13. Exercise Sheet in **Ordered Banach Spaces and Positive Operators**

For the exercise classes on July 11 and 12, 2023

Exercise 1 (Complexifications of function spaces).

(a) Endow $E := \mathbb{R}^2$ with the 1-norm and identify the complex space $E_{\mathbb{C}} = \mathbb{R}^2 \times \mathbb{R}^2$ constructed in the proof of Proposition 7.1.3(a) with the space $\mathbb{C}^2$ in the canonical way.
Show that the norm on $E_{\mathbb{C}}$ constructed in the proof of Proposition 7.1.3(a) does not coincide with the 1-norm on $\mathbb{C}^2$.

(b) Let $K \neq \emptyset$ be a compact metric space (or, more generally, a compact Hausdorff space) and endow the space $E := \mathrm{C}(K)$ with the sup norm. Identify the complex space $E_{\mathbb{C}} = \mathrm{C}(K) \times \mathrm{C}(K)$ constructed in the proof of Proposition 7.1.3(a) with the space $\mathrm{C}(K;\mathbb{C})$ in the canonical way.
Show that the norm on $E_{\mathbb{C}}$ constructed in the proof of Proposition 7.1.3(a) coincides with the sup norm on $\mathrm{C}(K;\mathbb{C})$.

Exercise 2 (Monotonicity of the spectral radius). Let E be an ordered Banach space and let $S, T \in \mathcal{L}(E)$ satisfy $0 \leq S \leq T$.

(a) Prove that if E_+ is generating and normal, then $\mathrm{r}(S) \leq \mathrm{r}(T)$.

(b) Show by means of a counterexample that the conclusion from part (a) does not longer hold, in general, if E_+ is only assumed to be generating.
Hint: Adapt Example 7.2.2.

Exercise 3 (A compact operator acting on the self-adjoint operators). Let $H \neq \{0\}$ be a complex Hilbert space and endow $\mathcal{L}(H)_{\mathrm{sa}}$ with the Loewner order and the operator norm. Fix a bounded linear operator $K \in \mathcal{L}(H)$. Consider the positive operator $T : \mathcal{L}(H)_{\mathrm{sa}} \to \mathcal{L}(H)_{\mathrm{sa}}$, $A \mapsto K^*AK$.

(a) Show that the complex extension $T_{\mathbb{C}}$ of T to the complexification $\mathcal{L}(H)$ of $\mathcal{L}(H)_{\mathrm{sa}}$ is also given by $T_{\mathbb{C}}A = K^*AK$ for all $A \in \mathcal{L}(H)$.

(b) Show that $\mathrm{r}(T) = \mathrm{r}(K)^2$.
Hint: Use the Beurling formula for the spectral radius.

(c) Assume now that K is compact and that $\mathrm{r}(K) > 0$. Using the compactness of K one can show that T is also compact (but this is not part of this exercise).
So there is, by part (b) of this exercise together with the Krein–Rutman theorem, a non-zero element $0 \leq A \in \mathcal{L}(H)_{\mathrm{sa}}$ such that $TA = \mathrm{r}(K)^2 A$.
Can you construct such an A in terms of the eigenvectors of K^*?

Exercise 4 (The spectral radius on C^1). Is there a positive linear operator T on $C^1([-1,1])$ such that $\mathrm{r}(T) \notin \sigma(T)$?
Warning: I do not know the answer and I have not thought about it, so I do not know whether the question is easy or difficult.