4. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on May 2 and 3, 2023

Exercise 1 (Duality of half the ice-cream cone).

(a) Let E be a finite-dimensional vector space let $C, D \subseteq E$ be closed wedges in E, and denote their dual wedges by $C', D' \subseteq E'$.
Show that the dual wedge of $C \cap D$ is the closure of $C' + D'$.

(b) Endow $\mathbb{R}^3$ with the generating cone

$$\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 \mid x_1, x_2 \geq 0 \text{ and } x_1^2 \geq x_2^2 + x_3^2\}.$$

Compute the dual cone.

(c) Sketch the cone from part (b) and its dual cone.

Exercise 2 (Non-closedness under linear maps). Find an example of a finite-dimensional real vector spaces E and F, a closed and generating cone E_+ in E, and a linear map $T : E \to F$ such that $T(E_+)$ is not closed in F.

Exercise 3 (Positive operators with respect to the standard cone). Endow $\mathbb{R}^c$ and $\mathbb{R}^d$ with the standard cones. What are the interior points of $\mathcal{L}(\mathbb{R}^c; \mathbb{R}^d)_+$?

Exercise 4 (Positive operator with the respect to the Loewner order). Endow $E := \mathbb{C}^{d\times d}_{\mathrm{sa}}$ with the Loewner order.

(a) For every $C \in \mathbb{C}^{d\times d}$ consider the positive operator $T_C : E \to E$ that is given by $T_C A = C^* A C$ for all $A \in E$.
For which $C \in \mathbb{C}^{d\times d}$ is T_C an interior point of $\mathcal{L}(E; E)_+$?

(b) Prove or disprove that for all $A, B \in E_+$ the matrix $AB + BA$ is also in E_+.

(c) Fix $C \in E_+$ and let $S_C : E \to E$ be given by $S_C A = AC + CA$ for all $A \in E$. When is S_C an interior point of $\mathcal{L}(E; E)_+$?

(d) Fix $C \in E_+$ and consider the map $R_C : E \to E$ that is given by $R_C A = \mathrm{tr}(A) C$ for each $A \in E$.
Show that R_C is positive for every $C \in E_+$. Under which conditions is R_C an interior point of $\mathcal{L}(E; E)_+$?