6. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on May 16 and 17, 2023

Exercise 1 (The Loewner order on the compact operators, again). Let H be an infinite-dimensional separable complex Hilbert space and let E denote the space of all self-adjoint compact linear operators on H, endowed with the Loewner order. From Exercise 5(a) on Sheet 5 you know that E_+ is a generating cone.

(a) Give an example of functions γ^+, γ^- with the properties in Theorem 3.4.1.

(b) Is E_+ normal?

Exercise 2 (Some function spaces).

(a) Let (Ω, μ) be a measure space and let $p \in [1, \infty]$. Endow $L^p(\Omega, \mu)$ with the pointwise almost everywhere order and its usual norm. Find mappings $\gamma^+, \gamma^- : L^p(\Omega, \mu) \to L^p(\Omega, \mu)_+$ with the properties stated in Theorem 3.4.1.

(b) Let E be a pre-ordered Banach space and let M be a non-empty compact metric space.[1] Let $\mathrm{C}(M; E)$ denote the space of all continuous functions from M to E, endowed with the norm given by

$$\|f\|_\infty := \max_{x \in M} \|f(x)\|_E$$

for each $f \in \mathrm{C}(M; E)$ and with the pointwise order. It is not difficult to show that $\mathrm{C}(M; E)$ is a pre-ordered Banach space.
When is the wedge $\mathrm{C}(M; E)_+$ a cone? When is it normal? When is it generating?

Exercise 3 (Lipschitz continuous decomposition?). Prove or disprove: For every pre-ordered Banach space E whose wedge is generating, there exist Lipschitz continuous maps $\gamma^+, \gamma^- : E \to E_+$ such that $x = \gamma^+(x) - \gamma^-(x)$ for each $x \in E$.

Exercise 4 (A space of holomorphic functions). Denote the open unit disk in $\mathbb{C}$ by $\mathbb{D}$. Let $\mathcal{H}^\infty(\mathbb{D})$ be the Banach space of all bounded holomorphic functions $\mathbb{D} \to \mathbb{C}$ with the supremum norm and set $E := \{ f \in \mathcal{H}^\infty(\mathbb{D}) \mid f(\frac{1}{n}) \in \mathbb{R} \text{ for all } 2 \leq n \in \mathbb{N} \}$. This is a closed subset of $\mathcal{H}^\infty(\mathbb{D})$ and thus a real Banach space with respect to the supremum norm over $\mathbb{D}$. Define $E_+ := \{ f \in E \mid f(\frac{1}{n}) \geq 0 \text{ for all } 2 \leq n \in \mathbb{N} \}$. This turns E into a pre-ordered Banach space.

(a) Show that E_+ is a cone.

(b) Is the cone E_+ normal? Is it generating?

[1] Or, more generally, a non-empty compact Hausdorff space.

Exercise 5 (Norm boundedness of order bounds). Let E be a pre-ordered Banach space and let I be a non-empty set. Let $\ell^{\mathrm{ob}}(I;E)$ denote the vector space of all functions $f : I \to E$ for which there exists a vector $h \in E_+$ such that $f(i) \in [-h,h]$ for all $i \in I$.[2] For each $f \in \ell^{\mathrm{ob}}(I;E)$ we define

$$\|f\|_{\mathrm{ob}} := \inf \left\{ \|h\|_E \;\middle|\; h \in E_+ \text{ and } f(i) \in [-h,h] \text{ for all } i \in I. \right\}$$

We endow the vector space $\ell^{\mathrm{ob}}(I;E)$ with the pointwise order.

(a) When is the wedge $\ell^{\mathrm{ob}}(I;E)_+$ a cone? When is it generating?

(b) When is $\|\cdot\|_{\mathrm{ob}}$ a norm? When is it a complete norm?

(c) Assume now that the wedge E_+ in E is normal and generating. Let D be a Banach space and let $(T_i)_{i\in I}$ be a family of bounded linear operators $D \to E$. Assume that for each $x \in D$ the orbit $\{T_i x \mid i \in I\}$ in E is order bounded.
Show that there exists a constant $M \geq 0$ with the following property: for each $x \in D$ there exists $h \in E_+$ such that $\|h\|_E \leq M \|x\|_D$ and $T_i x \in [-h,h]$ for each $i \in I$.

[2]Here, the acronym "ob" stands for "order bounded".