8. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on June 6 and 7, 2023

with Solutions

Exercise 1 (Properties of wedges in duality).

(a) Give an example of an ordered Banach space E such that E_+ is not normal but E'_+ is total in E'.

(b) Give an example of an ordered Banach space E such that (E_+ is not normal and) E'_+ is not total in E'.

Solution:

(a) Consider the Sobolev space $H^1((0, 2\pi))$ defined by

$$E := H^1((0, 2\pi)) = \{f \in L^2((0, 2\pi); \mathbb{R}) \mid \exists g \in L^2((0, 2\pi)) : \\ \forall \varphi \in C_c^\infty((0, 2\pi); \mathbb{R}) : (\varphi \mid g)_{L^2} = -(\varphi' \mid f)_{L^2}\}.$$

Here g is called the *weak derivative of* f and will be denoted by f'. Endowed with the norm

$$\|f\|_{H^1}^2 := \|f\|_{L^2}^2 + \|f'\|_{L^2}^2$$

and with the cone E_+ containing the functions f with $f(x) \geq 0$ for almost all $x \in (0, 2\pi)$, the space H^1 becomes an ordered Banach space.

Now consider the sequence $(f_n)_{n\in\mathbb{N}}$ defined by

$$f_n(x) := \sin(nx), \qquad \text{for all } x \in (0, 2\pi),\ n \in \mathbb{N}.$$

Then $f_n \in [0, \mathbb{1}]$ for all $n \in \mathbb{N}$ and

$$\begin{aligned}\|f_n\|_{H^1}^2 &= \int_0^{2\pi} \sin(nx)^2 \,\mathrm{d}x + \int_0^{2\pi} n^2 \cos(nx)^2 \,\mathrm{d}x \\ &= n^{-1} \int_0^{2n\pi} \sin(x)^2 \,\mathrm{d}x + n \int_0^{2n\pi} \cos(x)^2 \,\mathrm{d}x \\ &= \int_0^{2\pi} \sin(x)^2 \,\mathrm{d}x + n^2 \int_0^{2\pi} \cos(x)^2 \,\mathrm{d}x \to \infty.\end{aligned}$$

Hence, the set $\{f_n \mid n \in \mathbb{N}\}$ is order bounded but not norm bounded. By Theorem 3.5.5 (i) $\Rightarrow$ (ii), it follows that E_+ is not normal.

To see that E'_+ is total, notice that, since E_+ is a cone, Theorem 4.3.2 implies that E'_+ is weak* total. As the weak* and the weak topology coincide for duals of Hilbert spaces, it follows that E_+ is even weakly total, which already implies that E_+ is total.

(b) Consider $E = C^1([0,1])$ with the pointwise order and the usual C^1-norm. By Theorem 4.2.6 (ii) $\Rightarrow$ (i) it is easily checked that every positive functional on E can be extended to a positive functional on $C([0,1])$. Hence, every positive functional φ on $C^1([0,1])$ is of the form

$$\varphi(f) = \int f \, \mathrm{d}\mu$$

for some finite and positive measure $\mu : \mathcal{B}([0,1]) \to [0,\infty)$.

We claim that the continuous functional

$$\psi : C^1([0,1]) \to \mathbb{R}, \quad \psi(f) = f'(0)$$

is not in the norm closure of $E_+ - E_+$. Suppose to show a contradiction that ψ is in the norm closure of $E_+ - E_+$. Let $\varepsilon \in (0, 1/8)$ and $\varphi^+, \varphi^- \in E_+$ such that

$$\sup_{\|f\|_{C^1} \leq 1} |\langle \psi - \varphi, f \rangle| = \|\psi - \varphi\|_{C^1} < \varepsilon,$$

where $\varphi = \varphi^+ - \varphi^-$. Since $\|\psi\| = 1$ there exists $g \in E$ with $\|g\|_{C^1} = 1$ and $|\psi(g)| \geq 1 - \varepsilon$. Then $|\varphi(g)| \geq |\psi(g)| - |\varphi(g) - \psi(g)| \geq |\psi(g)| - \varepsilon \geq 1 - 2\varepsilon$. Let $(f_n)_{n \in \mathbb{N}}$ be a sequence in E with $\|f_n\| \leq 1 + \frac{1}{2n^2}$ such that[1]

$$f_n'(0) = 0, \quad f_n(x) = g(x) \qquad \text{for all } x \in [1/n, 1],\ n \in \mathbb{N}.$$

Then, clearly, $f_n \to g$ pointwisely, so by the bounded convergence theorem it follows that $|\varphi(f_n)| \to |\varphi(g)| \geq 1 - 2\varepsilon$. In particular, we obtain that

$$\begin{aligned} 1/2 < 1 - 3\varepsilon < |\varphi(f_n)| &= |f_n'(0) - \varphi(f_n)| = |\langle \psi - \varphi, f_n \rangle| \\ &\leq \sup_{\|f\|_{C^1} \leq 1 + \frac{1}{2n^2}} |\langle \psi - \varphi, f \rangle| < \varepsilon + \frac{\varepsilon}{2n^2} \end{aligned}$$

for all n large enough. This is a contradiction.

Exercise 2 (Multplicative functionals). Endow $C^1([0,1])$ with the pointwise order and the C^1-norm. Let $\varphi : C^1([0,1]) \to \mathbb{R}$ be linear and multiplicative (meaning that $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in C^1([0,1])$.

(a) Prove that φ is positive if and only if φ is continuous.
Hint: Be careful that the square root of a positive function in $C^1([0,1])$ might not be in $C^1([0,1])$, in general.

(b) *A bit more challenging*: Prove that φ is always positive and continuous.
Hint: If you know about automatic continuity of characters on Banach algebras, you can use this to solve the exercise. But it is also possible to solve it directly (i.e., without any Banach algebra theory).

Solution:

[1] Such a sequence exists. Consider for example a sequence given by $f_n'(x) = x \cdot g'(\frac{1}{n})$ for all $x \in [0, \frac{1}{n}]$ and $f_n'(x) = g'(x)$ for all $x \in [\frac{1}{n}, 1]$. Together with the condition $f_n(\frac{1}{n}) = g(\frac{1}{n})$ this sequence is then uniquely determined. It is easily checked that $\|f_n\|_{C^1} \leq 1 + 1/2n^2$, since $\|g\|_{C^1} = 1$.

(a) *Claim:* We first claim that $C^1([0,1])$ is generating.

Proof. To see this notice that $f^+ = f + \|f\| \cdot \mathbb{1}$ and $f^- = \|f\| \cdot \mathbb{1}$ are both in $C^1([0,1])_+$ and satisfy $f = f^+ - f^-$. □

We now prove the exercise.

"⇒": If φ is positive, it follows from the claim and Theorem 4.4.1 that φ is continuous.

"⇐": Let φ be continuous and $f \in C^1([0,1])_+$. Since the square root function is continuously differentiable on the interval $(0,1)$, we have $x \mapsto \sqrt{g(x)} \in C^1([0,1])$ if $g(x) > 0$ for all $x \in [0,1]$. So we let $\varepsilon > 0$ and consider $g_\varepsilon(x) := \varepsilon + f(x)$ for all $x \in [0,1]$. Then $g_\varepsilon(x) > 0$ for all $x \in [0,1]$ and

$$\varphi(g_\varepsilon) = \varphi(\sqrt{g_\varepsilon})^2 \geq 0$$

and $g_\varepsilon \to f$ in $C^1([0,1])$ as $\varepsilon \downarrow 0$, we obtain from the continuity of φ that $\varphi(f) \geq 0$. This shows the positivity of φ.

(b) We first prove the claim using the automatic continuity of characters in Banach algebras. First note that if we define a product on $C^1([0,1])$ by pointwise multiplication, it becomes an algebra. If we endow $C^1([0,1])$ with the norm[2]

$$\|f\| = \|f\|_\infty + \|f'\|_\infty, \qquad \text{for all } f \in C^1([0,1]),$$

then $C^1([0,1])$ becomes a Banach algebra.

The only thing that is not easily seen is the submultiplicativity of the norm. This follows as such: We have for $f, g \in C^1([0,1])$ that

$$\begin{aligned} \|fg\| &= \|fg\|_\infty + \|(fg)'\|_\infty \\ &\leq \|f\|_\infty \|g\|_\infty + \|f\|_\infty \|g'\|_\infty + \|f'\|_\infty \|g\|_\infty \\ &\leq \|f\|_\infty \|g\| + \|f'\|_\infty \|g\| \\ &= \|f\| \|g\|. \end{aligned}$$

We may assume without restricting generality that $\varphi \neq 0$. Then there exists $f \in C^1([0,1])$ such that $\varphi(f) \neq 0$. Thus, $\varphi(f) = \varphi(\mathbb{1} \cdot f) = \varphi(\mathbb{1})\varphi(f)$, which implies that $\varphi(\mathbb{1}) = 1$, so φ is a unital algebra homomorphism, which is always continuous.

To prove the claim without any Banach algebra theory, we need to show that we do not need the continuity in the proof of "⇐" in (a). Indeed, it suffices to prove that

$$\varphi(g_\varepsilon) \to \varphi(f)$$

without using the continuity of φ. But this is obvious, since

$$\varphi(g_\varepsilon) - \varphi(f) = \varphi(f + \varepsilon \cdot \mathbb{1}) - \varphi(f) = \varphi(f) - \varepsilon\varphi(\mathbb{1}) - \varphi(f) = \varepsilon\varphi(\mathbb{1}) \to 0,$$

as $\varepsilon \downarrow 0$.

[2] Instead of the usual but equivalent C^1-norm defined by $\|f\|_{C^1} = \max\{\|f\|_\infty, \|f'\|_\infty\}$ for each $f \in C^1([0,1])$.

Exercise 3 (Disjointness).
(a) Let $k \geq 0$ be an integer and consider the ordered Banach space $E := C^k([-1,1])$. Let $0 \leq f, g \in E$. Show that the following assertions are equivalent.

(i) The vectors f and g are disjoint.

(ii) The vectors f and g are D-disjoint.

(iii) We have $fg = 0$.

(b) Endow $\mathbb{R}^d$ with the ice-cream cone $\mathbb{R}^d_+$ and let $x, y \in \partial\mathbb{R}^d_+$ be non-zero. Show that x and y are not disjoint.

Solution:

(a) Let $0 \leq f, g \in E$.

"(i) $\Rightarrow$ (ii)": This follows immediately from Proposition 5.1.3 and holds for every two element in every ordered vector space.

"(ii) $\Rightarrow$ (iii)": By D-disjointness we have $[0,f] \cap [0,g] = \{0\}$. Since f and g are bounded function, we may assume after rescaling that $0 \leq f, g \leq \mathbb{1}$. So in particular $0 \leq f(x)g(x) \leq f(x), g(x)$ for all $x \in [-1,1]$. So $0 \leq fg \leq f, g$. This implies that $fg \in [0,f] \cap [0,g] = \{0\}$. Hence, $fg = 0$.

"(iii) $\Rightarrow$ (i)": Suppose that $fg = 0$. Let $\tilde{l}$ be a lower bound of $\{f, g\}$. Then $\tilde{l}(x) \leq f(x)$ and $\tilde{l}(x) \leq g(x)$ for all $x \in [-1,1]$. Notice that $l = 0$ is a lower bound of $\{g, f\}$. We need to show that $\tilde{l} \leq l = 0$. Suppose to show a contradiction that there exists $x \in [-1,1]$ such that $\tilde{l}(x) > 0$. Then

$$0 < \tilde{l}(x)^2 \leq \tilde{l}(x)g(x) \leq f(x)g(x) = 0,$$

which is a contradiction. It follows that $\tilde{l} \leq l \leq 0$. Hence $l = 0$ is the infimum of $\{f, g\}$ and it follows that f and g are disjoint.

(b) Notice first that the claim trivially true if $d = 1$, since there the ice-cream cone are just the nonnegative reals. It is false if $d = 2$, as can be seen very easily (for example by determining the set $\{x, y\}^{\mathrm{lb}}$ and showing that it coincides with $-E_+$.). So we may assume that $d \geq 3$.

Let $x, y \in \partial\mathbb{R}^d_+$ and denote by $\|\cdot\|$ the euclidean norm on the respective spaces $\mathbb{R}^d$ and $\mathbb{R}^{d-1}$. Since disjointness is invariant with respect to scaling with positive scalars (see Proposition 5.1.8 (e)), we may assume that $x_1 = y_1 = 1$. Let $P : \mathbb{R}^d \to \mathbb{R}^{d-1}$, $x = (x_1, \dots, x_d) \mapsto (x_2, \dots, x_d)$. Since x, y are in the boundary of the ice cream cone, $x_1 = \|Px\|$ and $y_1 = \|Py\|$. As $d \geq 3$, there exists $\tilde{z} \in \mathbb{R}^{d-1}$ with $\|\tilde{z}\| = 1$ and $\langle \tilde{z}, Px \rangle \geq 0$ as well as $\langle \tilde{z}, Py \rangle \geq 0$. Choose $\varepsilon > 0$ such that $\varepsilon^2 < \|\tilde{z}\|^2 = 1 \leq 2\varepsilon + \varepsilon^2$ and define $z \in \mathbb{R}^d$ by $z_1 = -\varepsilon$ and $Pz = \tilde{z}$.

We now prove the following two claims, which together imply that 0 is not the infimum of $\{x, y\}$.

Claim. We have $z \not\leq 0$.

Proof. Recall that a point v lies in $-E_+$ if $v_1 \leq 0$ and $v_1^2 \geq \|Pv\|^2$. For z we obtain

$$z_1^2 = \varepsilon^2 < \|\tilde{z}\|^2 = \|Pz\|^2 .$$

Hence, $z \nleq 0$. □

Claim. We have $z \leq x, y$.

Proof. We only show that $0 \leq x - z$. The proof for y is analogous. Clearly, $x_1 - z_1 = 1 - \epsilon \geq 0$ by the choice of ϵ. Furthermore

$$\begin{aligned} \|P(x-z)\|^2 = \|Px - \tilde{z}\|^2 &= \|Px\|^2 - 2\langle Px, \tilde{z}\rangle + \|\tilde{z}\|^2 \\ &\leq \|Px\|^2 + \|\tilde{z}\|^2 \leq 1 + 2\epsilon + \epsilon^2 \\ &= (1+\epsilon)^2 = (x_1 - z_1)^2. \end{aligned}$$

Hence, $x - z \geq 0$. □