4. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on May 2 and 3, 2023

with Solutions

Exercise 1 (Duality of half the ice-cream cone).

(a) Let E be a finite-dimensional vector space let $C, D \subseteq E$ be closed wedges in E, and denote their dual wedges by $C', D' \subseteq E'$.
Show that the dual wedge of $C \cap D$ is the closure of $C' + D'$.

(b) Endow $\mathbb{R}^3$ with the generating cone

$$\mathbb{R}^3_+ := \{x \in \mathbb{R}^3 \mid x_1, x_2 \geq 0 \text{ and } x_1^2 \geq x_2^2 + x_3^2\}.$$

Compute the dual cone.

(c) Sketch the cone from part (b) and its dual cone.

Solution:

(a) We show that $(C \cap D)' = \operatorname{cl}(C' + D')$.

"$\subseteq$": Note that for two closed wedges $W, V \subseteq E$, where E is finite-dimensional, we have[1]

$$W \subseteq V \qquad \text{if and only if} \qquad V' \subseteq W'$$

and that dual wedges are always closed. Thus it suffices to show the inclusion

$$\big(\operatorname{cl}(C' + D')\big)' = (C' + D')' \subseteq C \cap D = (C \cap D)''.$$

So let $x'' \in (C' + D')'$. By definition we have $\langle x'', x' \rangle \geq 0$ for all $x' \in C' + D'$, which implies $\langle x'', x_1' + x_2' \rangle \geq 0$ for all $x_1' \in C'$ and all $x_2' \in D'$. Since $0 \in C' \cap D'$ we obtain, in particular, that $\langle x'', x_1' \rangle \geq 0$ and $\langle x'', x_2' \rangle \geq 0$ for all $x_1' \in C'$ and all $x_2' \in D'$. Thus, $x'' \in C'' \cap D'' = C \cap D$. This shows the inclusion.

"$\supseteq$": Let $x' \in C' + D'$. Then there exists $x_1' \in C$ and $x_2' \in D$ such that $x' = x_1' + x_2'$. Hence, we have for all $x \in (C \cap D)$ that $\langle x', x \rangle = \langle x_1', x \rangle + \langle x_2', x \rangle \geq 0$. So $x' \in (C \cap D)'$. As the dual cone $(C \cap D)'$ is closed, the claimed inclusion follows.

(b) Notice that $\mathbb{R}^3_+$ is the intersection of the ice-cream cone

$$C := \{x \in \mathbb{R}^3 \mid x_1 \geq 0 \text{ and } x_1^2 \geq x_2^2 + x_3^2\}$$

[1] By finite-dimensionality, we have $W'' = W$ and $V'' = V$, and thus, it suffices to show the implication "$\Rightarrow$".

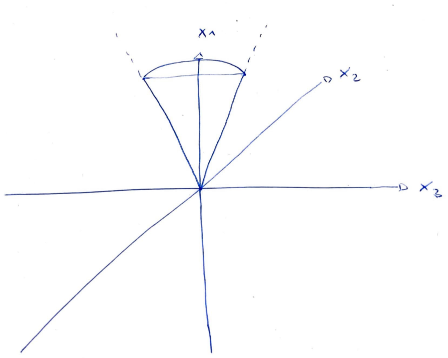

Figure 1: The cone $\mathbb{R}^3_+$ in Exercise 1 (b).

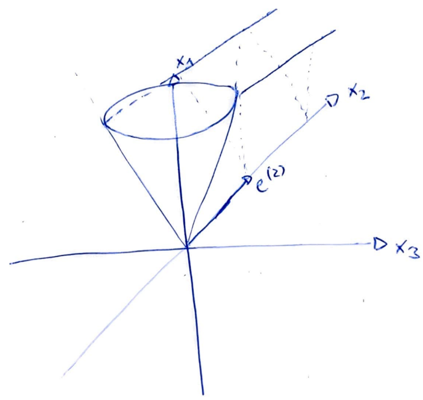

Figure 2: The dual of the cone $\mathbb{R}^3_+$ in Exercise 1 (b).

and the half space

$$D := \{x \in \mathbb{R}^3 \mid x_2 \geq 0\}.$$

So by (a) the dual of $\mathbb{R}^3_+$ is given by $C' + D'$. We compute $C' + D'$. By Exercise 4 (c) on Sheet 3 we know that the ice-cream cone is self-dual, so $C' = C$. Moreover, it follows from a simple computation that the dual of D is given by

$$D' = [0, \infty)e^{(2)}.$$

Notice also that $\mathbb{R}^3_+$ is not self-dual.

(c) A sketch of cone $\mathbb{R}^3_+$ can be found in Figure 1 and a sketch of the dual cone of $\mathbb{R}^3_+$ can be found in Figure 2.

Exercise 2 (Non-closedness under linear maps). Find an example of a finite-dimensional real vector spaces E and F, a closed and generating cone E_+ in E, and a linear map $T : E \to F$ such that $T(E_+)$ is not closed in F.

Solution: Let $E = \mathbb{R}^3$ and E_+ be the ice-cream cone in E. We know from the lecture that E_+ is closed and generating. Moreover, let $F = \mathbb{R}^2$ and consider the mapping

$$T : E \to F, \qquad (x_1, x_2, x_3) \mapsto (x_1 + x_2, x_3).$$

Claim. The image of E_+ under T is given by

$$T(E_+) = \{0\} \cup \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 > 0\}.$$

Proof. "$\subseteq$": Let $y \in T(E_+)$. Then if $x_3 = y_2 = 0$, then $x_1^2 \geq x_2^2$ implies that $y_1 \geq 0$. Hence, $y = 0$ or $y_1 > 0$. If $x_3 = y_2 \neq 0$, then $x_1^2 > x_2^2$ implies $y_1 > 0$. In any case $y = 0$ or $y_1 > 0$.
"$\supseteq$": Clearly, $0 \in T(E_+)$. If $y \in F$ with $y_1 > 0$, then choose $x \in E$ such that $x_3 := y_2$ and and $x_1 \geq 0$ large enough that $x_1^2 \geq (y_1 - x_1)^2 + x_3^2$. Such x_1 exists, since this inequality is equivalent to

$$0 \geq y_1^2 - 2x_1y_1 + y_2^2.$$

Now set $x_2 := y_1 - x_1$. Then clearly, $x \in E_+$ and $Tx = y$.

Exercise 3 (Positive operators with respect to the standard cone). Endow $\mathbb{R}^c$ and $\mathbb{R}^d$ with the standard cones. What are the interior points of $\mathcal{L}(\mathbb{R}^c; \mathbb{R}^d)_+$?

Solution: Recall from Propositon 2.3.2 that and operator $T : \mathbb{R}^c \to \mathbb{R}^d$ is in the interior of $\mathcal{L}(\mathbb{R}^c; \mathbb{R}^d)_+$ if and only if $T(\mathbb{R}^c_+ \setminus \{0\}) \subseteq \operatorname{Int}(\mathbb{R}^d_+)$.Clearly,

$$\operatorname{Int}(\mathbb{R}^d_+) = \{x \in \mathbb{R}^d \mid x_1, \ldots, x_d > 0\}.$$

Let T be in the interior of $\mathcal{L}(\mathbb{R}^c; \mathbb{R}^d)_+$. Then $Te^{(k)} \in Int(\mathbb{R}^d_+)$ for all $k \in \{1, \ldots, c\}$, where $e^{(k)}$ denotes the canonical unit vectors in $\mathbb{R}^c$. Since $Te^{(k)}$ is the k-th column of T it it follows that the k-th column only has positive entries. Since this is true for every k it follows that T only has positive entries.
Conversely, let T only have positive entries and take $x \in \mathbb{R}^c_+ \setminus \{0\}$. Then

$$Tx = \sum_{k=1}^{c} x_i Te^{(k)}.$$

Since all x_i are non-negative and at least one is positive, it follows that Tx only has positive entries; and thus, $Tx \in \operatorname{Int}(\mathbb{R}^d_+)$.

Exercise 4 (Positive operator with the respect to the Loewner order). Endow $E := \mathbb{C}^{d\times d}_{\mathrm{sa}}$ with the Loewner order.

(a) For every $C \in \mathbb{C}^{d\times d}$ consider the positive operator $T_C : E \to E$ that is given by $T_C A = C^*AC$ for all $A \in E$.
For which $C \in \mathbb{C}^{d\times d}$ is T_C an interior point of $\mathcal{L}(E; E)_+$?

(b) Prove or disprove that for all $A, B \in E_+$ the matrix $AB + BA$ is also in E_+.

(c) Fix $C \in E_+$ and let $S_C : E \to E$ be given by $S_C A = AC + CA$ for all $A \in E$. When is S_C an interior point of $\mathcal{L}(E; E)_+$?

(d) Fix $C \in E_+$ and consider the map $R_C : E \to E$ that is given by $R_C A = \mathrm{tr}(A)C$ for each $A \in E$.
Show that R_C is positive for every $C \in E_+$. Under which conditions is R_C an interior point of $\mathcal{L}(E;E)_+$?

Solution:

(a) Recall from Exercise 2 (b) on Sheet 3 that the interior points of E_+ are precisely the positive definite matrices. Moreover, recall from Proposition 2.3.2 that T_C is an interior point of $\mathcal{L}(E;E)_+$ if and only if

$$T_C(E_+ \setminus \{0\}) \subseteq \mathrm{Int}(E_+).$$

We differentiate two cases:

If $d = 1$, then $E \cong \mathbb{R}$ and $E_+ \setminus \{0\} = \mathrm{Int}(E_+)$. So $T_C \in \mathrm{Int}(\mathcal{L}(E;E)_+)$ if and only if $C \in \mathbb{C} \setminus \{0\}$.

If $d > 1$, then let $C \in \mathbb{C}^{d\times d}$ and $A \in E_+ \setminus \{0\}$ be such that A is positive semidefinite but not positive definite.[2] Then for every $C \in \mathbb{C}^{d\times d}$ the operator $T_C A = C^* AC$ is positive semidefinite but not positive definite. Hence, $T_C A \notin \mathrm{Int}(E_+)$. It follows that T_C is not in the interior of $\mathcal{L}(E;E)_+$ for any $C \in \mathbb{C}^{d\times d}$.

(b) Let

$$A := \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \qquad \text{and} \qquad C := \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

It is easily checked that $A, C \in E_+$ and

$$AC + CA = \begin{pmatrix} -1 & 3 \\ -3 & 8 \end{pmatrix} + \begin{pmatrix} -1 & -3 \\ 3 & -8 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

So $AC + CA \notin E_+$.

(c) Let $A, C \in E_+$.

If $d = 1$, then $E \cong \mathbb{R}$ and $E_+ \setminus \{0\} = \mathrm{Int}(E_+)$. So $S_C \in \mathrm{Int}(\mathcal{L}(E;E)_+)$ if and only if $C > 0$.

If $d > 1$, then let $C \in E_+$ and $A \in E_+ \setminus \{0\}$ be such that A is positive semidefinite but not positive definite.[3] Then there exists $0 \neq x \in \mathbb{C}^2$ such that $Ax = 0$. Thus,

$$(x \mid (AC + CA)x) = (Ax \mid Cx) + (Cx \mid Ax) = 0.$$

So $AC + CA$ can not be positive definite, so $AC + CA \notin \mathrm{Int}(E_+)$. Now by Proposition 2.3.2 the operator S_C is not in the interior of $\mathcal{L}(E;E)_+$ for any $C \in E_+$.

(d) Notice that a positive semidefinite matrix A has $\mathrm{tr}(A) = 0$ if and only if $A = 0$. Indeed, if $A = 0$ then clearly $\mathrm{tr}(A) = 0$. Conversely, recall that there exists a unitary matrix $U \in \mathbb{C}^{d\times d}$ such that UAU^* is diagonal. Since $\mathrm{tr}(UAU^*) = \mathrm{tr}(A) = 0$ it follows that $UAU^* = 0$. Hence, $A = 0$.

It follows that $R_C A \in \mathrm{Int}(E_+)$ for all $A \in E_+ \setminus \{0\}$ if and only if $C \in \mathrm{Int}(E_+)$.

[2] Note that one need $d > 1$ for such a matrix to exist.
[3] Note that one need $d > 1$ for such a matrix to exist.