11. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on June 27 and 28, 2023

with Solutions

Exercise 1 (Projection bands). Endow $\mathbb{R}^4$ with the cone

$$\mathbb{R}^4_+ := \{ x \in \mathbb{R}^4 \mid x_2 \geq 0 \text{ and } x_1 \geq |x_3| + |x_4| \}.$$

Write $\mathbb{R}^4$ as the direct sum of two non-trivial projection bands.

Solution: Clearly, $\mathbb{R}^4_+$ is closed, so by Proposition 2.1.2, the cone is Archimedean. Hence, by Proposition 5.1.16 it suffices to find a non-trivial band projection P. Consider for example the projection

$$P : \mathbb{R}^4 \to \mathbb{R}^4, \quad (x_1, \dots, x_4) \mapsto (0, x_2, 0, 0).$$

Then

$$(I - P) : \mathbb{R}^4 \to \mathbb{R}^4, \quad (x_1, \dots, x_4) \mapsto (x_1, 0, x_3, x_4)$$

and both operator P and $I - P$ are positive. Now, by Proposition 5.1.16, the ranges $B_1 := P\mathbb{R}^4$ and $B_2 := (I-P)\mathbb{R}^4$ are non-trivial projection bands with $B_1 + B_2 = \mathbb{R}^4$.

Exercise 2 (Order units). Consider the function $u : [-1, 1] \to \mathbb{R}$, $t \mapsto 1 - t^2$. In which of the following ordered Banach spaces (each of them endowed with the pointwise order) is u an order unit?

(a) $\mathrm{C}\left([-1, 1]\right)$

(b) $\mathrm{C}^1\left([-1, 1]\right)$

(c) $E := \left\{ f \in \mathrm{C}\left([-1, 1]\right) \mid f(-1) = f(1) = 0 \right\}$

(d) $F := \left\{ f \in \mathrm{C}^1\left([-1, 1]\right) \mid f(-1) = f(1) = 0 \right\}$

Solution:

(a) By Theorem 6.1.2 every order unit is an interior point of the positive cone. Now let $\varepsilon > 0$. Then $u - \varepsilon \cdot \mathbb{1}$ has values that are negative. So u is not in the interior of $\mathrm{C}([-1, 1])_+$; and thus, u is no order unit.

(b) The same argument as in (a) applies, as $\|\mathbb{1}\|_{\mathrm{C}^1} = \|\mathbb{1}\|_\infty$.

(c) As before in (a) we show that u is not in the interior of the cone. Let $\varepsilon > 0$ and $\delta > 0$ small enough that $u(x) < \varepsilon$ for all $x \in [-1, -1+\delta]$. By Urysohn's lemma there is a continuous function $f : [-1, 1] \to [0, 1]$ that satisfies $f(-1) = f(1) = 0$ and $f(-1+\delta) = 1$. Then $u - \varepsilon f$ has a negative value at the point $-1+\delta$. Hence, u is not in the interior of $\mathrm{C}([-1, 1])$; and thus, u is no order unit.

(d) We claim that u is an order unit for F. Indeed, let $f \in F$ with $\|f\|_{\mathrm{C}^1} \leq 1/2$, then $\|f\|_\infty, \|f'\|_\infty \leq 1/2$, and thus, for all $x \in [-1/2, 1/2]$ we have $u(x) - f(x) = 1 - x^2 - f(x) \geq 1 - 1/4 - 1/2 > 0$ and for $x \in [-1, -1/2]$

$$u(x) - f(x) = \int_{-1}^{x} \underbrace{-2t - f'(t)}_{\geq 1 - 1/2 \geq 0} \,\mathrm{d}t \geq 0,$$

and similarly for $x \in [1/2, 1]$

$$u(x) - f(x) = \int_{1}^{x} \underbrace{-2t - f'(t)}_{\geq 1 - 1/2 \geq 0} \,\mathrm{d}t = \int_{x}^{1} \underbrace{2t + f'(t)}_{\geq 1 - 1/2 \geq 0} \,\mathrm{d}t \geq 0.$$

So it follows that $f \leq u$. Now Theorem 6.1.2 (iii) $\Rightarrow$ (i) implies that u is an order unit.

Exercise 3 (Order unit in the spaces of self-adjoint compact operators?). Let H be an infinite-dimensional separable complex Hilbert space and endow the space $\mathcal{K}(H)_{\mathrm{sa}}$ of self-adjoint compact linear operators on H with the Loewner order. Show that there does not exist an order unit in $\mathcal{K}(H)_{\mathrm{sa}}$.

Solution: Suppose to show a contradiction that there is an order unit $0 \leq A \in \mathcal{K}(H)_{\mathrm{sa}}$. Then by a spectral decomposition there exists a null sequence $(\lambda_n)_{n\in\mathbb{N}}$ of nonnegative reals and an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of H such that

$$A = \sum_{n\in\mathbb{N}} \lambda_n (e_n \otimes e_n).$$

We may suppose that $\lambda_n > 0$ for all $n \in \mathbb{N}$, otherwise it is easily seen that $\varepsilon(e_n \otimes e_n) \not\leq A$ for all $\varepsilon > 0$ and all $n \in \mathbb{N}$ with $\lambda_n = 0$.
Now let $(\alpha)_{n\in\mathbb{N}}$ be a sequence of nonnegative real with $\alpha_n \to 0$ and $\alpha_n/\lambda_n \to \infty$. Then

$$B := \sum_{n\in\mathbb{N}} \alpha_n (e_n \otimes e_n)$$

is a compact operator (as a operator norm limit of finite-dimensional operators) that satisfies $\varepsilon B \not\leq A$ for all $\varepsilon > 0$. This contradicts the fact that A is an order unit.

Exercise 4 (Interior points in an infinite-dimensional ice cream cone). Endow ℓ^2 with the ice cream cone

$$\ell^2_+ := \{x \in \ell^2 \mid x_1 \geq 0 \text{ and } x_1^2 \geq \sum_{k=2}^{\infty} x_k^2\}.$$

Does ℓ^2_+ have an interior point?

Solution: Define $Px := (0, x_2, x_3, \dots)$ for all $x \in \ell^2$. Let $x \in \ell^2_+$ with $x_1 > \|Px\|_2$ and set $\varepsilon := x_1 - \|Px\|$. Then for $y \in \ell^2$ with $\|x - y\| < \varepsilon/3$ we have

$$|x_1 - y_1| < \varepsilon/3 \quad \text{and} \quad \|Px - Py\| < \varepsilon/3.$$

Hence,

$$\begin{aligned} y_1 &\geq x_1 - |x_1 - y_1| > x_1 - \tfrac{\varepsilon}{3} = \|Px\| + \tfrac{2\varepsilon}{3} \\ &\geq \|Py\| - \|Py - Px\| + \tfrac{2\varepsilon}{3} > \|Py\| + \tfrac{\varepsilon}{3} > \|Py\|\,. \end{aligned}$$

It follows that an open neighborhood of x is contained in ℓ^2_+. It follows that x is an interior point of ℓ^2_+.

Exercise 5 (And now something completely different). Endow the space $E := \left\{ f \in \mathrm{C}^1\left([-1,1]\right) \mid f(0) = 0 \right\}$ with the pointwise order. Is the positive cone generating?[1]

Solution: No, the cone is not generating. Consider the function $f : [-1,1] \to \mathbb{R}$, $x \mapsto x$ and suppose that there exist $f^+, f^- \in E_+$ such that $f^+ - f^- = f$. Then $f^+(x) = x + f^-(x) \geq x$ for all $x \in [-1,1]$. Hence, for $h > 0$ we get

$$\lim_{h \downarrow 0} \frac{f^+(h) - f^+(0)}{h} \geq \lim_{h \downarrow 0} \frac{h - 0}{h} = 1$$

and in case $h < 0$ we obtain

$$\lim_{h \uparrow 0} \frac{f^+(h) - f^+(0)}{h} \leq \lim_{h \uparrow 0} \frac{0 - 0}{h} = 0.$$

This is a contradiction to the assumption that $f^+ \in E_+$.

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[1] In case that you are wondering how this is related to the current contents of the lecture: it isn't.