3. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on April 25 and 26, 2023

Exercise 1 (Masquerade of cones, continued). Are the following two ordered vector spaces isomorphic?

(1) The space $\mathbb{R}^4$ with the ice cream cone.

(2) The space of all self-adjoint complex 2×2-matrices with the Loewner order.

Exercise 2 (Loewner order). Let $d \in \mathbb{N}$ and let E denote the space of all self-adjoint complex $d \times d$-matrices, endowed with the Loewner order.

(a) Show that, as claimed in Example 2.1.6(b), the set $B := \{a \in E_+ \mid \operatorname{tr} a = 1\}$ is a base of E_+.

(b) Show that, as claimed in Example 2.1.6(b), the interior points of E_+ are precisely the positive definite self-adjoint matrices.

(c) Let $b \in \mathbb{C}^{d \times d}$. Show that the mapping $E \ni a \mapsto bab^* \in E$ is positive. When is it an order isomorphism?

(d) For every $a \in E$ let $\tau_a \in E'$ be given by $\langle \tau_a, b \rangle := \operatorname{tr}(a^* b)$ for all $b \in E$. Show that

$$\psi \colon E \to E', \qquad a \mapsto \tau_a$$

is an isomorphism of (pre-)ordered vector spaces (where E' is endowed with the dual wedge).

Exercise 3 (Masquerade, Third Act). Endow the space E of self-adjoint complex 3×3-matrices with the Loewner order and denote its positive cone by E_+.

(a) Set $n := \dim E$. Compute n.

(b) Show that E_+ has a face that is not a half-line and not one of the sets $\{0\}$ and E_+.

(c) Is the ordered vector space (E, E_+) isomorphic to $\mathbb{R}^n$ with the ice cream cone?

Exercise 4 (The ice-cream cone, again). Let $d \in \mathbb{N}$ and endow $\mathbb{R}^d$ with the ice-cream cone $\mathbb{R}^d_+$.

(a) Determine the interior points of $\mathbb{R}^d_+$.

(b) Find a base of $\mathbb{R}^d_+$.

(c) Let us identify $\mathbb{R}^d$ with its own dual space in the canonical way. Show that, under this identification, the dual wedge of $\mathbb{R}^d_+$ is also the ice-cream cone in $\mathbb{R}^d$.