11. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on June 27 and 28, 2023

Exercise 1 (Projection bands). Endow $\mathbb{R}^4$ with the cone

$$\mathbb{R}^4_+ := \left\{x \in \mathbb{R}^4 \mid \; x_2 \geq 0 \text{ and } x_1 \geq |x_3| + |x_4| \,.\right\}$$

Write $\mathbb{R}^4$ as the direct sum of two non-trivial projection bands.

Exercise 2 (Order units). Consider the function $u : [-1, 1] \to \mathbb{R}$, $t \mapsto 1 - t^2$. In which of the following ordered Banach spaces (each of them endowed with the pointwise order) is u an order unit?

(a) $\mathrm{C}\left([-1, 1]\right)$

(b) $\mathrm{C}^1\left([-1, 1]\right)$

(c) $E := \Big\{f \in \mathrm{C}\left([-1, 1]\right) \mid \; f(-1) = f(1) = 0\Big\}$

(d) $F := \Big\{f \in \mathrm{C}^1\left([-1, 1]\right) \mid \; f(-1) = f(1) = 0\Big\}$

Exercise 3 (Order unit in the spaces of self-adjoint compact operators?). Let H be an infinite-dimensional separable complex Hilbert space and endow the space $\mathcal{K}(H)_{\mathrm{sa}}$ of self-adjoint compact linear operators on H with the Loewner order. Show that there does not exist an order unit in $\mathcal{K}(H)_{\mathrm{sa}}$.

Exercise 4 (Interior points in an infinite-dimensional ice cream cone). Endow ℓ^2 with the ice cream cone

$$\ell^2_+ := \left\{x \in \ell^2 \mid \; x_1 \geq 0 \text{ and } x_1^2 \geq \sum_{k=2}^{\infty} x_k^2\right\}.$$

Does ℓ^2_+ have an interior point?

Exercise 5 (And now something completely different). Endow the space $E := \left\{f \in \mathrm{C}^1\left([-1, 1]\right) \mid \; f(0) = 0\right\}$ with the pointwise order. Is the positive cone generating?[1]

[1] In case that you are wondering how this is related to the current contents of the lecture: it isn't.