5. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on May 9 and 10, 2023

with Solutions

Exercise 1 (Non-ideally convex wedges).

(a) Find an example of a Banach space E and an Archimedean wedge E_+ which is generating but not ideally convex.
Hint: Start with an arbitrary infinite-dimensional Banach space E and a discontinuous linear functional $\varphi : E \to \mathbb{R}$.

(b) Find an example of a Banach space E and an Archimedean cone E_+ which is generating but not ideally convex.
Hint: Start with an arbitrary infinite-dimensional Banach space E and a Hamel basis of E.

Solution:

(a) Let E be an arbitrary infinite-dimensional Banach space and $\varphi : E \to \mathbb{R}$ a discontinuous linear functional. Define a wedge

$$E_+ := \{x \in E \mid \varphi(x) \geq 0\}.$$

Clearly, E_+ is indeed a wedge. Moreover, E_+ is generating, since for every $x \in E$ we have $x \in E_+$ or $-x \in E_+$.

We show that E_+ is Archimedean. Let $x, y \in E_+$ be such that $x \leq \frac{1}{n}y$ for all $n \in \mathbb{N}$, then $y - n \cdot x \geq 0$ for all $n \in \mathbb{N}$. Hence,

$$0 \leq \varphi(y - n \cdot x) = \varphi(y) - n \cdot \varphi(x)$$

for all $n \in \mathbb{N}$ and $x \in E_+$ implies that $\varphi(x) = 0$. In particular, $x \leq 0$, so E_+ is Archimedean.

Suppose to show a contradiction that E_+ is ideally convex. Since φ is discontinuous; and thus, unbounded, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in E that converges to 0 with $\varphi(x_n) = 1$ for all $n \in \mathbb{N}$. So $(x_n)_{n\in\mathbb{N}}$ is a increasing (and even decreasing) sequence in E_+, since $\varphi(x_{n+1} - x_n) = 0$ for all $n \in \mathbb{N}$. By Proposition 3.2.3 (i) $\Rightarrow$ (v) it follows that $x_n \leq 0$ for all $n \in \mathbb{N}$. This is a contradiction, since $\varphi(x_n) = 1$ for all $n \in \mathbb{N}$.

(b) Let E be any infinite-dimensional Banach space and $B \subseteq E$ be a Hamel basis of E. Define the cone

$$E_+ := \left\{ \sum_{k=1}^{n} \alpha_k b_k \;\middle|\; \alpha_1, \dots, \alpha_n \geq 0,\, b_1, \dots, b_n \in B,\, n \in \mathbb{N}_0 \right\}.$$

Clearly, E_+ is a wedge. Since the finite sum representation of each element in E with respect to a Hamel basis is unique, it follows that each element $x \in E_+ \cap -E_+$ must be 0. Hence, E_+ is a cone. That E_+ is generating follows from the observation that $E_+ - E_+$ is exactly the span of the basis B.

For $x \in E$ and each $b \in B$ denote by $\alpha_b(x)$ the unique real number such that

$$x = \sum_{b \in B} \alpha_b(x)\, b.$$

Notice α_b is a linear functional, $\alpha_b(x) \neq 0$ for at most finitely many $b \in B$ and that $x \leq y$ if and only if $\alpha_b(x) \leq \alpha_b(y)$ for all $b \in B$.

To show that E_+ is Archimedean, let $x, y \in E_+$ such that $x \leq \frac{1}{n} y$. Then repeating the argument in (a), we obtain

$$0 \leq \alpha_b(y - n \cdot x) = \alpha_b(y) - n \cdot \alpha_b(x)$$

for all $b \in B$ and all $n \in \mathbb{N}$. Hence, $\alpha_b(x) = 0$ for all $b \in \mathbb{N}$, which implies that $x = 0 \leq 0$. So E_+ is Archimedean.

Suppose to show a contradiction that E_+ is ideally convex. Let $(b_k)_{k \in \mathbb{N}}$ be a pairwise distinct sequence in B and $(\alpha_k)_{k \in \mathbb{N}}$ be a sequence in $(0, \infty)$ such that $\|\alpha_k b_k\| < \frac{1}{2^k}$. Define $x_n := \sum_{k=1}^{n} \alpha_k b_k$ for every $n \in \mathbb{N}$. Then $(x_n)_{n \in \mathbb{N}}$ is an increasing sequence that converges some $x \in E$. By Proposition 3.2.3 (i) $\Rightarrow$ (v) it follows that $x_n \leq x$ for all $n \in \mathbb{N}$; and hence, $\alpha_b(x_n) \leq \alpha_b(x)$ for all $b \in B$ and $n \in \mathbb{N}$. This implies that $\alpha_b(x) > 0$ for infinitely many $b \in B$. This is a contradiction. It follows that E_+ is not ideally convex.

Exercise 2 (Vector-valued ℓ^p-spaces). Let E be a pre-ordered Banach space and let $p \in [1, \infty]$. For each sequence $x = (x_n)_{n \in \mathbb{N}}$ in E we define $\|x\|_p \in [0, \infty]$ as

$$\|x\|_p := \begin{cases} \left(\sum_{n=1}^{\infty} \|x_n\|^p \right)^{1/p} & \text{if } p \in [1, \infty), \\ \sup_{n \in \mathbb{N}} \|x_n\| & \text{if } p = \infty. \end{cases}$$

Define $\ell^p(\mathbb{N}; E)$ to be the set of all sequences x in E, indexed over $\mathbb{N}$, that satisfy $\|x\|_p < \infty$. One can show that this is a vector space with the pointwise operations and a Banach space when endowed with the norm $\|\cdot\|_p$. Let us equip the Banach space $\ell^p(\mathbb{N}; E)$ with the wedge

$$\ell^p(\mathbb{N}; E)_+ := \Big\{ x \in \ell^p(\mathbb{N}; E) \mid \; x_n \in E_+ \text{ for each } n \in \mathbb{N} \Big\}.$$

Prove that $\ell^p(\mathbb{N}; E)_+$ is closed. When is $\ell^p(\mathbb{N}; E)_+$ a cone? When is it generating?

Solution: Consider the coordinate mappings

$$c_n : \ell^p(\mathbb{N}; E) \to E, \qquad x = (x_n)_{n \in \mathbb{N}} \mapsto x_n.$$

and notice that $\|c_n x\|_E \leq \|x\|_{\ell^p}$ for each $x \in \ell^p(\mathbb{N}; E)$ and each $p \in [0, \infty]$. The cone $\ell^p(\mathbb{N}; E)_+$ is exactly the intersection

$$\ell^p(\mathbb{N}; E)_+ = \bigcap_{n \in \mathbb{N}} c_n^{-1}(E_+).$$

Since E_+ is closed and c_n continuous, this is an intersection of closed sets, and thus, itself closed.
Claim. The set $\ell^p(\mathbb{N};E)_+$ is a cone if and only if E_+ is a cone.
Proof. Notice that $\ell^p(\mathbb{N};E)_+$ is always a wedge.
"$\Rightarrow$": Let $\ell^p(\mathbb{N};E)_+$ be a cone. Consider $x \in E_+ \cap -E_+$. Then the element $(\frac{1}{2^n}x)_{n\in\mathbb{N}}$ is in $\ell^p(\mathbb{N};E)_+$ and $-\ell^p(\mathbb{N};E)_+$. Hence, $(\frac{1}{2^n}x)_{n\in\mathbb{N}} = 0$, and thus, $x = 0$. So E_+ is a cone.
"$\Leftarrow$": Let E_+ be a cone. Pick $x = (x_n)_{n\in\mathbb{N}}$ such that $-x, x \in \ell^p(\mathbb{N};E)_+$. Then in particular $-x_n, x_n \in E_+$ for all $n \in \mathbb{N}$. Hence, $x_n = 0$ for all $n \in \mathbb{N}$, which implies that $x = 0$. So $\ell^p(\mathbb{N};E)_+$ is a cone.
Claim. The set $\ell^p(\mathbb{N};E)_+$ is generating if and only if E_+ is generating.
Proof. "$\Rightarrow$": Let $\ell^p(\mathbb{N};E)_+$ be generating and let $x \in E$. Then there exist $(x_n^+)_{n\in\mathbb{N}}$ and $(x_n^-)_{n\in\mathbb{N}}$ such that $(x_n^+)_{n\in\mathbb{N}} - (x_n^-)_{n\in\mathbb{N}} = (\frac{1}{2^n}x)_{n\in\mathbb{N}}$. Hence, $\frac{1}{2^n}x = x_n^+ - x_n^-$ and $x_n^+, x_n^- \in E_+$ for all $n \in \mathbb{N}$. So E_+ is generating.
"$\Leftarrow$": Let E_+ be generating and $x = (x_n)_{n\in\mathbb{N}} \in \ell^p(\mathbb{N};E)$. Since E_+ is generating there exists a constant $M > 0$ and elements x_n^+ and x_n^- in E_+ such that $x_n = x_n^+ - x_n^-$ and $\|x_n^+\|, \|x_n^-\| \leq M\|x_n\|$ for all $n \in \mathbb{N}$. In particular, the sequences $x^+ := (x_n^+)_{n\in\mathbb{N}}$ and $x^- := (x_n^-)_{n\in\mathbb{N}}$ are in $\ell^p(\mathbb{N};E)_+$ and satisfy $x = x^+ - x^-$. Thus $\ell^p(\mathbb{N};E)_+$ is generating.

Exercise 3 (Properties of convex sets). Let E be a normed space over $\mathbb{R}$ and let $C \subseteq E$ be convex.

(a) Show that if x is an interior point of C and $y \in \partial C$, then $y+\lambda(x-y) = \lambda x+(1-\lambda)y$ is also an interior point of C for all $\lambda \in (0,1]$.

(b) Show that if E is finite-dimensional, $0 \in C$, and C spans E, then C has non-empty interior.

(c) Show that if E is finite-dimensional, then C is ideally convex.

Solution:

(a) Let $\lambda \in (0,1]$. Then clearly, $\operatorname{Int}(C) \subseteq \lambda \operatorname{Int}(C) + (1-\lambda)C \subseteq C$. Since the interior of C is the largest open set contained in C, it suffices to show that $\lambda \operatorname{Int}(C) + (1-\lambda)C$ is open. But this follows from

$$\lambda\operatorname{Int}(C) + (1-\lambda)C = \bigcup_{x\in C} \lambda \operatorname{Int}(C) + (1-\lambda)x,$$

since this is a union of open sets. In particular it follows that $y + \lambda(x-y) = \lambda x + (1-\lambda)y$ is an interior point of C, whenever $x \in \operatorname{Int}(C)$ and $y \in C$.

(b) Since every spanning set of E contains a basis and C spans E there exist linearly independent $b_1, \dots, b_d \in C$ such that

$$E = \{\alpha_1 b_1 + \cdots + \alpha_d b_d \mid \alpha_1, \dots, \alpha_d \in \mathbb{R}\},$$

where $d = \dim(E)$. Now the simplex

$$[b_1, \dots, b_d] = \{\alpha_1 b_1 + \cdots + \alpha_d b_d \mid \alpha_1, \dots, \alpha_d \geq 0,\ \alpha_1 + \cdots + \alpha_d \leq 1\}$$

is contained in C and has non-empty interior. An open set in the simplex is for instance the set determined by the coefficients $\alpha_i \in (0, \frac{1}{d})$ for all $i \in \{1, \dots, d\}$.

(c) Let $(x_n)_{n\in\mathbb{N}_0}$ be a norm bounded sequence in C and $(\lambda_n)_{n\in\mathbb{N}_0}$ be a sequence in $[0,\infty)$ with $\sum_{n=0}^{\infty}\lambda_n = 1$. We may assume w.l.o.g. that $\lambda_n > 0$ for all $n \in \mathbb{N}_0$ and that $x_0 = 0$ (otherwise we shift the sequence and the set C by $-x_0$).

Let F be the span of $(x_n)_{n\in\mathbb{N}_0}$ and let $d := \dim(F)$. Then we may assume, after potentially reordering the sequences, that $x_1,\dots,x_d$ are a basis of F. Thus we find a linear and bijective map $T : F \to \mathbb{R}^d$ that maps each $x_1,\dots,x_d$ to a canonical unit vector $e_1,\dots,e_d$, respectively. Notice that since $\lambda_0 > 0$ and $x_0 = 0$ it follows that

$$T\left(\left(\sum_{n=0}^{d}\lambda_n\right)^{-1}\sum_{n=0}^{d}\lambda_n x_n\right) = \left(\sum_{n=0}^{d}\lambda_n\right)^{-1}\sum_{n=1}^{d}\lambda_n e_n$$

lies in the interior of the standard simplex of $\mathbb{R}^d$. Hence, $\left(\sum_{n=0}^{d}\lambda_n\right)^{-1}\sum_{n=0}^{d}\lambda_n x_n$ lies in the interior of C with respect to the relative topology induced by F.

Moreover, by convexity the point $\left(\sum_{n=d+1}^{\infty}\lambda_n\right)^{-1}\sum_{n=d+1}^{\infty}\lambda_n x_n$ lies in the closure of C (also with respect to the subspace topology of F).

Now $\sum_{n=0}^{\infty}\lambda_n x_n$ is but a convex combination of a point in the interior and a point in the closure of C with weights strictly between 0 and 1. By (a) it follows that $\sum_{n=0}^{\infty}\lambda_n x_n \in C$. So C is ideally convex.

Exercise 4 (Continuous decomposition in the ice-cream cone). Endow $\mathbb{R}^d$ with the ice cream cone. Give an explicit example of functions γ^+,γ^- with the properties stated in Theorem 3.4.1.

Solution: Define $Px := (0, x_2, \dots, x_d)$ for all $x \in \mathbb{R}^d$. Let

$$\gamma^+ : \mathbb{R}^d \to \mathbb{R}^d_+, \qquad x \mapsto \begin{cases} (\|Px\|_2, x_2, \dots, x_d), & x_1 < 0, \\ (\|Px\|_2 + x_1, x_2, \dots, x_d) & x_1 \geq 0, \end{cases}$$

and

$$\gamma^- : \mathbb{R}^d \to \mathbb{R}^d_+, \qquad x \mapsto \begin{cases} (-x_1 + \|Px\|_2, 0, \dots, 0), & x_1 < 0, \\ (\|Px\|_2, 0, \dots, 0), & x_1 \geq 0. \end{cases}$$

Then γ^+ and γ^- are well-defined, positively homogeneous and satisfy

$$\|\gamma^+(x)\|_2, \|\gamma^-(x)\|_2 \leq \|x\|_2 + \|Px\|_2 \leq 2\|x\|_2$$

for all $x \in E$. The continuity is also clear. Moreover, $\gamma^+(x) - \gamma^-(x) = x$.

Exercise 5 (The Loewner order on the self-adjoint operators). Let H be an infinite-dimensional separable Hilbert space[1] over $\mathbb{C}$ and let $\mathcal{K}(H)_{\mathrm{sa}}$ denote the

[1] Actually, neither the infinite dimension nor the separability is relevant for any of the properties in (a)–(c); and for part (d), only the infinite dimension is relevant. But infinite dimension and separability simplifies the notation in the solution a bit.

space of all self-adjoint compact linear operators on H; this is a Banach space over $\mathbb{R}$ with respect to the operator norm.
Similarly as on $\mathcal{L}(H)_{\text{sa}}$ we define the *Loewner cone* $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ on $\mathcal{K}(H)_{\text{sa}}$ to consist of all positive semidefinite operators in $\mathcal{K}(H)_{\text{sa}}$.

(a) Show that $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ is a closed generating cone in $\mathcal{K}(H)_{\text{sa}}$.

(b) Show that $\mathcal{K}(H)_{\text{sa}}$ has empty interior.

(c) For each closed vector subspace V of H define

$$F_V := \Big\{A \in \big(\mathcal{K}(H)_{\text{sa}}\big)_+ \mid A \text{ vanishes on } V\Big\}.$$

Prove that each such set F_V is a closed face of $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ and that, conversely, every closed face of $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ is of the form F_V for a closed vector subspace V of H.

Hint for the converse part: For a closed face F, define $V := \bigcap_{A \in F} \ker A$. It might be helpful to prove that[2]

$$W := \{x \in H \mid x \otimes x \in F\}$$

is a closed vector subspace of H.

(d) *Extra challenge:*
Show that the Loewner cone $\big(\mathcal{L}(H)_{\text{sa}}\big)_+$ in $\mathcal{L}(H)_{\text{sa}}$ has a closed face that is not of the form

$$\Big\{A \in \big(\mathcal{L}(H)_{\text{sa}}\big)_+ \mid A \text{ vanishes on } V\Big\}$$

for any closed vector subspace V of H.

Solution:

(a) We show that $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ is closed. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ that converges to $A \in \mathcal{K}(H)_{\text{sa}}$. Then

$$0 \le (x \mid A_n x) \to (x \mid Ax)$$

for all $x \in H$. So A is also in $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ and therefore the cone is closed.

To show that $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ is generating, let $A \in \mathcal{K}(H)_{\text{sa}}$. Clearly $\sigma(A) \subseteq \mathbb{R}$. By spectral theory there exists a orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and a sequence of real numbers $(\lambda_n)_{n\in\mathbb{N}}$ such that

$$A = \sum_{n=1}^{\infty} \lambda_n \,(e_n \otimes e_n),$$

where $(e_n \otimes e_n)x := (e_n \mid x)e_n$ is the orthogonal rank-1-projection onto the span of e_n. By setting

$$A^+ := \sum_{n=1}^{\infty} \max(\lambda_n, 0)\,(e_n \otimes e_n),$$

[2] For each $x \in H$ the operator $x \otimes x : H \to H$ is defined by $(x \otimes x)z = (x \mid z)\,x$ for each $z \in H$.

and

$$A^- := \sum_{n=1}^{\infty} \max(-\lambda_n, 0)\,(e_n \otimes e_n),$$

We obtain that $A = A^+ + A^-$ and that $A^+, A^- \in \big(\mathcal{K}(H)_{\mathrm{sa}}\big)_+$. Hence, the cone is generating.

(b) Let $A \in \big(\mathcal{K}(H)_{\mathrm{sa}}\big)_+$ and let $(\lambda_n)_{n\in\mathbb{N}}$ be as in (a). Then also by spectral theory $(\lambda_n)_{n\in\mathbb{N}}$ converges to 0. So for every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that $|\lambda_n| < \varepsilon$. Then $S := -\varepsilon(e_n \otimes e_n)$ satisfies $\|S\| = \varepsilon$ and $(e_n \mid (A+S)e_n) = \lambda_n + \varepsilon < 0$. So $A + S$ is not positive definite. It follows that A is not in the interior of $\big(\mathcal{K}(H)_{\mathrm{sa}}\big)_+$, and hence, the interior is empty.

(c) We first show that for each closed vector subspace V of H the set F_V is a closed face of $\big(\mathcal{K}(H)_{\mathrm{sa}}\big)_+$. Clearly F_V is a wedge, since every linear combination of operator A and B that vanish on V also vanishes on V. Now let $A \in F_V$ and $B \in [0, A]$. We show that $B \in F_V$. Then by Proposition 1.4.3 (ii) $\Rightarrow$ (i) it follows that F_V is a face.

By choice of A and B it follows that A vanishes on V and $A - B \geq 0$. Suppose there exists $v \in V$ such that $(v \mid Bv) \neq 0$. Then $0 \leq (v \mid (A-B)v) = (v \mid Bv) < 0$. This is a contradiction.

Conversely, let F be a closed face in $\mathcal{K}(H)_{\mathrm{sa}}$ and define as in the hint the set closed subspace $V := \bigcap_{A\in F} \ker A$. Then clearly $F \subseteq F_V$, since every operator in F vanishes on V.

For the converse direction we show the claim in the hint. Let $\alpha \geq 0$ and $x, y \in W$. Then $(\alpha x)\otimes(\alpha x) = |\alpha|\,(x\otimes x) \in F$. To see that $x+y \in W$ we use that the inequality $2\mathrm{Re}(ab) \leq |a|^2 + |b|^2$ holds for all $a, b \in \mathbb{C}$.[3] Hence, a simple calculation yields

$$\begin{aligned}
\big(z \mid ((x+y)\otimes(x+y))z\big) &= \big(z \mid (x\otimes x)z\big) + 2\mathrm{Re}\big((z \mid x)(y \mid z)\big) + \big(z \mid (y\otimes y)z\big) \\
&\leq 2|(x \mid z)|^2 + 2|(y \mid z)|^2 \\
&= 2(z \mid (x\otimes x)z) + 2(z \mid (y\otimes y)z)
\end{aligned}$$

for all $z \in H$. So it follows that $(x+y)\otimes(x+y) \leq 2(x\otimes x) + 2(y\otimes y)$ and thus, $(x+y)\otimes(x+y) \in F$ by Proposition 1.4.3 (i) $\Rightarrow$ (iii). In summary it follows that W is indeed a vector space. Since the mapping $H \ni x \mapsto x\otimes x$ is continuous and W is its preimage under the set closed set F it follows that W is even closed.

With the claim proved, we now show that $W \oplus V = H$. Clearly, $x \in W \cap V$ implies that $(x\otimes x)x = \|x\|^2 x = 0$. Hence, $x = 0$. To conclude, we show that $W^\perp := \{x \in H \mid \forall y \in W : (y \mid x) = 0\} \subseteq V$. Suppose there is $x \in W^\perp$ with $x \notin V$. Then there is an element $A \in F$ such that $Ax \neq 0$. Since by the spectral theorem

$$A = \sum_{n=1}^{\infty} \lambda_n (e_n \otimes e_n)$$

for appropriate sequences $(e_n)_{n\in\mathbb{N}}$ of eigenvectors of A and $(\lambda_n)_{n\in\mathbb{N}}$ in $[0,\infty)$. So there exists a $n \in \mathbb{N}$ some non-zero eigenvalue $\lambda_n > 0$ such that $(e_n \mid x) \neq 0$. Since F is a face and $\lambda_n(e_n \otimes e_n)$ is clearly dominated by A it follows that $e_n \otimes e_n \in W$,

[3] This can be seen by showing that $|a|^2 - 2\mathrm{Re}(ab) + |b|^2 = |a - \bar{b}|^2$ for all $a, b \in \mathbb{C}$.

and thus, $e_n \in W$. This is a contradiction, since e_n is not orthogonal to x and x was assumed to be in $W^\perp$. It follows that $W^\perp \subseteq V$ and thus, $W \oplus V = H$.

To finalize the proof, we show now that $F_V \subseteq F$. Take $A \in F_V$. Then harnessing the power of the spectral theorem once again we obtain

$$A = \sum_{n=1}^{\infty} \lambda_n (e_n \otimes e_n)$$

for appropriate sequences $(e_n)_{n\in\mathbb{N}}$ of eigenvectors of A and $(\lambda_n)_{n\in\mathbb{N}}$ in $[0,\infty)$. Then clearly $\lambda_n \neq 0$ implies that $(e_n \mid x) = 0$ for all $x \in \ker A$, and thus, $e_n \in V^\perp = W$. Hence, $e_n \otimes e_n \in F$, whenever $\lambda_n \neq 0$. By closedness of F it follows that $A \in F$.

(d) *Claim.* The Loewner cone in $\mathcal{K}(H)_{\text{sa}}$ is a face of the Loewner cone in $\mathcal{L}(H)_{\text{sa}}$ that is not of the form F_V for any closed subspace $V \subseteq H$.

Proof. Let $B \in \big(\mathcal{K}(H)_{\text{sa}}\big)_+$ and $0 \leq B \leq A$. We show that A is also compact, and thus, $A \in B \in \big(\mathcal{K}(H)_{\text{sa}}\big)_+$.

Clearly,

$$\|A^{1/2}\|^2 = |(A^{1/2}x \mid A^{1/2}x)| \leq |(B^{1/2}x \mid B^{1/2}x)| = \|B^{1/2}\|^2 \tag{1}$$

Since $B^{1/2}$ is also compact by the spectral theorem, the sequence $(B^{1/2}x_n)_{n\in\mathbb{N}}$ has a convergent subsequence, whenever $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in H. Since, convergent sequences are Cauchy, the inequality (1) now implies that $(A^{1/2}x_n)_{n\in\mathbb{N}}$ also has a Cauchy (and thus convergent) subsequence. This shows that B is compact.

To see that $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ is not of the form F_V notice that $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ contains all rank-1-operators, so there exists no non-trivial subspace of H on which all operators in $\big(\mathcal{K}(H)_{\text{sa}}\big)_+$ vanish.