BERGISCHE UNIVERSITÄT WUPPERTAL

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TECHNISCHE UNIVERSITÄT DRESDEN

9. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on June 13 and 14, 2023

with Solutions

Exercise 1 (Non-disjointness in the Loewner order).

(a) Endow the space $\mathbb{C}^{2\times 2}_{\text{sa}}$ of self-adjoint 2×2-matrices with the Loewner order. Show that any two non-zero positive elements a, b in this space are not disjoint.

(b) Let X be an ordered vector space and let $V \subseteq X$ be a vector subspace which we endow with the order inherited from X.[1] Let $v, w \in V$. Show that if v and w are disjoint within the ordered vector space X, then they are also disjoint within the ordered vector space V.
Conversely, give an example of spaces X and V and elements $v, w \in V$ such that v and w are disjoint within the space V but not within the space X.

(c) Let H be an infinite-dimensional seperable complex Hilbert space and endow the space $\mathcal{K}(H)_{\text{sa}}$ of compact self-adjoint operators on H with the Loewner order. Let $x, y \in H \setminus \{0\}$. Show that $x \otimes x$ and $y \otimes y$ are not disjoint.
Hint: Let $G \subseteq H$ denote the span of x and y and let $V \subseteq \mathcal{K}(H)_{\text{sa}}$ consist of those operators that leave G invariant and vanish on the orthogonal complement of G.

(d) In the setting of part (c), show that two non-zero elements of the positive cone are never disjoint.

Solution:

(a) It follows from the definition of the infimum that disjointness is preserved by order isomorphisms. Recall from Exercise 3 on Sheet 2 that the Loewner cone in $\mathbb{C}^{2\times 2}_{\text{sa}}$ is order isomorphic to the ice cream cone in $\mathbb{R}^3$. Hence, the statement follows immediately from Example 5.1.4 (b) (together with Exercise 3 (b) on Sheet 8).

(b) *Claim.* For any two elements $x, y \in V$ the set of upper bounds of x and y in V is the intersection of the set of upper bound in X with V.

Proof. If z is an upper bound of x and y in V, then it is clearly also an upper bound of x and y in X and $z \in V$. Conversely, if $z \in V$ and z is an upper bound of x and y, then z is clearly an upper bound of x and y in V. □

Now let $v, w \in V$ be disjoint in X. Then by Proposition 5.1.5 (i) ⇒ (ii) the set of upper bound in X of $v + w$ and $-v - w$ and the set of upper bound in X of $v - w$ and $w - v$ coincide. By intersecting both set with V, the above claim yields that the

[1] In other words, $V_+ = V \cap X_+$.

set of upper bound in V of $v+w$ and $-v-w$ and the set of upper bound in V of $v-w$ and $w-v$ coincide. Now Proposition 5.1.5 (ii) $\Rightarrow$ (i) yields that v and w are disjoint in V.

We now give the example. Let $X = \mathbb{R}^3$ endowed with the ice cream cone and $V = \{x \in \mathbb{R}^3 \mid x_3 = 0\}$ endowed with the cone $V_+ = X_+ \cap V$. Then V is order isomorphic to $\mathbb{R}^2$ endowed with the ice cream cone. By Exercise 3 (b) on Sheet 8 we know that any non-zero element $v, w \in \partial V_+$ are not disjoint in X but are always disjoint in V.

(c) If x and y are linearly dependent, it follows easily that $x \otimes x$ and $y \otimes y$ are also linearly dependent. Hence, the operators are not disjoint.

So let $x, y \in H$ be linearly independent. Consider the mapping[2]

$$i : \mathbb{C}^{2\times 2}_{\mathrm{sa}} \to \mathcal{K}(H)_{\mathrm{sa}}, \quad \begin{pmatrix} a_1 & a_3 \\ a_3^* & a_2 \end{pmatrix} \mapsto a_1(x\otimes x) + a_2(y \otimes y) + a_3(x \otimes y) + a_3^*(y \otimes x).$$

It follows from

$$\begin{aligned} &\left(\begin{pmatrix} (x \mid z) \\ (y \mid z) \end{pmatrix} \begin{pmatrix} a_1 & a_3 \\ a_3^* & a_2 \end{pmatrix} \middle| \begin{pmatrix} (x \mid z) \\ (y \mid z) \end{pmatrix} \right) \\ &= \big((a_1(x\otimes x) + a_2(y \otimes y) + a_3(x \otimes y) + a_3^*(y \otimes x))z \big| z \big), \end{aligned}$$

which holds for all $z \in H$, and the fact that by linear independence of x and y the mapping

$$H \to \mathbb{C}^2, \quad z \mapsto \begin{pmatrix} (x \mid z) \\ (y \mid z) \end{pmatrix}$$

is surjective, that the mapping i is bi-positive, and thus, injective (see Proposition 1.6.4). In particular it follows that we may view $\mathbb{C}^{2\times 2}_{\mathrm{sa}}$ as an ordered subspace of $\mathcal{K}(H)_{\mathrm{sa}}$ that contains the operators $x \otimes x$ and $y \otimes y$. Thus, the non-disjointness of these operator follows from (b) and (a).

(d) Let $A, B \in \mathcal{K}(H)_{\mathrm{sa}} \setminus \{0\}$. Then by the spectral theorem for self-adjoint compact operators there exist two non-zero vectors $x, y \in H$ such that $A \geq x \otimes x \geq 0$ and $B \geq y \otimes y \geq 0$. By (c) the operators $x \otimes x$ and $y \otimes y$ are not disjoint. Hence, there exists a lower bound $C \leq x \otimes x, y \otimes y$ such that $C \not\leq 0$. But C is also a lower bound of A and B. Hence, A and B are not disjoint.

Exercise 2 (The space of test functions). Let $\mathrm{C}_\mathrm{c}^\infty(0,1)$ denote the real vector space of all infinitely differentiable functions $f : (0,1) \to \mathbb{R}$ whose support $\overline{\{x \in (0,1) : f(x) \neq 0\}}$ is a compact subset of $(0,1)$ (i.e., $\mathrm{C}_\mathrm{c}^\infty(0,1)$ is the space of *test functions* on $(0,1)$). Endow $\mathrm{C}_\mathrm{c}^\infty(0,1)$ with the pointwise order.

(a) Show that the positive cone in $\mathrm{C}_\mathrm{c}^\infty(0,1)$ is generating. Is it Archimedean?

(b) When are two elements of $\mathrm{C}_\mathrm{c}^\infty(0,1)$ disjoint? When are two positive elements of $\mathrm{C}_\mathrm{c}^\infty(0,1)$ D-disjoint?

Solution:

[2]Recall that for $x, y \in H$ we have defined $(x \otimes y)z := (y \mid z)x$ for all $z \in H$.

(a) Let $f \in \mathrm{C}_\mathrm{c}^\infty(0,1)$ and let $0 < a < b < 1$ define an interval $(a,b) \subsetneq (0,1)$ that contains the support of f. Now choose any $g \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R})$ with $0 \le g \le 1$ and $\int_0^1 g(x)\,\mathrm{d}x = 1$ that is supported on $(-1,1)$. Then

$$\tilde{g}(x) := \int_a^b \frac{1}{\delta} g\left(\frac{x-t}{\delta}\right)\,\mathrm{d}t = \int_{\frac{a}{\delta}}^{\frac{b}{\delta}} g\left(\tfrac{x}{\delta} - t\right)\,\mathrm{d}t$$

for small $\delta > 0$ is supported in a small neighborhood of (a,b) and satisfies $\tilde{g} = 1$ on the support of f. Moreover, $\tilde{g} \in \mathrm{C}_\mathrm{c}^\infty(0,1)$.[3] Then the decomposition $f = f^+ - f^-$ is given by

$$f^+ = f + \|f\|_\infty g, \qquad f^- = \|f\|_\infty g,$$

where $f^+, f^- \ge 0$.

Let $f, g \in \mathrm{C}_\mathrm{c}^\infty(0,1)$ with $f \ge 0$ and $g \le \frac{1}{n} f$ for all $n \in \mathbb{N}$. Then pointwisely $g(x) \le \frac{1}{n} f(x)$, and thus, $g(x) \le 0$. It follows that $g \le 0$ and that the positive cone is Archimedean.

(b) *Claim.* Two positive elements in $\mathrm{C}_\mathrm{c}^\infty(0,1)$ are disjoint if and only if they are D-disjoint.

Proof. By Proposition 5.1.3 it suffices to show that D-disjointness implies disjointness. Let f, g be D-disjoint and suppose there exists a lower bound $l \in \mathrm{C}_\mathrm{c}^\infty(0,1)$ of f and g that does not satisfy $l \le 0$. As the order is pointwise, there exists $x \in (0,1)$ such that $l(x) > 0$. Hence, f and g are larger than 0 in a common open neighborhood of x. By a similar argument as in (a) we are able to find a non-zero function $0 \le h \in \mathrm{C}_\mathrm{c}^\infty(0,1)$ such that $h \le f, g$. This contradicts the D-disjointness of f and g. □

Claim. Two elements $f, g \in \mathrm{C}_\mathrm{c}^\infty(0,1)$ are disjoint if and only if their open supports $\{x \in (0,1) \mid f(x) \neq 0\}$ and $\{x \in (0,1) \mid g(x) \neq 0\}$ are disjoint.

Proof. Suppose that U is a non-empty subset of the open supports of f and g. Then by Proposition 5.1.8 (i) and by potentially making U smaller, we may we may assume that $0 < f(x) < g(x)$ for all $x \in U$. Let $0 \le h \in \mathrm{C}_\mathrm{c}^\infty(0,1)$ be non-zero with support in U. Let $u \in \{f-g, g-f\}^{\mathrm{ub}}$ and consider $\tilde{u} := u - \lambda h$ for a choice of $\lambda > 0$ such that $\tilde{u}$ is still an upper bound of $f - g$ and $g - f$ and there exists $x \in U$ such that $g(x) < f(x)\tilde{u}(x) < g(x)$. Then, in particular, $\tilde{u} \notin \{f+g, -f-g\}^{\mathrm{ub}}$. Hence, f and g are not disjoint.

Conversely, let f and g have disjoint open support. We show that f and g are disjoint. Let $u \in \{f+g, -f-g\}^{\mathrm{ub}}$. Then $u \ge f+g$ and $u \ge -f-g$. In particular, $u(x) \ge |f(x)|$ for all x in the support of f and $u(x) \ge |g(x)|$ for all x in the support of g (and $u(x) \ge 0$ for all x outside of the support of f and g). So $u \ge f-g, g-f$, and thus, $u \in \{f-g, g-f\}^{\mathrm{ub}}$.

The converse inclusion now follows from the fact that f and $-g$ also have disjoint open support. □

[3] The construction of $\tilde{g}$ is a standard argument in the literature. There $\tilde{g}$ is called a *mollification* of g. Mollifications are usually used to smoothen non-smooth functions. In our example we have smoothed the indicator function $\mathbb{1}_{(a,b)}$.

Exercise 3 (Holomorphic functions again). Let $E \subseteq \mathcal{H}^\infty(\mathbb{D})$ be the space from Exercise 4 on Sheet 6, endowed with the order defined there.

(a) Let $n_0 \geq 2$ be an integer. Show that there exists a function $h \in E$ that satisfies $h(\frac{1}{n_0}) = 1$ but $h(\frac{1}{n}) < 0$ for all integers $n \geq 2$ different from n_0.

(b) Let $f, g \in E_+$ be non-zero. Show that there exists an integer $n_0 \geq 2$ such that both functions f and g do not vanish at $\frac{1}{n_0}$.

(c) Let $f, g \in E_+$ be non-zero. Show that f and g are not disjoint.

Solution:

(a) Consider the function

$$h : \mathbb{D} \to \mathbb{C}, \quad z \mapsto -a(x - \tfrac{1}{n_0})^2 + 1,$$

where $a \in \mathbb{R}$ is to be determined. Then for the right choice of a the function h satisfies the desired properties. Indeed, $h(\frac{1}{n_0}) = 1$ and has the zeros $x_{1,2} = \frac{1}{n_0} \pm \sqrt{1/a}$, which lie in the interval $(\frac{1}{n_0+1}, \frac{1}{n_0-1})$ for an appropriate choice of $a \in \mathbb{R}$.

(b) For every non-zero function $f \in \mathcal{H}^\infty(\mathbb{C})$ there exists an integer $n_1 \geq 2$ such that $f(\frac{1}{n_0}) \neq 0$. Otherwise the identity theorem there exists a sequence converging to 0 on which the the function is 0. By continuity the function is 0 in 0. Hence, the identity theorem implies that $f = 0$. Let $n_1 \geq 2$ be the integer with this property for $f \in E_+$ and $n_2 \geq 2$ be the integer with this property for $g \in E_+$. Set $n_0 := \max\{n_1, n_2\}$.

(c) Let $f, g \in E_+$ be non-zero and let $n_0 \geq 2$ be an integer for which f and g do not vanish at $\frac{1}{n_0}$. Let $0 < \alpha := \min\{f(\frac{1}{n_0}), g(\frac{1}{n_0})\}$. Let h be a function with the properties listed in (a). Then αh is a lower bound of f and g. Indeed, for $n \in \mathbb{N}\backslash\{n_0\}$ we have $f(\frac{1}{n}), g(\frac{1}{n}) \geq 0$ and $h(\frac{1}{n}) \leq 0$, and $f(\frac{1}{n_0}), g(\frac{1}{n_0}) > \alpha = h(\frac{1}{n_0})$. But $h \not\leq 0$. So 0 is not the infimum of $\{f, g\}$.

Exercise 4 (Anti-lattices). Let E be an ordered vector space whose cone is generating. The space E is called an *anti-lattice* if the following holds for all vectors x, y: if $\{x, y\}$ has a supremum in E, then $x \geq y$ or $y \geq x$.
Prove that E is an anti-lattice if and only if there is no pair of non-zero disjoint elements of E_+.[4]

Hints: To show "$\Rightarrow$", prove that any two disjoint elements $x, y \in E_+$ have supremum $x + y$. To show "$\Leftarrow$", take two elements x, y which have a supremum and shift $-x$ and $-y$ such that you get two elements of E_+ that have an infimum. Then subtract the infimum from both elements.

[4] So $\mathbb{R}^d$ with the ice-cream cone is an anti-lattice if $d \geq 3$ or $d = 1$ (Exercise 3(b) on Sheet 8). Moreover, the self-adjoint compact operators on a complex Hilbert space form an anti-lattice with respect to the Loewner order (Exercise 1(d) on the present sheet) and the space E in Exercise 3 is also an anti-lattice (part (c) of that Exercise).

Solution: "$\Rightarrow$": Suppose that there is a pair of non-zero disjoint elements $x, y \in E_+$. We show first that $x + y$ is the supremum of $\{x, y\}$. Clearly $x + y \geq x, y$. So $x + y$ is an upper bound of x and y. If u is an upper bound of x and y, then $u \geq x - y$ and $u \geq y - x$. Hence, by disjointness $u \geq x + y, -x - y$. Thus, $x + y$ is indeed the supremum of $\{x, y\}$.
However, we neither have $x \geq y$ nor $y \geq x$, since then the infimum of $\{x, y\}$ is would not equal 0, which would contradict the disjointness of x and y.

"$\Leftarrow$": Assume that E is not an anti-lattice. So there exist two elements $x, y \in E$ such that $\{x, y\}$ has the supremum $s \in E$ and neither $x \geq y$ nor $y \geq x$. Now the elements $s - y$ and $s - x$ are in E_+ and non-zero, as otherwise we have $s = x$ or $s = y$ contradicting the assumption that neither $x \geq y$ nor $y \geq x$ holds. It suffices to show that $s - y$ and $s - x$ are disjoint, but this follows immediately from the fact that the infimum of the shifted set $\{-x + s, -y + s\}$ is 0, since $-s$ is the infimum of $\{-x, -y\}$. This concludes the proof.