10. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on June 20 and 21, 2023

Exercise 1 (Vector lattices and bands therein). A *vector lattice* is an ordered vector space E such that for all $x, y \in E$ the supremum $\sup\{x, y\} =: x \vee y$ exists.
Throughout this exercise let E be a vector lattice. For each $x \in E$ one defines $|x| := x \vee -x$.
An *ideal* in E is a vector subspace I of E such that $|y| \in I$ for every $y \in I$ and such that $0 \leq x \leq y$ implies $x \in I$ for all $x \in E$ and all $y \in I$.

(a) Show that the cone E_+ is generating.

(b) Show that for all $x, y \in E$ the infimum $\inf\{x, y\} =: x \wedge y$ exists.

(c) Let I be an ideal in E. Show that the induced cone $I_+ := I \cap E_+$ is generating in I.

Assume now in addition that the cone E_+ is Archimedean.

(d) Show that every band in E is an ideal.

(e) Show that a vector subspace I of E is an ideal if and only if its cone $I_+ := I \cap E_+$ is a face of E_+.

(f) Show that two elements $x, y \in E$ are disjoint if and only if $|x| \wedge |y| = 0$.

Exercise 2 (Closed ideals and bands in L^p). Let $p \in [1, \infty)$.

(a) Show that $L^p(\mathbb{R})$,[1] endowed with the pointwise almost everywhere order, is a vector lattice.

(b) Determine all closed ideals in $L^p(\mathbb{R})$.
Hint: Use Exercise 4 on Sheet 2.

(c) Show that every band in $L^p(\mathbb{R})$ is closed.

(d) Determine all bands in $L^p(\mathbb{R})$.
Hint: Use parts (b) and (c) as well as Example 5.1.17.

Exercise 3 (Closed ideals and bands in $\mathrm{C}(K)$). Let $K \neq \emptyset$ be a compact metric space (or, more generally, a compact Hausdorff space).

(a) Show that $\mathrm{C}(K)$, endowed with the pointwise order, is a vector lattice.

(b) Show that the closed ideals in $\mathrm{C}(K)$ are precisely the sets

$$I_S := \{f \in \mathrm{C}(K) \mid f|_S = 0\}$$

[1] All results in this exercise remain true for L^p over a general measure space. We focus on $\mathbb{R}$ just to keep the notation and intuition a bit simpler.

for closed subsets S of K.

(c) Similarly as for L^p-spaces in Exercise 2(c) one can show that every band on $\mathrm{C}(K)$ is closed.[2]
Show that the bands in $\mathrm{C}(K)$ are precisely the sets I_S where $S \subseteq K$ is regularly closed (as claimed in Exercise 5.1.11).

(d) Show that the projection bands in $\mathrm{C}(K)$ are precisely the sets I_S where $S \subseteq K$ is clopen.

Exercise 4 (A space of differentiable functions). Let us consider the space $\mathrm{C}^1([-1,1])$, endowed with the pointwise order. Here is an argument:

"The space $\mathrm{C}^1([-1,1])$ is not a vector lattice.
Indeed, let $f \in \mathrm{C}^1([-1,1])$ be given by $f(x) = x$ for all $x \in [-1,1]$. Then $f(x) \vee -f(x) = |f(x)| = |x|$ for every $x \in [-1,1]$, so the function $f \vee -f$ is not in $\mathrm{C}^1([-1,1])$".

(a) Is the argument correct?

(b) Is $\mathrm{C}^1([-1,1])$ a vector lattice?

[2] You do not need to do this here.