5. Exercise Sheet in

Ordered Banach Spaces and Positive Operators

For the exercise classes on May 9 and 10, 2023

Exercise 1 (Non-ideally convex wedges).

(a) Find an example of a Banach space E and an Archimedean wedge E_+ which is generating but not ideally convex.
Hint: Start with an arbitrary infinite-dimensional Banach space E and a discontinuous linear functional $\varphi : E \to \mathbb{R}$.

(b) Find an example of a Banach space E and an Archimedean cone E_+ which is generating but not ideally convex.
Hint: Start with an arbitrary infinite-dimensional Banach space E and a Hamel basis of E.

Exercise 2 (Vector-valued ℓ^p-spaces). Let E be a pre-ordered Banach space and let $p \in [1, \infty]$. For each sequence $x = (x_n)_{n\in\mathbb{N}}$ in E we define $\|x\|_p \in [0, \infty]$ as

$$\|x\|_p := \begin{cases} \Big(\sum_{n=1}^{\infty} \|x_n\|^p \Big)^{1/p} & \text{if } p \in [1, \infty), \\ \sup_{n\in\mathbb{N}} \|x_n\| & \text{if } p = \infty. \end{cases}$$

Define $\ell^p(\mathbb{N}; E)$ to be the set of all sequences x in E, indexed over $\mathbb{N}$, that satisfy $\|x\|_p < \infty$. One can show that this is a vector space with the pointwise operations and a Banach space when endowed with the norm $\|\cdot\|_p$. Let us equip the Banach space $\ell^p(\mathbb{N}; E)$ with the wedge

$$\ell^p(\mathbb{N}; E)_+ := \Big\{ x \in \ell^p(\mathbb{N}; E) \mid \ x_n \in E_+ \text{ for each } n \in \mathbb{N} \Big\}.$$

Prove that $\ell^p(\mathbb{N}; E)_+$ is closed. When is $\ell^p(\mathbb{N}; E)_+$ a cone? When is it generating?

Exercise 3 (Properties of convex sets). Let E be a normed space over $\mathbb{R}$ and let $C \subseteq E$ be convex.

(a) Show that if x is an interior point of C and $y \in \partial C$, then $y+\lambda(x-y) = \lambda x+(1-\lambda)y$ is also an interior point of C for all $\lambda \in (0, 1]$.

(b) Show that if E is finite-dimensional, $0 \in C$, and C spans E, then C has non-empty interior.

(c) Show that if E is finite-dimensional, then C is ideally convex.

Exercise 4 (Continuous decomposition in the ice-cream cone). Endow $\mathbb{R}^d$ with the ice cream cone. Give an explicit example of functions γ^+, γ^- with the properties stated in Theorem 3.3.1.

Exercise 5 (The Loewner order on the self-adjoint operators). Let H be an infinite-dimensional separable Hilbert space[1] over $\mathbb{C}$ and let $\mathcal{K}(H)_{\mathrm{sa}}$ denote the space of all self-adjoint compact linear operators on H; this is a Banach space over $\mathbb{R}$ with respect to the operator norm.
Similarly as on $\mathcal{L}(H)_{\mathrm{sa}}$ we define the *Loewner cone* $\big(\mathcal{K}(H)_{\mathrm{sa}}\big)_+$ on $\mathcal{K}(H)_{\mathrm{sa}}$ to consist of all positive semidefinite operators in $\mathcal{K}(H)_{\mathrm{sa}}$.

(a) Show that $\big(\mathcal{K}(H)_{\mathrm{sa}}\big)_+$ is a closed generating cone in $\mathcal{K}(H)_{\mathrm{sa}}$.

(b) Show that $\mathcal{K}(H)_{\mathrm{sa}}$ has empty interior.

(c) For each closed vector subspace V of H define

$$F_V := \Big\{A \in \big(\mathcal{K}(H)_{\mathrm{sa}}\big)_+ \;\big|\; A \text{ vanishes on } V\Big\}.$$

Prove that each such set F_V is a closed face of $\big(\mathcal{K}(H)_{\mathrm{sa}}\big)_+$ and that, conversely, every closed face of $\big(\mathcal{K}(H)_{\mathrm{sa}}\big)_+$ is of the form F_V for a closed vector subspace V of H.

Hint for the converse part: For a closed face F, define $V := \bigcap_{A\in F} \ker A$. It might be helpful to prove that[2]

$$W := \{x \in H \mid x \otimes x \in F\}$$

is a closed vector subspace of H.

(d) *Extra challenge:*
Show that the Loewner cone $\big(\mathcal{L}(H)_{\mathrm{sa}}\big)_+$ in $\mathcal{L}(H)_{\mathrm{sa}}$ has a closed face that is not of the form

$$\Big\{A \in \big(\mathcal{L}(H)_{\mathrm{sa}}\big)_+ \;\big|\; A \text{ vanishes on } V\Big\}$$

for any closed vector subspace V of H.

[1] Actually, neither the infinite dimension nor the separability is relevant for any of the properties in (a)–(c); and for part (d), only the infinite dimension is relevant. But infinite dimension and separability simplifies the notation in the solution a bit.

[2] For each $x \in H$ the operator $x \otimes x : H \to H$ is defined by $(x \otimes x)z = (x \mid z)\, x$ for each $z \in H$.