6. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on May 16 and 17, 2023

with Solutions

Exercise 1 (The Loewner order on the compact operators, again). Let H be an infinite-dimensional separable complex Hilbert space and let E denote the space of all self-adjoint compact linear operators on H, endowed with the Loewner order. From Exercise 5(a) on Sheet 5 you know that E_+ is a generating cone.

(a) Give an example of functions γ^+, γ^- with the properties in Theorem 3.4.1.

(b) Is E_+ normal?

Solution:

(a) The decomposition into γ^+ and γ^- is constructed analogous to the decomposition in Exercise 5 (a) on Sheet 5. We repeat it anyway. Recall that by the spectral theorem for self-adjoint compact operators for each $A \in \mathcal{K}(H)_{\mathrm{sa}}$ there exists an orthonormal basis $(a_n)_{n\in\mathbb{N}}$ of eigenvectors of A and a sequence $(\alpha_n)_{n\in\mathbb{N}}$ of eigenvalues of A (which are real) that converges to 0 such that

$$A = \sum_{n=1}^{\infty} \alpha_n \, (a_n \otimes a_n)$$

We then set

$$\gamma^+(A) = \sum_{n=1}^{\infty} \max(\alpha_n, 0)\,(e_n \otimes e_n), \quad \gamma^-(A) = \sum_{n=1}^{\infty} \max(-\alpha_n, 0)\,(e_n \otimes e_n).$$

It is also easy to see that the operator A^+ and A^+ obtained through this construction do not depend on the explicit choices of sequences $(a_n)_{n\in\mathbb{N}}$ and $(\alpha_n)_{n\in\mathbb{N}}$.

We show that this defines the desired decomposition. The positive homogeneity it clear. The norm bound follows from

$$\begin{aligned}\|\gamma^+(A)\|^2 &= \sup_{\|x\|=1} (\gamma^+(A)x \mid \gamma^+(A)x) \\ &= \sup_{n\in\mathbb{N}} \max(\alpha^2, 0) \le \sup_{n\in\mathbb{N}} |\alpha_n^2| = \sup_{\|x\|=1} (Ax \mid Ax) = \|A\|^2.\end{aligned}$$

So we have

$$\|\gamma^+(A)\| \le \|A\|, \qquad \text{and analogously,} \qquad \|\gamma^-(A)\| \le \|A\|,$$

as desired.

We show that γ^+ and γ^- are continuous. To this end let

$$A_k = \sum_{n=1}^{\infty} \alpha_n^{(k)} \, (a_n^{(k)} \otimes a_n^{(k)}), \qquad A = \sum_{n=1}^{\infty} \alpha_n \, (a_n \otimes a_n),$$

with $A_k \to A$ for $k \to \infty$. Clearly, the set $M := \{\alpha_n^{(k)}, \alpha_n \mid n, k \in \mathbb{N}\}$ is bounded. Take a sequence of polynomials $(p_i)_{i\in\mathbb{N}}$ in $\mathbb{C}[X]$ that converges uniformly on M to $\max(\,\cdot\,, 0)$. Then, by orthogonality of the sequences $(a_n^{(k)})_{n\in\mathbb{N}}$ and $(a_n)_{n\in\mathbb{N}}$ it follows that

$$p_i(A_k) = \sum_{n=1}^{\infty} p_i(\alpha_n^{(k)}) \, (a_n^{(k)} \otimes a_n^{(k)}), \qquad p_i(A) = \sum_{n=1}^{\infty} p_i(\alpha_n) \, (a_n \otimes a_n),$$

Moreover, for all $k \in \mathbb{N}$, the uniform convergence of the polynomials on M implies that for every $\epsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$ we have

$$\|p_i(A_k) - \gamma^+(A_k)\|, \, \|p_i(A) \to \gamma^+(A)\| < \epsilon.$$

in operator norm. Furthermore, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$ we have

$$\|p_{i_0}(A_k) - p_{i_0}(A)\| < \epsilon.$$

Altogether, we obtain

$$\begin{aligned} \|\gamma^+(A_k) - \gamma^+(A)\| &\leq \|\gamma^+(A_k) - p_{i_0}(A_k)\| \\ &\quad + \|p_{i_0}(A_k) - p_{i_0}(A)\| + \|p_{i_0}(A) - \gamma^+(A)\| < 3\epsilon. \end{aligned}$$

Hence, γ^+ is continuous. The proof for γ^- is analogous.

(b) Let $0 \leq A \leq B$. Then $0 \leq (x \mid Ax) \leq (x \mid Bx)$ for all $x \in H$. Notice that the roots $A^{1/2}$ and $B^{1/2}$ are also self-adjoint. Hence

$$\begin{aligned} \left\|A^{1/2}\right\| &= \sup_{\|x\|=1} \sqrt{(A^{1/2}x \mid A^{1/2}x)} = \sup_{\|x\|=1} \sqrt{(x \mid Ax)} \\ &\leq \sup_{\|x\|=1} \sqrt{(x \mid Bx)} = \sup_{\|x\|=1} \sqrt{(B^{1/2}x \mid B^{1/2}x)} = \left\|B^{1/2}\right\|. \end{aligned}$$

Now the statement follows by noticing that $\|A^2\| = \|A\|^2$ for all $A \in \mathcal{K}(H)_{\mathrm{sa}}$. The inequality "$\leq$" follows from the submulitplicativity of the norm. The converse inequality follows from

$$\|A^2\| \geq \|A^2 x\| \geq (x \mid A^2 x) = (Ax \mid Ax) \qquad \text{for all } x \in H.$$

Taking the supremum over all $x \in H$ with $\|x\| = 1$ yields the claim.

Exercise 2 (Some function spaces).

(a) Let (Ω, μ) be a measure space and let $p \in [1, \infty]$. Endow $L^p(\Omega, \mu)$ with the pointwise almost everywhere order and its usual norm. Find mappings $\gamma^+, \gamma^- : L^p(\Omega, \mu) \to L^p(\Omega, \mu)_+$ with the properties stated in Theorem 3.4.1.

(b) Let E be a pre-ordered Banach space and let M be a non-empty compact metric space.[1] Let $\mathrm{C}(M;E)$ denote the space of all continuous functions from M to E, endowed with the norm given by

$$\|f\|_\infty := \max_{x\in M} \|f(x)\|_E$$

for each $f \in \mathrm{C}(M;E)$ and with the pointwise order. It is not difficult to show that $\mathrm{C}(M;E)$ is a pre-ordered Banach space.
When is the wedge $\mathrm{C}(M;E)_+$ a cone? When is it normal? When is it generating?

Solution:

(a) Set

$$\begin{aligned} \gamma^+ &: L^p(\Omega,\mu) \to L^p(\Omega,\mu)_+, \quad f \mapsto \max(f,0), \\ \gamma^- &: L^p(\Omega,\mu) \to L^p(\Omega,\mu)_+, \quad f \mapsto \max(-f,0). \end{aligned}$$

Then γ^+ and γ^- are positive homogeneous and continuous. Moreover, for $f \in L^p(\Omega,\mu)$ we have

$$\|\gamma^+(f)\|,\ \|\gamma^-(f)\| \leq \|f\|.$$

(b) Since the constant functions are in $\mathrm{C}(M;E)$, it follows that E is a cone, whenever $\mathrm{C}(M;E)_+$ is a cone. Conversely, if E is a cone and $f, -f \in \mathrm{C}(M;E)_+$. Then, pointwisely, $f(x) \in E_+ \cap -E_+$. So $f(x) = 0$ for all $x \in M$ which implies $f = 0$.

Since the constant functions are in $\mathrm{C}(M;E)$, it follows that E is normal, whenever $\mathrm{C}(M;E)$ is normal. Conversely, if E is normal and $0 \leq f \leq g$ are in $\mathrm{C}(M;E)$ it follows that $0 \leq f(x) \leq g(x)$ for all $x \in M$ and thus, $\|f(x)\| \leq \|g(x)\|$. Hence, it follows that $\|f\| \leq \|g\|$.

Since the constant functions are in $\mathrm{C}(M;E)$, it follows that E is generating, whenever $\mathrm{C}(M;E)$ is generating. If E_+ is generating, let $\gamma^+, \gamma^- : E \to E_+$ be as in Theorem 3.4.1. For $f \in \mathrm{C}(M;E)$ set $f^+(x) := \gamma^+(f(x))$ and $f^-(x) := \gamma^+(f(x))$ for all $x \in M$. Then there exists $M \geq 1$ such that

$$\|f^+\| = \sup_{x\in M} \|f^+(x)\| \leq \sup_{x\in M} \|f(x)\|$$

So $f^+ \in \mathrm{C}(M;E)$ and analogously $f^- \in \mathrm{C}(M;E)$. Hence, $\mathrm{C}(M;E)_+$ is generating.

Exercise 3 (Lipschitz continuous decomposition?). Prove or disprove: For every pre-ordered Banach space E whose wedge is generating, there exist Lipschitz continuous maps $\gamma^+, \gamma^- : E \to E_+$ such that $x = \gamma^+(x) - \gamma^-(x)$ for each $x \in E$.

Solution: This is in fact an open problem. If you have solved it, you should probably write a paper about it.

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[1] Or, more generally, a non-empty compact Hausdorff space.

Exercise 4 (A space of holomorphic functions). Denote the open unit disk in $\mathbb{C}$ by $\mathbb{D}$. Let $\mathcal{H}^\infty(\mathbb{D})$ be the Banach space of all bounded holomorphic functions $\mathbb{D} \to \mathbb{C}$ with the supremum norm and set $E := \left\{ f \in \mathcal{H}^\infty(\mathbb{D}) \mid f(\frac{1}{n}) \in \mathbb{R} \text{ for all } 2 \le n \in \mathbb{N} \right\}$. This is a closed subset of $\mathcal{H}^\infty(\mathbb{D})$ and thus a real Banach space with respect to the supremum norm over $\mathbb{D}$. Define $E_+ := \{ f \in E \mid f(\frac{1}{n}) \ge 0 \text{ for all } 2 \le n \in \mathbb{N}\}$. This turns E into a pre-ordered Banach space.

(a) Show that E_+ is a cone.

(b) Is the cone E_+ normal? Is it generating?

Solution:

(a) That E_+ is a wedge is clear. If $f, -f \in E_+$ it follows that $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$. In other words, f is 0 on a set with accumulation point. Hence, the identity theorem for holomorphic functions implies that $f = 0$.

(b) We first show that E_+ is not normal. By means of Theorem 3.5.5 (i) $\Rightarrow$ (iii) it suffices to show that the order interval $[0, \mathbb{1}]$ is not norm bounded. Here $\mathbb{1}$ denotes the constant function with the value 1, which is clearly in E_+. Since sin is entire and not constant, there exists a sequence $(z_n)_{n\in\mathbb{N}}$ with $|\sin(z_n)| \to \infty$. Clearly, the family $\mathbb{D} \ni z \to \sin(2|z_n|z)$ is in $[0, \mathbb{1}]$, since on the real axis these functions are bounded by 1. However, $\|z \mapsto \sin(2|z_n|z)\| \ge \left|\sin\left(2|z_n|\frac{z_n}{2|z_n|}\right)\right| \to \infty$. So $[0, \mathbb{1}]$ is not norm bounded.

We now show that E_+ is generating. Let $f \in E$. Then by continuity $f(0) \in \mathbb{R}$. Let $f(z) = \sum_{k=0}^\infty \alpha_k z^k$. Since f is bounded, this series converges absolutely on $\mathbb{D}$. Clearly, $\alpha_0 = f(0) \in \mathbb{R}$. Now $f_1 : z \mapsto \frac{1}{z}(f(z) - \alpha_0) = \sum_{k=0}^\infty \alpha_{k+1} z^k$ is also bounded and holomorphic on the open unit disk and satisfies $f_1(\frac{1}{n}) = n(f(\frac{1}{n}) - \alpha_0) \in \mathbb{R}$ for all $2 \le n \in \mathbb{N}$. So $f_1 \in E$, and thus, $\alpha_1 = f_1(0) \in \mathbb{R}$. By induction we can prove that $\alpha_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$.

Notice that if $f(z) = \sum_{k=0}^\infty \alpha_k z^k$ converges absolutely on $\mathbb{D}$ and $\alpha_k \ge 0$, then $f(\frac{1}{n}) = \sum_{k=0}^\infty \frac{\alpha_k}{n^k} \ge 0$.

So any function $f \in E$ can be decomposed into

$$f^+(z) = \sum_{k=0}^\infty \max(\alpha_n, 0) z^k, \qquad f^-(z) = \sum_{k=0}^\infty \max(-\alpha_n, 0) z^k$$

with $f^+, f^- \in E_+$ and $f = f^+ - f^-$. It follows that E_+ is generating.

Exercise 5 (Norm boundedness of order bounds). Let E be a pre-ordered Banach space and let I be a non-empty set. Let $\ell^{\mathrm{ob}}(I; E)$ denote the vector space of all functions $f : I \to E$ for which there exists a vector $h \in E_+$ such that $f(i) \in [-h, h]$ for all $i \in I$.[2] For each $f \in \ell^{\mathrm{ob}}(I; E)$ we define

$$\|f\|_{\mathrm{ob}} := \inf \left\{ \|h\|_E \mid h \in E_+ \text{ and } f(i) \in [-h, h] \text{ for all } i \in I \right\}.$$

We endow the vector space $\ell^{\mathrm{ob}}(I; E)$ with the pointwise order.

[2] Here, the acronym "ob" stands for "order bounded".

(a) When is the wedge $\ell^{\mathrm{ob}}(I;E)_+$ a cone? When is it generating?

(b) When is $\|\cdot\|_{\mathrm{ob}}$ a norm? When is it a complete norm?

(c) Assume now that the wedge E_+ in E is normal and generating. Let D be a Banach space and let $(T_i)_{i\in I}$ be a family of bounded linear operators $D \to E$. Assume that for each $x \in D$ the orbit $\{T_i x \mid i \in I\}$ in E is order bounded.
Show that there exists a constant $M \geq 0$ with the following property: for each $x \in D$ there exists $h \in E_+$ such that $\|h\|_E \leq M \|x\|_D$ and $T_i x \in [-h, h]$ for each $i \in I$.

Solution:

(a) Since E contains all the constant functions, the wedge E_+ is a cone, whenever $\ell^{\mathrm{ob}}(I;E)_+$ is a cone. Conversely, if E_+ is a cone and $f, -f \in \ell^{\mathrm{ob}}(I;E)_+$, then $f(i) = 0$ for all $i \in I$, and thus, $f = 0$. So $\ell^{\mathrm{ob}}(I;E)$ is also a cone.

The cone $\ell^{\mathrm{ob}}(I;E)_+$ is always generating, since for $f \in \ell^{\mathrm{ob}}(I;E)$, there exists $h \in E_+$ such that $f(i) \in [-h,h]$ for all $i \in I$, and thus, $f^+(i) := f(i) + h$ and $f^-(i) := h$ define functions in $\ell^{\mathrm{ob}}(I;E)_+$ with $f = f^+ - f^-$.

(b) *Claim.* The mapping $\|\cdot\|_{\mathrm{ob}}$ is a norm if and only if E_+ is a cone.

Proof. Clearly, $\|0\|_{\mathrm{ob}} = 0$ and $\|f\|_{\mathrm{ob}} \geq 0$ for all $f \in \ell^{\mathrm{ob}}(I;E)$. It also follows readily that $\|\cdot\|_{\mathrm{ob}}$ is positively homogeneous. To show the triangle inequality let $f, g \in \ell^{\mathrm{ob}}(I;E)$ and notice that

$$\begin{aligned}&\{\|h\|_E \mid h \in E_+ \text{ and } f(i)+g(i) \in [-h,h] \text{ for all } i \in I\}\\ &\supseteq \{\|h_f + h_g\|_E \mid h_f, h_g \in E_+ \text{ and } f(i) \in [-h_f, h_f],\ g(i) \in [-h_g, h_g] \text{ for all } i \in I\}\end{aligned}$$

So taking the infima of both sets it follows from the triangle inequality of $\|\cdot\|_{\mathrm{ob}}$ that

$$\|f+g\|_{\mathrm{ob}} \leq \|f\|_{\mathrm{ob}} + \|g\|_{\mathrm{ob}}\,.$$

It follows that $\|\cdot\|_{\mathrm{ob}}$ is always a semi-norm.

If E_+ is no cone, then the interval $[0,0] = E_+ \cap -E_+$ contains a non-zero element x. By definition we have $\|f\|_{\mathrm{ob}} = 0$ for $f(i) := x \neq 0$. Hence, $\|\cdot\|_{\mathrm{ob}}$ is not positive definite.

Conversely, if E_+ is a cone, and $f \in E$ with $\|f\|_{\mathrm{ob}} = 0$, then for every $n \in \mathbb{N}$ there exists $h_n \in E_+$ such that $\|h_n\|_E < \frac{1}{n^3}$ and $f(i) + h_n \in [0, 2h_n]$ for all $i \in I$. Set $g := \sum_{n=1}^\infty n h_n$. Since E is an ordered Banach space, $g \in E_+$ and

$$nf(i) \leq n(f(i) + h_n) \leq 2nh_n \leq 2g,$$

which implies

$$f(i) \leq 2\tfrac{1}{n} g$$

for all $n \in \mathbb{N}$ and all $i \in I$. Since a E_+ is closed, it is also Archimedean and we obtain that $f(i) \leq 0$ for all $i \in I$. With the same argument for $-f$ instead of f we obtain that $-f(i) \leq 0$ for all $i \in I$. Hence, $f(i) \in [0,0] = \{0\}$ for all $i \in I$. It follows that $f = 0$.

Claim. The norm $\|\cdot\|_{\mathrm{ob}}$ is complete if and only if E_+ is normal.

Proof. "$\Leftarrow$": Let E_+ be normal. Then for $f \in \ell^{\mathrm{ob}}(I;E)$ there exists for each $\epsilon > 0$ an element $h \in E_+$ such that $0 \le \|h\|_E - \|f\|_{\mathrm{ob}} < \epsilon$ and $f(i) + h \in [0, 2h]$ for all $i \in I$. Normality of E_+ now implies the existence of $M \ge 1$, which is independent on ϵ, such that $\|f(i)\|_E - \|h\|_E \le \|f(i) + h\|_E \le 2M \|h\|_E$. In particular,

$$\|f(i)\|_E \le (2M+1)\|h\|_E \le (2M+1)(\|f\|_{\mathrm{ob}} + \epsilon).$$

Since ϵ was arbitrarily chosen and M does not depend on ϵ we have that $\|f(i)\|_E \le (2M+1)\|f\|_{\mathrm{ob}}$ for all $i \in I$. Now suppose $\sum_{n=1}^{\infty} f_n$ with $f_n \in \ell^{\mathrm{ob}}(I;E)$ is absolutely convergent. By the above norm inequality it follows that $\sum_{n=1}^{\infty} f_n(i)$ converges absolutely in E for each $i \in I$. By completeness of E, the series $\sum_{n=1}^{\infty} f_n(i)$ thus converges in E for all $i \in I$. It remains to show that $\sum_{n=1}^{\infty} f_n \in \ell^{\mathrm{ob}}(I;E)$.

To this end, notice that there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in E_+ with $\|h_n\|_E - \|f_n\|_E < \frac{1}{2^n}$ and $f_n(i) \in [-h_n, h_n]$ for all $i \in I$. Then the series $g := \sum_{n=1}^{\infty} h_n$ also converges absolutely in E; hence it is convergent. It follows that

$$\sum_{n=1}^{\infty} f_n(i) \in [-g, g]$$

for all $i \in I$. So it follows that $\sum_{n=1}^{\infty} f_n \in \ell^{\mathrm{ob}}(I;E)$.

"$\Rightarrow$": Let $\|\cdot\|_{\mathrm{ob}}$ be a complete norm. To show that E_+ is normal. Let $0 \le x \le y$.

Consider the subspace $D := E_+ - E_+$ with the norm $\|\cdot\|_D$ from Lemma 3.2.5 and recall that $\|x\|_D = \|x\|_E$ for all $x \in E_+$. Recall also that $\|\cdot\|_D$ is complete, since E_+ is closed, and thus, ideally convex.

Let V be the subspace of $\ell^{\mathrm{ob}}(I;E)$ that contains the constant functions. We will show that V is closed in $\|\cdot\|_{\mathrm{ob}}$. As then the mapping

$$\iota : (D, \|\cdot\|_D) \to (V, \|\cdot\|_{\mathrm{ob}}), \qquad x \mapsto (x)_{i\in I}$$

is bijective and continuous between Banach spaces. Indeed, let $x \in D$. By definition of D and $\|\cdot\|_D$ (see Lemma 3.2.5) there exist $y, z \in E_+$ with $x = y - z$ and $\|y\|_D + \|z\|_D \le 2\|x\|_D$. Set $h := y + z$. Then $x \in [-h, h]$, and thus, $\|(x)_{i\in I}\|_{\mathrm{ob}} \le \|h\|_D \le 2\|x\|_D$. Now the open mapping theorem implies that ι^{-1} is continuous, and thus, the existence of a bound $M \ge 0$ such that

$$\|x\|_D \le M \|(x)_{i\in I}\|_{\mathrm{ob}}.$$

Now it follows that every order bounded set in D is norm bounded in D.[3] Hence, by Theorem 3.5.5 (ii) $\Rightarrow$ (i) the cone E_+ is normal in D. Since, as mentioned above, the norms $\|\cdot\|_D$ and $\|\cdot\|_E$ coincide on E_+ it follows that E_+ is also normal in E.

It remains to show that V is closed in $\|\cdot\|_{\mathrm{ob}}$. To this end let $(f_n)_{n\in\mathbb{N}}$ be a sequence in V, which converges to $f \in \ell^{\mathrm{ob}}(I;E)$. Then there exists for each $k \in \mathbb{N}$ an element $n_k \in \mathbb{N}$ such that for all $n \ge n_k$ we have $\|f_n - f\|_{\mathrm{ob}} < \frac{1}{k^3}$. So in particular there exists for each $k \in \mathbb{N}$ an order bound $h_k \in E_+$ with $\|h_k\| < \frac{2}{k^3}$ and $f_n(i) - f(i) \in [-h_k, h_k]$ for all $i \in I$ and all $n \ge n_k$. Since f_n is constant we have for all $i_1, i_2 \in I$ and $n \ge n_k$ that

$$f(i_1) - f(i_2) = f(i_1) - f_n(i_1) + f_n(i_2) - f(i_2) \in [-2h_k, 2h_k].$$

[3] Indeed, since E_+ is generating in D, every set $S \subseteq D$ contained in an order interval $[h_1, h_1]$ is also contained in an order interval of the form $[-h, h]$ for some $h \in E_+$ (see proof of (c)) and thus $\iota(S)$ is bounded in $\|\cdot\|_{\mathrm{ob}}$. So the above inequality shows that S is bounded in $\|\cdot\|_D$.

Set $g := \sum_{n=1}^{\infty} kh_k \in E_+$ and notice that $k(f(i_1) - f(i_2)) \leq 2g$. Dividing by k, the Archimedean property of E_+ yields that $f(i_1) - f(i_2) \leq 0$. Since $i_1, i_2 \in I$ where arbitrarily chosen, it follows that $f(i_1) - f(i_2) = 0$ for all $i_1, i_2 \in I$. This means that f is indeed constant.

(c) Define the operator

$$T : C \to \ell^{\mathrm{ob}}(I; E), \qquad x \mapsto (T_i x)_{i \in I}.$$

We show that T is well-defined. Let $x \in C$. By order boundedness of the orbit, there exists $h_1, h_2 \in E$ such that $T_i x \in [h_1, h_2]$ for all $i \in I$. Since E_+ is generating, we can write $h_j = x_j - y_j$ for some $x_j, y_j \in E_+$ for all $j = 1, 2$. Then $[h_1, h_2] \subseteq [-(x_2 + y_1), x_2 + y_1]$. Thus, there exists $\tilde{h} := x_2 - y_1 \in E_+$ such that $T_i x \in [-\tilde{h}, \tilde{h}]$ for all $i \in I$. This implies that $Tx \in \ell^{\mathrm{ob}}(I; E)$.

We now show that T is closed. Let $(x_n)_{n \in \mathbb{N}}$ converge to 0 in C and $(Tx_n)_{n \in \mathbb{N}}$ converge to y in $\ell^{\mathrm{ob}}(I; E)$. We have to show that $0 = Tx = y$. Recall from the proof of (b) that normality of E_+ implies the existence of $C \geq 0$ such that

$$\|f(i)\|_E \leq C \|f\|_{\mathrm{ob}}$$

for all $f \in \ell^{\mathrm{ob}}(I; E)$ and all $i \in I$. Hence,

$$\|T_i x_n - y(i)\|_E \leq C \|Tx_n - y\|_{\mathrm{ob}} \to 0$$

as $n \to \infty$ for all $i \in I$. This implies that $y = 0$. Hence, T is a closed operator. Now the closed graph theorem implies that there exists $K \geq 0$ such that

$$\|Tx\|_{\mathrm{ob}} \leq K \|x\|_C .$$

We may assume without restriction that $x \neq 0$. By definition of $\|\cdot\|_{\mathrm{ob}}$ there exists $h \in E_+$ (which depends on x) such that $\|h\|_E - \|Tx\|_{\mathrm{ob}} < K \|x\|_C$ and $T_i x \in [-h, h]$. In summary, setting $M := 2K$ we obtain

$$\|h\|_E \leq M \|x\|_C .$$

This proves the claim.