12. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on July 4 and 5, 2023

with Solutions

Exercise 1 (Holomorphic functions, again). Consider the space $\mathcal{H}^\infty(\mathbb{D})$ and the cone on it that was introduced in Exercise 4 on Sheet 6. Is there an equivalent norm on $\mathcal{H}^\infty(\mathbb{D})$ that is additive on the cone?
Note: This can be answered in a single line.

Solution: From Exercise 4 on Sheet 6, we know that the cone on $\mathcal{H}^\infty(\mathbb{D})$ is not normal. As normality is preserved under endowing the space with equivalent norms, it follows from Proposition 6.2.3 that no equivalent norm is additive. (So we could answer this question in a single line, if we had wider lines.)

Exercise 2 (Centred cones). Let E be a real Banach space, let $x \in E$ and $x' \in E'$ and assume that $\langle x', x\rangle = 1$. Let $Q \in E \to E$ denote the projection onto $\ker x'$ along the span of x, i.e., $Qz = z - \langle x', z\rangle x$ for all $z \in E$.[1] Define

$$E_+ \coloneqq \left\{ z \in E \mid \ \langle x', z\rangle \geq \|Qz\| \right\}.$$

(a) Prove that E_+ is a closed cone in E. We call it the *centered cone* in E with parameters x and x'.

(b) Show that the ice cream cone in $\mathbb{R}^d$ can be represented as a centered cone.

(c) Prove that E_+ has non-empty interior.

(d) Prove that there exists an equivalent norm on E that is additive on E_+.

Solution:

(a) That E_+ is close follows from the fact that E_+ is the preimage of the interval $[0, \infty)$ under the continuous mapping $E \ni z \mapsto \langle x', z\rangle - \|Qz\|$. For $\alpha, \beta \geq 0$ and $z_1, z_2 \in E_+$ we have

$$\langle x', \alpha z_1 + \beta z_2\rangle = \alpha\langle x', z_1\rangle + \beta\langle x', z_2\rangle \geq \alpha \|Qz_1\| + \beta \|Qz_2\| \geq \|Q(\alpha z_1 + \beta z_2)\|.$$

So E_+ is a wedge. Let $z \in E_+ \cap (-E_+)$. Then

$$\begin{aligned} \langle x', z\rangle \geq \|Qz\| = \left\|z - \langle x', z\rangle x\right\| &\geq \left\|x'\right\|^{-1} \langle x', z - \langle x', z\rangle x\rangle \\ &= \left\|x'\right\|^{-1} \left(\langle x', z\rangle - \langle x', z\rangle\langle x', x\rangle\right) = 0 \end{aligned}$$

[1] Why is Q a projection?

and similarly

$$\langle x', -z \rangle \geq \|Qz\| \geq 0.$$

So $\langle x', z \rangle = \|Qz\| = 0$. In particular, $0 = Qz = z - \langle x', z \rangle x = z$, which shows that E_+ is indeed a cone.

(b) Let $d \geq 2$ and endow $\mathbb{R}^d$ with the Euclidean norm. Choose x' to be the first canonical unit vector in $(\mathbb{R}^d)' = \mathbb{R}^d$ x be the first canonical unit vector in $\mathbb{R}^d$. Then $Q : \mathbb{R}^d \to \mathbb{R}^d$ is the projection

$$Qz = z - \langle x', z \rangle x = (0, z_2, \dots, z_d)$$

and $\mathbb{R}^d_+ = \{z \in \mathbb{R}^d \mid z_1 = \langle x', z \rangle \geq \|Qz\|_2\}$ is the ice cream cone in $\mathbb{R}^d$.

(c) This is proved in analogy to the proof of Exercise 4 on Sheet 11. We adapt the proof there: We claim that x is an interior point of E_+, since $\langle x', x \rangle = 1$ and $\|Qx\| = 0$. By continuity there exists $\delta > 0$ small enough such that for every $y \in E_+$ with $\|x - y\| < \delta$ we have

$$|\langle x', x - y \rangle| < 1/3 \quad \text{and} \quad \|Qx - Qy\| < 1/3.$$

Hence,

$$\begin{aligned} \langle x', y \rangle &\geq \langle x', x \rangle - |\langle x', x - y \rangle| > \langle x', x \rangle - \tfrac{1}{3} = \|Qx\| + \tfrac{2}{3} \\ &\geq \|Qy\| - \|Qy - Qx\| + \tfrac{2}{3} > \|Qy\| + \tfrac{1}{3} > \|Qy\|. \end{aligned}$$

It follows that an open neighborhood of x is contained in E_+. It follows that x is an interior point of E_+.

(d) Let $z \in E_+$. Then

$$\begin{aligned} \|z\| &= \|(I - Q)z + Qz\| \leq \|(I - Q)z\| + \|Qz\| \\ &\leq \big\|\langle x', z \rangle x\big\| + \langle x', z \rangle = \langle x', z \rangle (\|x\| + 1). \end{aligned}$$

By dividing the expression by $1 + \|x\|$ it follows from Theorem 6.2.5 (iii) $\Rightarrow$ (i) that there exists an equivalent norm on E that is additive on E_+.

Exercise 3 (Automatic convergence of increasing sequences). Let E be an ordered Banach space.

(a) Assume that the norm is additive on E_+. Prove that every increasing norm bounded sequence in E is convergent.

(b) Does the same claim as in (a) remain true if we only assume that there exists in equivalent norm on E that is additive on E_+?

(c) Give an example of an ordered Banach space where there exists an increasing norm bounded sequence that is not convergent.

(d) *In case that you know what a net is:*
Assume that every increasing norm bounded sequence in E is convergent. Prove that every increasing norm bounded net in E is convergent.

Solution:

(a) Let $(x_n)_{n\in\mathbb{N}}$ be an increasing norm bounded sequence in E. Then the sequences of differences $(x_{n+1} - x_n)_{n\in\mathbb{N}}$ is a sequence in E_+. By additivity of the norm on E_+ we obtain

$$\|x_N - x_1\| = \sum_{n=1}^{N-1} \|x_{n+1} - x_n\|$$

and as the sequence $(\|x_N\| - x_1)_{N\in\mathbb{N}}$ is bounded and increasing, it converges. In particular, this implies that the series $(\sum_{n=1}^{\infty} x_{n+1} - x_n)_{N\in\mathbb{N}}$ is absolutely convergent. By completeness it follows that this series converges to some point $y \in E_+$. As $(x_N - x_1)_{N\in\mathbb{N}}$ is the sequence of partial sums of the series it follows that $(x_N - x_1)_{N\in\mathbb{N}}$ converges to y; and thus $(x_N)_{N\in\mathbb{N}}$ converges to $x := y + x_1$.

(b) Yes, since a sequence that is bounded and convergence with respect to some norm is also bounded and convergent with respect to every equivalent norm.

(c) Consider the Banach space $\mathrm{C}([0,1])$ endowed with the supremum norm and the pointwise order. Then the sequence $(f_n)_{n\in\mathbb{N}}$ defined by

$$f_n : [0,1] \to \mathbb{R}, \quad f_n(x) := 1 - x^n.$$

Then it is easily seen that $f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and that the sequence is norm bounded. However, the sequence is not convergent in supremum norm, as pointwisely the sequence converges to function that is not continuous.

(d) Suppose to show a contradiction that $(x_\lambda)_{\lambda\in\Lambda}$ is an increasing and norm bounded net that does not converge. Then, by completeness, the net is not Cauchy. In particular, there exists $\varepsilon > 0$ and for each λ_0 there exist $\lambda_1, \lambda_2 \geq \lambda_0$ such that $\|x_{\lambda_1} - x_{\lambda_2}\| \geq \varepsilon$. In particular, it follows from the triangle inequality that

$$\|x_{\lambda_1} - x_{\lambda_0}\| \geq \varepsilon/2 \quad \text{or} \quad \|x_{\lambda_2} - x_{\lambda_0}\| \geq \varepsilon/2.$$

Now we choose an increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ inductively as follows: For $n = 1$ let λ_1 be arbitrary and we obtain from the above the existence of $\lambda_2 \geq \lambda_1$ such that $\|x_{\lambda_2} - x_{\lambda_1}\| \geq \varepsilon/2$. For $n > 1$ having chosen $\lambda_n \geq \lambda_{n-1}$ such that $\left\|x_{\lambda_n} - x_{\lambda_{n-1}}\right\| \geq \varepsilon/2$, we find again by the above argument a $\lambda_{n+1} \geq \lambda_n$ such that $\left\|x_{\lambda_{n+1}} - x_{\lambda_n}\right\| \geq \varepsilon/2$.

In total, we obtain an increasing sequence $(x_{\lambda_n})_{n\in\mathbb{N}}$ such that $\left\|x_{\lambda_{n+1}} - x_{\lambda_n}\right\| \geq \varepsilon/2$.[2] Then $(x_{\lambda_n})_{n\in\mathbb{N}}$ is an increasing and norm bounded sequence that is not Cauchy. So the sequence does not converge, which contradicts (a). It follows that the net must be convergent.

[2]Notice that this sequence is not necessarily a subnet of $(x_\lambda)_{\lambda\in\Lambda}$.

Exercise 4 (Dini's theorem in ordered Banach spaces). Let E be an ordered Banach space.

(a) Assume that E_+ is normal. Let (x_n) be an increasing sequence in E that converges weakly to a point $x \in E$.
Prove that (x_n) is even norm convergent to x.

(b) Let K be a compact metric space.[3] Let (f_n) be an increasing sequence in $\mathrm{C}(K)$ that converges pointwise to a function $f \in \mathrm{C}(K)$. Dini's theorem says that the convergence is automatically uniform.
What does this have to do with part (a)?

(c) Assume that E_+ is normal and that E is reflexive. Prove that every increasing norm bounded sequence in E is convergent.

Solution:

(a) This is proved in analogy to Dini's theorem: Let $(x_n)_{n\in\mathbb{N}}$ be increasing and weakly convergent to $x \in E$ and let $\varepsilon > 0$. Then for each $n \in \mathbb{N}$ define the sets of positive functionals

$$U_n := \{x' \in E'_+ \mid \langle x', x - x_n\rangle < \varepsilon\}.$$

Since $\langle x', x - x_n\rangle \downarrow 0$ for each $x' \in E'_+$, the family $\mathcal{U} := \{U_n \mid n \in \mathbb{N}\}$ is an increasing and weak*-open covering of the positive elements in the closed unit ball $B_{\le 1}(0)$ in E'. By the theorem of Banach–Alaoglu the closed unit ball in E' is weak*-compact, and so is $E'_+ \cap B_{\le 1}(0)$. Hence, $\mathcal{U}$ contains (by monotonicity of the cover) there exists an element $n_0 \in \mathbb{N}$ such that for all $n \ge n_0$ covers $B_{\le 1}(0) \cap E'_+ \subseteq U_n$. In particular, it follows that

$$\sup_{x' \in B_{\le 1}(0)\cap E'_+} |\langle x', x - x_n\rangle| \le \sup_{x' \in U_n} |\langle x', x - x_n\rangle| \le \varepsilon$$

for all $n \ge n_0$.

From the normality of the cone E_+ and Theorem 4.3.2 it follows that the dual cone E'_+ is generating. Hence, by Theorem 3.4.1 there exist a real number $K \ge 1$ and (positively homogeneous) mappings $\gamma^+, \gamma^- : E' \to E'_+$ such that $x' = \gamma^+(x') - \gamma^-(x')$ and $\|\gamma^+(x')\|, \|\gamma^-(x')\| \le K \|x'\|$ for all $x' \in E'$. This implies that

$$\begin{aligned}\sup_{x' \in B_{\le 1}(0)} |\langle x', x - x_n\rangle| &\le K \sup_{x' \in B_{\le 1}(0)} \left(|\langle \tfrac{1}{K}\gamma^+(x'), x - x_n\rangle| + |\langle \tfrac{1}{K}\gamma^-(x'), x - x_n\rangle|\right) \\ &\le 2K\varepsilon\end{aligned}$$

for all $n \ge n_0$. Hence, $(x_n)_{n\in\mathbb{N}}$ converges uniformly to x.

(b) Recall that the supremum norm on $\mathrm{C}(K)$ is monotone; and thus, the cone $\mathrm{C}(K)_+$ is normal. If $(f_n)_{n\in\mathbb{N}}$ is increasing and converges pointwisely to $f \in \mathrm{C}(K)$, then the sequence is bounded. So by Lebesgue's theorem of bounded convergence

$$\int_K f_n \,\mathrm{d}\mu \to \int_K f \,\mathrm{d}\mu$$

[3] Or, more generally, a compact Hausdorff space.

for every finite and signed measure μ on the Borel-σ-algebra of K. As by Riesz's representation theorem every continuous functional $\nu \in \mathrm{C}(K)'$ is of the form

$$\langle \nu, g \rangle = \int_K g \,\mathrm{d}\mu$$

for some finite and signed measure μ. Thus, it follows that $(f_n)_{n\in\mathbb{N}}$ converges weakly to f. Now (a) yields the uniform convergence of the sequence to f.

(c) Let $(f_n)_{n\in\mathbb{N}}$ be increasing and norm bounded. Let $\mu \in E'_+$. Then $(\langle \mu, f_n \rangle)_{n\in\mathbb{N}}$ is monotone and bounded; and thus, convergent. By normality of E_+ and Theorem 4.3.2 it follows that E'_+ is generating. Hence, $(\langle \mu, f_n \rangle)_{n\in\mathbb{N}}$ converges for all $\mu \in E'$. Then $g : E' \to \mathbb{R}$ given by

$$\langle g, \mu \rangle := \lim_{n\to\infty} \langle \mu, f_n \rangle$$

defines a continuous linear functional on E'. Hence, by reflexivity, $g \in E$ and $(f_n)_{n\in\mathbb{N}}$ converges weakly to g. Now from (a) it follows that $(f_n)_{n\in\mathbb{N}}$ converges in norm to f.
