7. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on May 23 and 24, 2023

with Solutions

Exercise 1 (Positive extension of functionals). Consider the ordered Banach space $E := \mathrm{C}^1([0, 2\pi])$ (with the pointwise order and the C^1-norm).

(a) Let $V \subseteq E$ denote the span of $\{\mathbb{1}, \sin\}$ and let the functional $v' \in V'$ be given by

$$\langle v', \alpha\, \mathbb{1} + \beta \sin \rangle = \alpha$$

for all $\alpha, \beta \in \mathbb{R}$. Is v' positive? Can v' be extended to a functional $x' \in E'_+$?

(b) Let $W := \{w \in E \mid\ w(0) = 0\}$ and let $w' \in W'$ be given by

$$\langle w', w \rangle := \frac{\mathrm{d}}{\mathrm{d}x} w(x) \Big|_{x=0}$$

for all $w \in W$. Is w' positive? Can w' be extended to a functional $x' \in E'_+$?

Solution:

(a) *Claim:* The functional $v' : V \to \mathbb{R}$ is positive and it can be extended to a positive linear functional in V'.

Proof. Positivity: We have $\langle v', v \rangle = v(0)$ for each $v \in V$, which is clearly positive if v is positive everywhere on $[0, 2\pi]$.

Positive and continuous extension: The functional $\delta_0 \in E'$ that is given by $\langle \delta_0, f \rangle = f(0)$ for each $f \in E$ is positive and extends v.

Alternatively, the following abstract argument works to get a positive and continuous extension of v': As $\mathbb{1} \in V$, the subspace V is majorizing in E. Hence, Corollary 4.2.2 shows that there exists a positive linear functional $\varphi : E \to \mathbb{R}$ that extends v'. To show continuity of φ, observe that that every f in the unit ball of E satisfies $-\mathbb{1} \le f \le \mathbb{1}$ and thus, $|\varphi(f)| \le \varphi(\mathbb{1})$.[1] ☐

(b) *Claim:* The functional $w' : W \to \mathbb{R}$ is positive, but it cannot be extended to a positive linear functional in E'.

[1] We will see in Section 4.4 that positive linear functionals on ordered Banach spaces with generating cone are automatically continuous.

Proof. Positivity: For every $w \in W_+$ we have

$$\langle w', w\rangle = \lim_{x\downarrow 0} \frac{w(x) - w(0)}{x - 0} = \lim_{x\downarrow 0} \frac{w(x)}{x} \geq 0$$

since $w(x) \geq 0$ for every $x \in [0, 2\pi]$. So w' is indeed positive.

Non-existence of positive and continuous extension: To see that w' cannot be extended to an element $x' \in E'_+$ we use Theorem 4.2.4 (i) $\Leftrightarrow$ (ii): Let $n \in \mathbb{N}$. There exists a function $w_n \in W$ that satisfies $0 \leq w_n \leq \mathbb{1}$ and whose derivative at 0 is equal to n.

Thus, w_n is dominated by the element $\mathbb{1}$ of the unit ball of E, but one has $\langle w', w_n\rangle = n$. Hence, assertion (ii) in Theorem 4.2.4 is not satisfied for any constant $c \geq 0$. $\square$

Exercise 2 (Distance to the cone).

(a) Let (Ω, μ) be a measure space, let $p \in [1, \infty]$, and endow $L^p := L^p(\Omega, \mu)$ with its usual norm and the pointwise almost everywhere order. Show that

$$\operatorname{dist}(f, -L^p_+) = \left\|f^+\right\|$$

for each $f \in L^p$, where $f^+ \in L^p$ is defined by the formula $f^+(\omega) := f(\omega) \vee 0$ for almost all $\omega \in \Omega$.

(b) Let H be an infinite-dimensional, separable, complex Hilbert space[2] and let E denote the space of all self-adjoint compact linear opeators on H, endowed with the Loewner order. Show that

$$\operatorname{dist}(A, -E_+) = \left\|A^+\right\|$$

for each $A \in E$.
Here, A^+ is defined by means of the functional calculus, i.e., if $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues of A and (u_n) is an orthonormal basis of H that consists of corresponding eigenvectors, then

$$A^+ := \sum_{n=1}^{\infty} \lambda_n^+ \, u_n \otimes u_n$$

with $\lambda_n^+ = \lambda_n \vee 0$ (where the series converges unconditionally with respect to the operator norm).

Solution:

(a) Let $f \in L^p$.

"$\leq$": The function $f - f^+$ is in $-L^p_+$ and its distance to f is $\|f^+\|$.

"$\geq$": Let $g \in -L^p_+$. Then $f - g \geq f$, so $(f - g)^+ \geq f^+$ and hence,

$$\|f - g\| \geq \left\|(f - g)^+\right\| \geq \left\|f^+\right\|,$$

which proves the claim.

[2] Again, the assumptions that H be infinite-dimensional and separable are actually not needed here; it is here to simplify the notation.

(b) Let $A \in E$.

"$\leq$": It follows from the definition of A^+ that $B := A - A^+$ is in $-E_+$. Moreover, B has distance $\|A^+\|$ from A.

"$\geq$": If all eigenvalues of A are strictly negative, then $A^+ = 0$ and the claimed inequality is clear. So assume that at least one eigenvalue of A is ≥ 0. Since A is compact, the sequence of eigenvalues converges to 0, so there exists a maximal eigenvalue $\lambda \geq 0$ of A. Let $u \in H$ be a corresponding eigenvector of norm 1. The definition of A^+ gives $\lambda = \|A^+\|$. Thus, or every $C \in -E_+$ we have

$$\|A - C\| \geq ((A - C)u \mid u) \geq (Au \mid u) = \lambda\,(u \mid u) = \lambda = \left\|A^+\right\|,$$

which shows the desired inequality.

Exercise 3 (Distance to the cone and positive extension of functionals). Endow $[-1,1]$ with the Borel σ-algebra and the Lebesgue measure and endow the space $L^1 := L^1([-1,1])$ with its usual norm and the pointwise almost everywhere order. Consider the functions $v, w \in L^1_+$ that are given by

$$v(x) = 1 + x \qquad \text{and} \qquad w(x) = 1 - x$$

for almost all $x \in [-1,1]$ and let $V \subseteq L^1$ denote the linear span of $\{v, w\}$. Let $v' \in V'$ be given by

$$\langle v', \alpha v + \beta w\rangle = \alpha$$

for all $\alpha, \beta \in \mathbb{R}$.

(a) Show that a vector $\alpha v + \beta w \in V$ (with $\alpha, \beta \in \mathbb{R}$) is positive if and only if $\alpha, \beta \geq 0$. Conclude that the functional v' is positive.

(b) Show that v' cannot be extended to a positive and continuous linear functional on all of L^1.

Hint: First show that, for $g \in L^\infty([-1,1])$, the functional $f \mapsto \int_{-1}^1 f(x)g(x)\,\mathrm{d}x$ on L^1 is positive if and only if $g(x) \geq 0$ for almost all $x \in [-1,1]$.

(c) It follows from part (b) and from Theorem 4.2.6 that there exists a sequence (v_n) in V such that

$$\operatorname{dist}(v_n, V_+) \to \infty, \qquad \text{while } \operatorname{dist}(v_n, E_+) \text{ remains bounded}$$

as $n \to \infty$. Find an explicit example of such a sequence (v_n).

Hint: First show, for instance by distinguishing different cases for the signs of α and β, that $\|\alpha v + \beta w\| \geq \max\{|\alpha|, |\beta|\}$ for all $\alpha, \beta \in \mathbb{R}$.

Solution:

(a) *Claim:* Let $\alpha, \beta \in \mathbb{R}$. Then $\alpha v + \beta w \geq 0$ if and only if $\alpha, \beta \geq 0$.

Proof. "⇐": This implication is clear since $v, w \geq 0$.

"⇒": Let $\alpha v + \beta w \geq 0$. It follows that

$$0 \leq \alpha \underbrace{\int_{-1}^{-1+\varepsilon} v(x)\,\mathrm{d}x}_{=\varepsilon^2/2} + \beta \underbrace{\int_{-1}^{-1+\varepsilon} w(x)\,\mathrm{d}x}_{=2\varepsilon-\varepsilon^2/2}$$

for all small $\varepsilon > 0$. Dividing by ε and letting $\varepsilon \downarrow 0$ yields $\beta \geq 0$.

The same argument, but with integration from $1 - \varepsilon$ to 1, shows that $\alpha \geq 0$.

Alternatively, the following abstract argument works also to prove that $\alpha, \beta \geq 0$. Notice that the identity mapping $V \to \mathrm{C}([-1, 1])$ is well-defined and bi-positive. Then $0 \leq \alpha v(0) + \beta w(0) = \beta$ shows that $\beta \geq 0$ and $0 \leq \alpha v(1) + \beta w(0) = \alpha$ show that $\alpha \geq 0$. □

Claim: The functional $v' \in V'$ is positive.

Proof. Let $\alpha, \beta \in \mathbb{R}$ and assume that $\alpha v + \beta w \geq 0$. As we have just shown it follows that $\alpha, \beta \geq 0$. Hence $\langle v', \alpha v + \beta w\rangle = \alpha \geq 0$. □

(b) *We first show the claim in the hint.*

Proof. Fix $n \in \mathbb{N}$ and consider the set $M_n := \{x \in [-1, 1] : g(x) \leq -\frac{1}{n}\}$. Then $\mathbb{1}_{M_n} \in L^1_+$ and hence,

$$0 \leq \int_{-1}^{1} \mathbb{1}_{M_n}(x) g(x)\,\mathrm{d}x \leq -\frac{1}{n} |M_n|,$$

where $|M_n|$ denotes the Lebesgue measure of M_n. Thus, $|M_n| = 0$. Hence, the set $\{x \in [-1, 1] : g(x) < 0\} = \bigcup_{n\in\mathbb{N}} M_n$ has Lebesgue measure 0. □

Claim: The functional $v' \in V'$ cannot be extended to a positive continuous linear functional on L^1.

Proof. Assume that there exists a positive continuous linear functional x' on L^1. There exists a function $g \in L^\infty$ such that

$$\langle x', f\rangle = \int_{-1}^{1} f(x) g(x)\,\mathrm{d}x$$

for all $f \in L^1$. By the hint that we have shown above, we know that $g(x) \geq 0$ for almost all $x \in [-1, 1]$.

Since $\langle x', v\rangle = \langle v', v\rangle = 1$, the function g is not the 0 element of L^∞. Hence,

$$0 < \int_{-1}^{1} w(x) g(x)\,\mathrm{d}x = \langle x', w\rangle = \langle v', w\rangle = 0,$$

which is a contradiction. □

(c) *We first prove the claim in the hint.*

Proof. If $\alpha, \beta \geq 0$, then we obtain, since both v and w are positive functions[3], that

$$\|\alpha v + \beta w\| = \alpha \|v\| + \beta \|w\| = 2(\alpha + \beta) \geq |\alpha| \vee |\beta| \,.$$

If $\alpha, \beta \leq 0$ we can multiply the function $\alpha v + \beta w$ with -1 to get the same estimate.

Now let α and β have different sign. We distinguish two cases:

First case: $|\alpha| \leq |\beta|$.
By multiplying with -1 if necessary we may assume that $\alpha \leq 0$ and $\beta \geq 0$, so $\beta \geq -\alpha \geq 0$. On the interval $[-1, 0]$ the function w dominates v, so $\beta w(x) \geq -\alpha v(x)$ for all $x \in [-1, 0]$. Hence,

$$\|\alpha v + \beta w\| \geq \int_{-1}^{0} \alpha v(x) + \beta w(x) \, \mathrm{d}x = \frac{1}{2}\alpha + \frac{3}{2}\beta \geq \beta = |\alpha| \vee |\beta| \,,$$

where the last inequality follows from $\alpha \geq -\beta$.

Second case: $|\alpha| \geq |\beta|$.
By multiplying with -1 if necessary we may then assume that $\beta \leq 0$ and $\alpha \geq 0$, so $0 \leq -\beta \leq \alpha$. The same argument as in the first case, but now on the interval $[0, 1]$, then shows that $\|\alpha v + \beta w\| \geq \alpha = |\alpha| \vee |\beta|$. ☐

Construction of the sequence (v_n)*:* Set, for instance, $v_n := -nv + n^2 w$ for each $n \in \mathbb{N}$.

Proof of the claimed properties of (v_n). Every element u of V_+ can, according to part (a), be written as $u = \alpha v + \beta w$ for numbers $\alpha, \beta \geq 0$. The distance of $-u$ to v_n for any $n \in \mathbb{N}$ is

$$\|-u - v_n\| = \left\|(-\alpha - n)v + (-\beta + n^2)w\right\| \geq |-\alpha - n| \vee \left|-\beta + n^2\right| \geq \alpha + n \geq n,$$

where the first inequality follows from the hint and the latter two inequalities both use that $\alpha \geq 0$. Hence,

$$\operatorname{dist}(v_n, -V_+) \geq n$$

for each $n \in \mathbb{N}$.

On the other hand, fix $n \in \mathbb{N}$. The function v_n satisfies $v_n(x) \leq 0$ if and only if $x \geq \frac{n-1}{n+1} = 1 - \frac{2}{n+1} =: 1 - \delta_n$. With the notation $v_n^- := (-v_n)^+$ we thus get from Exercise 2 (b) that

$$\operatorname{dist}(v_n, L^1_+) = \left\|v_n^-\right\| = \int_{1-\delta_n}^{1} -v_n(x) \, \mathrm{d}x = n\big(2\delta_n - \frac{\delta_n^2}{2}\big) - n^2 \frac{\delta_n^2}{2} = \frac{2n}{n+1} \leq 2.$$

This shows that $\operatorname{dist}(v_n, L^1_+)$ remains bounded as $n \to \infty$. ☐

[3]And since L^1 is an AL-space; in particular, the norm is additive on the positive cone.