice given in Exercise 16 and define the operator

$$(Tf)(t) := \int_0^t g(s)f(s)ds$$

with a given $g \in C[0,1]$. Calculate $\|T\|$. For which g is T positive?

Exercise 21. Show that: if E is a Banach lattice and $S, T : E \to E$ positive operators, then

$$r(S+T) \geq \max\{r(S), r(T)\}.$$

Exercise 22. Give an example of a Banach lattice E, a positive operator T, and an invariant ideal $J \subset E$ such that there is $\lambda \in \rho(T)$ for which J is not $R(\lambda, T)$-invariant.