7. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on May 23 and 24, 2023

Exercise 1 (Positive extension of functionals). Consider the ordered Banach space $E \coloneqq C^1([0, 2\pi])$ (with the pointwise order and the C^1-norm).

(a) Let $V \subseteq E$ denote the span of $\{\mathbb{1}, \sin\}$ and let the functional $v' \in V'$ be given by

$$\langle \varphi, \alpha \, \mathbb{1} + \beta \sin \rangle = \alpha$$

for all $\alpha, \beta \in \mathbb{R}$. Is v' positive? Can v' be extended to a functional $x' \in E'_+$?

(b) Let $W \coloneqq \{w \in E \mid \; w(0) = 0\}$ and let $w' \in W'$ be given by

$$\langle w', w \rangle \coloneqq \frac{\mathrm{d}}{\mathrm{d}x} w(x) \Big|_{x=0}$$

for all $w \in W$. Is w positiv? Can w' be extended to a functional $x' \in E'_+$?

Exercise 2 (Distance to the cone).

(a) Let (Ω, μ) be a measure space, let $p \in [1, \infty]$, and endow $L^p \coloneqq L^p(\Omega, \mu)$ with its usual norm and the pointwise almost everywhere order. Show that

$$\operatorname{dist}(f, -L^p_+) = \left\| f^+ \right\|$$

for each $f \in L^p$, where $f^+ \in L^p$ is defined by the formula $f^+(\omega) \coloneqq f(\omega) \vee 0$ for almost all $\omega \in \Omega$.

(b) Let H be an infinite-dimensional, separable, complex Hilbert space[1] and let E denote the space of all self-adjoint compact linear opeators on H, endowed with the Loewner order. Show that

$$\operatorname{dist}(A, -E_+) = \left\| A^+ \right\|$$

for each $A \in E$.

Here, A^+ is defined by means of the functional calculus, i.e., if $(\lambda_n)_{n\in\mathbb{N}}$ is the sequence of eigenvalues of A and (u_n) is an orthonormal basis of H that consists of corresponding eigenvectors, then

$$A^+ \coloneqq \sum_{n=1}^{\infty} \lambda_n^+ \, u_n \otimes u_n$$

[1] Again, the assumptions that H be infinite-dimensional and separable are actually not needed here; there is just here to simplify the notation.

with $\lambda_n^+ = \lambda_n \vee 0$ (where the series converges unconditionally with respect to the operator norm).

Exercise 3 (Distance to the cone and positive extension of functionals). Endow $[-1,1]$ with the Borel σ-algebra and the Lebesgue measure and endow the space $L^1 := L^1([-1,1])$ with its usual norm and the pointwise almost everywhere order. Consider the functions $v, w \in L^1_+$ that are given by

$$v(x) = 1 + x \qquad \text{and} \qquad w(x) = 1 - x$$

for almost all $x \in [-1,1]$ and let $V \subseteq L^1$ denote the linear span of $\{v, w\}$. Let $v' \in V'$ be given by

$$\langle v', \alpha v + \beta w \rangle = \alpha$$

for all $\alpha, \beta \in \mathbb{R}$.

(a) Show that a vector $\alpha v + \beta w \in V$ (with $\alpha, \beta \in \mathbb{R}$) is positive if and only if $\alpha, \beta \geq 0$. Conclude that the functional v' is positive.

(b) Show that v' cannot be extended to a positive and continuous linear functional on all of L^1.

Hint: First show that, for $g \in L^\infty([-1,1])$, the functional $f \mapsto \int_{-1}^1 f(x)g(x)\,\mathrm{d}x$ on L^1 is positive if and only if $g(x) \geq 0$ for almost all $x \in [-1,1]$.

(c) It follows from part (b) and from Theorem 4.2.6 that there exists a sequence (v_n) in V such that

$$\operatorname{dist}(v_n, V_+) \to \infty, \qquad \text{while } \operatorname{dist}(v_n, E_+) \text{ remains bounded}$$

as $n \to \infty$. Find an explicit example of such a sequence (v_n).

Hint: First show, for instance by distinguishing different cases for the signs of α and β, that $\|\alpha v + \beta w\| \geq \max\{|\alpha|, |\beta|\}$ for all $\alpha, \beta \in \mathbb{R}$.