3. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on April 25 and 26, 2023

with Solutions

Exercise 1 (Masquerade of cones, continued). Are the following two ordered vector spaces isomorphic?

(1) The space $\mathbb{R}^4$ with the ice cream cone.

(2) The space of all self-adjoint complex 2×2-matrices with the Loewner order.

Solution: We recall that the space of self-adjoint complex 2×2-matrices is given by

$$\mathbb{C}^{2\times 2}_{\mathrm{sa}} = \left\{ \begin{pmatrix} a_1 & a_3 + ib \\ a_3 - ib & a_2 \end{pmatrix} \in \mathbb{C}^{2\times 2} \;\middle|\; a_1, a_2, a_3, b \in \mathbb{R} \right\}.$$

Also recall from linear algebra that self-adjoint matrices have real spectrum. So as it is in the real and symmetric case, a 2×2-matrix $A \in \mathbb{C}^{2\times 2}_{\mathrm{sa}}$ is in the Loewner cone if and only if its eigenvalues are non-negative. This is the case if and only if $\det(A) \geq 0$ and $\operatorname{tr}(A) \geq 0$. We claim that the mapping defined by

$$i : \mathbb{R}^4 \to \mathbb{C}^{2\times 2}_{\mathrm{sa}}, \quad x = (x_1, x_2, x_3, x_4) \mapsto \begin{pmatrix} x_1 + x_2 & x_3 + ix_4 \\ x_3 - ix_4 & x_1 - x_2 \end{pmatrix}$$

is order isomorphic.

Bijectivity is straightforward: clearly, i is injective (by Proposition 1.6.4 this even follows from the bipositivity) and its domain and codomain both have dimension 4 (as a vector space over the reals). To show bipositivity, let $x \in \mathbb{R}^4$. Then

$$\begin{aligned} & i(x) \geq 0 \\ \Leftrightarrow\quad & \operatorname{tr} i(x) \geq 0 \quad \wedge \quad \det i(x) \geq 0 \\ \Leftrightarrow\quad & 2x_1 \geq 0 \quad \wedge \quad (x_1 + x_2)(x_1 - x_2) - (x_3^2 + x_4^2) \geq 0 \\ \Leftrightarrow\quad & x_1 \geq 0 \quad \wedge \quad x_1^2 \geq x_2^2 + x_3^2 + x_4^2 \\ \Leftrightarrow\quad & x \geq 0. \end{aligned}$$

Exercise 2 (Loewner order). Let $d \in \mathbb{N}$ and let E denote the space of all self-adjoint complex $d \times d$-matrices, endowed with the Loewner order.

(a) Show that, as claimed in Example 2.1.6(b), the set $B := \{a \in E_+ \mid \operatorname{tr} a = 1\}$ is a base of E_+.

(b) Show that, as claimed in Example 2.1.6(b), the interior points of E_+ are precisely the positive definite self-adjoint matrices.

(c) Let $b \in \mathbb{C}^{d\times d}$. Show that the mapping $E \ni a \mapsto bab^* \in E$ is positive. When is it an order isomorphism?

(d) For every $a \in E$ let $\tau_a \in E'$ be given by $\langle \tau_a, b\rangle := \operatorname{tr}(a^*b)$ for all $b \in E$. Show that

$$\psi\colon E \to E', \qquad a \mapsto \tau_a$$

is an isomorphism of (pre-)ordered vector spaces (where E' is endowed with the dual wedge).

Solution: In the following we denote by $(\cdot \mid \cdot)$ the standard sesquilinear product on $\mathbb{C}^d$ that is anti-linear in the first and linear in the second component.

(a) If $A \in \mathbb{C}^{d\times d}_{\mathrm{sa}}$ is positive-semidefinite, then

$$\operatorname{tr}(A) = \sum_{k=1}^{d} (e^{(k)} \mid Ae^{(k)}) \geq 0$$

shows that $\lambda := \operatorname{tr}(A) \geq 0$, $\operatorname{tr}\left(\frac{A}{\operatorname{tr}(A)}\right) = 1$ and $A = \lambda \frac{A}{\operatorname{tr}(A)}$. Here $e^{(k)}$ denotes the k-th canonical unit vector. This representation is unique. Indeed, let $A_1, A_2 \in \mathbb{C}^{d\times d}_{\mathrm{sa}}$ with $\operatorname{tr}(A_1) = \operatorname{tr}(A_2) = 1$ and $\lambda_1, \lambda_2 > 0$ with $\lambda_1 A_1 = \lambda_2 A_2$. Then $\lambda_1 = \lambda_1 \operatorname{tr}(A_1) = \operatorname{tr}(\lambda_1 A_1) = \operatorname{tr}(\lambda_2 A_2) = \lambda_2 \operatorname{tr}(A_2) = \lambda_2$; and hence, $A_1 = A_2$. It follows that B is a base.

(b) We first show that the positive definite matrices are in the interior of the cone E_+. Let A be positive definite. Then by compactness of the unit sphere in $\mathbb{C}^d$ the minimum

$$\epsilon := \min\{(x \mid Ax) \mid x \in \mathbb{C}^d,\ \|x\|_2 = 1\}$$

exists and is positive. Choose $S \in \mathbb{C}^{d\times d}_{\mathrm{sa}}$ with $\|S\|_2 < \frac{\epsilon}{2}$. Then, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}(x \mid (A+S)x) &= (x \mid Ax) + (x \mid Sx)\\ &\geq (x \mid Ax) + \|x\|_2 \|Sx\|_2 \geq (x \mid Ax) - \epsilon/2 \geq \epsilon/2\end{aligned}$$

for all x in the unit sphere of $\mathbb{C}^d$. Hence, A is in the interior of E_+.

Conversely, if $A \in E_+$ is not positive definite, then there exists $x \in \mathbb{C}^d$ with $\|x\| = 1$ and $(x \mid Ax) \leq 0$. Let $\epsilon > 0$ and set $Sy := -\epsilon(x \mid y)x = -\epsilon x x^* y$. Then clearly, $\|S\|_2 = \epsilon$ and

$$(x \mid (A+S)x) = (x \mid Ax) + (x \mid Sx) \leq (x \mid Sx) = -\epsilon \|x\|_2^2 = -\epsilon < 0.$$

Hence, A is not in the interior of E_+.

(c) Let $b \in \mathbb{C}^{d\times d}$ and $a \in E_+$. Then for all $x \in \mathbb{C}^d$ we have

$$(x \mid bab^*x) = (b^*x \mid a(b^*x)) \geq 0.$$

Hence, bab^* is also positive semidefinite.

Claim. The map is an order isomorphism if and only if b is invertible.

Proof. Notice that if b is invertible, then

$$E \ni a \mapsto b^{-1}a(b^{-1})^* = b^{-1}a(b^*)^{-1}$$

is the inverse of the mapping in the exercise statement and also positive.

Conversely, if b is not invertible, then b is not injective. So there exists $0 \neq z \in \mathbb{C}^d$ such that $(b^*x \mid z) = (x \mid bz) = 0$ for all $x \in \mathbb{C}^d$. Set $a := (z \mid \cdot)z = zz^*$. Then $a \neq 0$ is self-adjoint and $\operatorname{Im}(a) \subseteq \operatorname{span}\{z\}$. Hence,

$$(x \mid bab^*x) = (b^*x \mid ab^*x) = 0.$$

Thus the map $E \ni a \mapsto bab^*$ is not injective.

(d) We show injectivity of the mapping $a \mapsto \tau_a$. Let $a \in E$ such that $\tau_a = 0$. For $b = (e^{(i)} \mid \cdot)e^{(j)} = e^{(j)}(e^{(i)})^*$. Then, since $a = a^*$ and the trace is cyclic, $\operatorname{tr}(a^*b) = \operatorname{tr}\left(ae^{(j)}(e^{(i)})^*\right) = \operatorname{tr}\left((e^{(i)})^*ae^{(j)}\right) = a_{ij}$. Since, $\tau_a = 0$ it follows that $a_{ij} = 0$ for all $i, j \in \{1, \dots, d\}$. Hence, $a = 0$. This shows the injectivity.

As E is finite dimensional $\dim(E) = \dim(E')$, and thus, the surjectivity of $a \mapsto \tau_a$ follows.

To show positivity, let $a \in E_+$, $b \in E_+$. Since any positive semidefinite matrices has a positive semidefinite root, we can write $b = b^{1/2}b^{1/2}$ for $b^{1/2} \in E_+$. Thus, using that a and $b^{1/2}$ are self-adjoint, we obtain

$$\langle \tau_a, b\rangle = \operatorname{tr}(a^*b) = \operatorname{tr}\left(a^*b^{1/2}b^{1/2}\right) = \operatorname{tr}\left(b^{1/2}ab^{1/2}\right) = \operatorname{tr}\left(b^{1/2}a(b^{1/2})^*\right) \geq 0,$$

since by (c) $b^{1/2}a(b^{1/2})$ is positive semidefinite and the diagonal entries of positive semidefinite matrices are non-negative.

Conversely, suppose $\tau_a \in E'_+$. Then for every $x \in \mathbb{C}^d$ define $a = (x \mid \cdot)x = xx^*$ and notice that a is self-adjoint and $\mathbb{R} \ni \langle y, ay\rangle = (y^*x)(x^*y) = (y^*x)\overline{(y^*x)} \geq 0$. So a is also positive semidefinite. It follows that

$$0 \leq \langle \tau_a, b\rangle = \operatorname{tr}(x^*a^*x) = x^*a^*x = (x \mid ax)$$

Hence, $a \in E_+$.

Exercise 3 (Masquerade, Third Act). Endow the space E of self-adjoint complex 3×3-matrices with the Loewner order and denote its positive cone by E_+.

(a) Set $n := \dim E$. Compute n.

(b) Show that E_+ has a face that is not a half-line and not one of the sets $\{0\}$ and E_+.

(c) Is the ordered vector space (E, E_+) isomorphic to $\mathbb{R}^n$ with the ice cream cone?

Solution:

(a) It is easily checked that the complex self-adjoint 3×3-matrices are of the form

$$\begin{pmatrix} a_1 & a_{12}+ib_{12} & a_{13}+ib_{13} \\ a_{12}-ib_{12} & a_2 & a_{23}+ib_{23} \\ a_{13}-ib_{13} & a_{23}-ib_{23} & a_3 \end{pmatrix}$$

where $a_1, a_2, a_3, a_{12}, a_{13}, a_{23}, b_{12}, b_{13}, b_{23} \in \mathbb{R}$. So it follows that $n := \dim(E) = 9$.

(b) *Claim.* We claim that the set

$$F := \{A \in E_+ \mid A_{11} = 0\}$$

is a face of E_+ that is neither a half-line nor trivial.

Proof. Clearly, F is a wedge. Let $A \in F$ and $B \in [0, A]$. Then $(e^{(1)} \mid (A-B)e^{(1)}) \geq 0$, which shows that the entry B_{11} of B is non-positive. Since B is positive semi-definite, it follows that $B_{11} = 0$. Thus, $B \in F$. It follows from Proposition 1.4.3 (ii) $\Rightarrow$ (i) that F is a face. Moreover, F is $\neq E_+, \{0\}$ nor a half-line, since it contains the matrices

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{but not} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) *Claim.* No.

Proof. Suppose there is an order isomorphism $i : E \to \mathbb{R}^n$. Then i maps faces of E_+ to faces of the ice-cream cone $\mathbb{R}^n_+$ (this is proved as in the solution to Exercise 2 (b)). Moreover, $\{0\}$, the entire cone and half-lines are preserved under order isomorphisms. Let F be as in (b). It follows that $i(F)$ is a face $\neq \mathbb{R}_+, \{0\}$ of the ice-cream cone. By Example 1.4.7. (b) it follows that $i(F)$ must be a half-line. This is a contradiction, since F is no half-line, as proved in (b).

Exercise 4 (The ice-cream cone, again). Let $d \in \mathbb{N}$ and endow $\mathbb{R}^d$ with the ice-cream cone $\mathbb{R}^d_+$.

(a) Determine the interior points of $\mathbb{R}^d_+$.

(b) Find a base of $\mathbb{R}^d_+$.

(c) Let us identify $\mathbb{R}^d$ with its own dual space in the canonical way. Show that, under this identification, the dual wedge of $\mathbb{R}^d_+$ is also the ice-cream cone in $\mathbb{R}^d$.

Solution:

(a) *Claim.* The interior points of $\mathbb{R}^d_+$ are given by the set

$$\{x \in \mathbb{R}^d \mid x_1 \geq 0 \wedge x_1^2 > \sum_{n=2}^{d} x_n^2\}$$

Proof. Notice that

$$\mathbb{R}^d_+ = \underbrace{\{x \in \mathbb{R}^d \mid x_1 \geq 0 \wedge x_1^2 > \sum_{n=2}^{d} x_n^2\}}_{A:=} \cup \underbrace{\{x \in \mathbb{R}^d \mid x_1 \geq 0 \wedge x_1^2 = \sum_{n=2}^{d} x_n^2\}}_{B:=}$$

and that the union is disjoint. Then B is the boundary of $\mathbb{R}^d_+$. To see this, recall that a point is in the boundary of a set if and only if every neighborhood has non-trivial intersection with the set and its complement (if suffices to show this for ϵ-balls). Let $x \in B$ and $\epsilon > 0$. Then the elements x and

$$y = (x_1, x_2 + \operatorname{sign}(x_2)\epsilon/2, x_3, \dots)$$

satisfy $x \in \mathbb{R}^d_+$ and $y \in \mathbb{R}^d \backslash \mathbb{R}^d_+$ and lie in the ϵ-ball about x. It follows that $B \subseteq \partial\mathbb{R}^d_+$. Clearly, A is open, so $A \cap \partial\mathbb{R}^d_+ = \emptyset$. Thus, $\partial\mathbb{R}^d_+ \subseteq B$, and thus, $B = \partial\mathbb{R}^d_+$. This implies $\operatorname{int}(\mathbb{R}^d_+) = \mathbb{R}^d_+ \setminus \partial\mathbb{R}^d_+ = A$.

(b) *Claim.* The set

$$B := \{x \in \mathbb{R}^d_+ \mid \langle e^{(1)}, x\rangle = 1\}$$

is a base for $\mathbb{R}^d_+$.[1]

Proof. Clearly B is a convex subset of $\mathbb{R}^d_+$. Moreover, if $x \in \mathbb{R}^d_+ \setminus \{0\}$, it follows from the definition of the ice-cream cone that $x_1 > 0$. Hence, $\langle e^{(1)}, x\rangle = x_1 > 0$. Then for $\lambda = \langle e^{(1)}, x\rangle$ and $b := \lambda^{-1}x$, we have $x = \lambda b$ and $\langle e^{(1)}, b\rangle = 1$.

It is left to show uniqueness of λ and b. Suppose that $b_1, b_2 \in B$ and $\lambda_1, \lambda_2 > 0$ with $x = \lambda_1 b_1 = \lambda_2 b_2$. Then it follows from $\lambda_1 = \lambda_1\langle e^{(1)}, b_1\rangle = \lambda_2\langle e^{(1)}, b_2\rangle = \lambda_2$, and thus, $b_1 = \lambda_1^{-1}x = b_2$. It follows that B is indeed a base of $\mathbb{R}_+$.

(c) Recall that

$$\left(\mathbb{R}^d_+\right)' = \{x' \in \mathbb{R}^d \mid \forall x \in \mathbb{R}^d_+ : \langle x', x\rangle \geq 0\}.$$

We show that $\left(\mathbb{R}^d_+\right)' = \mathbb{R}^d_+$.

"$\supseteq$": Let $x, x' \in \mathbb{R}^d_+$ and denote by $Px := (0, x_2, x_3, \dots)$. Then it follows from the Cauchy-Schwarz inequality that

$$\langle x', x\rangle = x_1' \cdot x_1 + \langle Px', Px\rangle \geq \|Px'\|_2 \cdot \|Px\|_2 + \langle Px', Px\rangle \geq 0$$

Since x was arbitrary, it follows that $x' \in \left(\mathbb{R}^d_+\right)'$.

"$\subseteq$": Let $x' \in \left(\mathbb{R}^d_+\right)'$. Then $\langle x_1', e^{(1)}\rangle \geq 0$ shows that $x_1' \geq 0$, where $e^{(1)}$ denotes the first canonical unit vector.

Define x by $x_1 := \|Px'\|_2$ and $x_n := -x_n'$ for all $n \geq 2$. Notice that $Px = -Px'$ and $x_1 \geq \|Px\|_2 = \|-Px'\|_2$. Hence, $x \in \mathbb{R}^d_+$ and

$$x_1' \cdot \|Px'\|_2 = x_1' \cdot x_1 = \langle x', x\rangle - \langle Px', Px\rangle \geq \langle Px', Px'\rangle = \|Px'\|_2^2$$

If $\|Px'\|_2 > 0$, then $x_1' \geq \|Px'\|_2$. If $\|Px'\|_2 = 0$, then $x_1' \geq 0$ also implies $x_1' \geq \|Px'\|_2$. In any case $x' \in \mathbb{R}^d_+$ holds.

[1] Notice that B defines an affine hyperplane that is orthogonal to $e^{(1)}$ and shifted along the $e^{(1)}$ vector by 1. Since $e^{(1)}$ is the rotational symmetry axis of the ice-cream cone, B is a basis of $\mathbb{R}^d_+$. Now that we have an intuitive understanding what happens geometrically, we conclude with the rigorous proof.