8. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on June 6 and 7, 2023

Exercise 1 (Properties of wedges in duality).

(a) Give an example of an ordered Banach space E such that E_+ is not normal but E'_+ is total in E'.

(b) Give an example of an ordered Banach space E such that (E_+ is not normal and) E'_+ is not total in E'.

Exercise 2 (Multplicative functionals). Endow $C^1([0,1])$ with the pointwise order and the C^1-norm. Let $\varphi : C^1([0,1]) \to \mathbb{R}$ be linear and multiplicative (meaning that $\varphi(fg) = \varphi(f)\varphi(g)$ for all $f, g \in C^1([0,1])$).

(a) Prove that φ is positive if and only if φ is continuous.
Hint: Be careful that the square root of a positive function in $C^1([0,1])$ might not be in $C^1([0,1])$, in general.

(b) *A bit more challenging*: Prove that φ is always positive and continuous.
Hint: If you know about automatic continuity of characters on Banach algebras, you can use this to solve the exercise. But it is also possible to solve it directly (i.e., without any Banach algebra theory).

Exercise 3 (Disjointness).

(a) Let $k \geq 0$ be an integer and consider the ordered Banach space $E := C^k([-1,1])$ (compare Exercise 1 on Sheet 4). Let $0 \leq f, g \in E$. Show that the following assertions are equivalent.

(i) The vectors f and g are disjoint.

(ii) The vectors f and g are D-disjoint.

(iii) We have $fg = 0$.

(b) Endow $\mathbb{R}^d$ with the ice-cream cone $\mathbb{R}^d_+$ and let $x, y \in \partial\mathbb{R}^d_+$ be non-zero. Show that x and y are not disjoint.