12. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on July 4 and 5, 2023

Exercise 1 (Holomorphic functions, again). Consider the space $\mathcal{H}^\infty(\mathbb{D})$ and the cone on it that was introduced in Exercise 4 on Sheet 6. Is there an equivalent norm on $\mathcal{H}^\infty(\mathbb{D})$ that is additive on the cone?
Note: This can be answered in a single line.

Exercise 2 (Centred cones). Let E be a real Banach space, let $x \in E$ and $x' \in E'$ and assume that $\langle x', x \rangle = 1$. Let $Q \in E \to E$ denote the projection onto $\ker x'$ along the span of x, i.e., $Qz = z - \langle x', z \rangle x$ for all $z \in E$.[1] Define

$$E_+ \coloneqq \left\{ z \in E \mid \langle x', z \rangle \geq \|Qz\| \right\}.$$

(a) Prove that E_+ is a closed cone in E. We call it the *centered cone* in E with parameters x and x'.

(b) Show that the ice cream cone in $\mathbb{R}^d$ can be represented as a centered cone.

(c) Prove that E_+ has non-empty interior.

(d) Prove that there exists an equivalent norm on E that is additive on E_+.

Exercise 3 (Automatic convergence of increasing sequences). Let E be an ordered Banach space.

(a) Assume that the norm is additive on E_+. Prove that every increasing norm bounded sequence in E is convergent.

(b) Does the same claim as in (a) remain true if we only assume that there exists in equivalent norm on E that is additive on E_+?

(c) Give an example of an ordered Banach space where there exists an increasing norm bounded sequence that is not convergent.

(d) *In case that you know what a net is:*
Assume that every increasing norm bounded sequence in E is convergent. Prove that every increasing norm bounded net in E is convergent.

[1] Why is Q a projection?

Exercise 4 (Dini's theorem in ordered Banach spaces). Let E be an ordered Banach space.

(a) Assume that E_+ is normal. Let (x_n) be an increasing sequence in E that converges weakly to a point $x \in E$.
Prove that (x_n) is even norm convergent to x.

(b) Let K be a compact metric space.[2] Let (f_n) be an increasing sequence in $C(K)$ that converges pointwise to a function $f \in C(K)$. Dini's theorem says that the convergence is automatically uniform.
What does this have to do with part (a)?

(c) Assume that E_+ is normal and that E is reflexive. Prove that every increasing norm bounded sequence in E is convergent.

[2] Or, more generally, a compact Hausdorff space.