9. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on June 13 and 14, 2023

Exercise 1 (Non-disjointness in the Loewner order).

(a) Endow the space $\mathbb{C}^{2\times 2}_{\mathrm{sa}}$ of self-adjoint 2×2-matrices with the Loewner order. Show that any two non-zero positive elements a, b in this space are not disjoint.

(b) Let X be an ordered vector space and let $V \subseteq X$ be a vector subspace which we endow with the order inherited from X.[1] Let $v, w \in V$. Show that if v and w are disjoint within the ordered vector space X, then they are also disjoint within the ordered vector space V.
Conversely, give an example of spaces X and V and elements $v, w \in V$ such that v and w are disjoint within the space V but not within the space X.

(c) Let H be an infinite-dimensional seperable complex Hilbert space and endow the space $\mathcal{K}(H)_{\mathrm{sa}}$ of compact self-adjoint operators on H with the Loewner order. Let $x, y \in H \setminus \{0\}$. Show that $x \otimes x$ and $y \otimes y$ are not disjoint.
Hint: Let $G \subseteq H$ denote the span of x and y and let $V \subseteq \mathcal{K}(H)_{\mathrm{sa}}$ consist of those operators that leave G invariant and vanish on the orthogonal complement of G.

(d) In the setting of part (c), show that two non-zero elements of the positive cone are never disjoint.

Exercise 2 (The space of test functions). Let $\mathrm{C}^\infty_{\mathrm{c}}(0,1)$ denote the real vector space of all infinitely differentiable functions $f : (0,1) \to \mathbb{R}$ whose support $\overline{\{x \in (0,1) : f(x) \neq 0\}}$ is a compact subset of $(0,1)$ (i.e., $\mathrm{C}^\infty_{\mathrm{c}}(0,1)$ is the space of *test functions* on $(0,1)$). Endow $\mathrm{C}^\infty_{\mathrm{c}}(0,1)$ with the pointwise order.

(a) Show that the positive cone in $\mathrm{C}^\infty_{\mathrm{c}}(0,1)$ is generating. Is it Archimedean?

(b) When are two elements of $\mathrm{C}^\infty_{\mathrm{c}}(0,1)$ disjoint? When are two positive elements of $\mathrm{C}^\infty_{\mathrm{c}}(0,1)$ D-disjoint?

Exercise 3 (Holomorphic functions again). Let $E \subseteq \mathcal{H}^\infty(\mathbb{D})$ be the space from Exercise 4 on Sheet 6, endowed with the order defined there.

(a) Let $n_0 \geq 2$ be an integer. Show that there exists a function $h \in E$ that satisfies $h(\frac{1}{n_0}) = 1$ but $h(\frac{1}{n}) < 0$ for all integers $n \geq 2$ different from n_0.

(b) Let $f, g \in E_+$ be non-zero. Show that there exists an integer $n_0 \geq 2$ such that both functions f and g do not vanish at $\frac{1}{n_0}$.

(c) Let $f, g \in E_+$ be non-zero. Show that f and g are not disjoint.

[1] In other words, $V_+ = V \cap X_+$.

Exercise 4 (Anti-lattices). Let E be an ordered vector space whose cone is generating. The space E is called an *anti-lattice* if the following holds for all vectors x, y: if $\{x, y\}$ has a supremum in E, then $x \geq y$ or $y \geq x$.
Prove that E as an anti-lattice if and only if there is no pair of non-zero disjoint elements of E_+.[2]

Hints: To show "$\Rightarrow$", prove that any two disjoint elements $x, y \in E_+$ have supremum $x + y$. To show "$\Leftarrow$", take two elements x, y which have a supremum and shift $-x$ and $-y$ such that you get two elements of E_+ that have an infimum. Then subtract the infimum from both elements.

[2] So $\mathbb{R}^d$ with the ice-cream cone is an anti-lattice if $d \geq 3$ or $d = 1$ (Exercise 3(b) on Sheet 8). Moreover, the self-adjoint compact operators on a complex Hilbert space form an anti-lattice with respect to the Loewner order (Exercise 1(d) on the present sheet) and the space E in Exercise 3 is also an anti-lattice (part (c) of that Exercise).