2. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on April 18 and 19, 2023

with Solutions

Exercise 1 (Extreme rays).

(a) Determine the extreme rays of the standard cone in ℓ^p for $p \in [1, \infty]$.

(b) Determine the extreme rays of the standard cone in $L^p([0,1])$ for $p \in [1, \infty]$.

Solution:

(a) *Claim.* The extreme rays of $(\ell^p)_+$ are the sets $[0, \infty)\, e^{(k)}$, where $e^{(k)}$ are the canonical unit vectors.

Proof. Fix $k \in \mathbb{N}$. To prove that $[0, \infty)\, e^{(k)}$ is an extreme ray we show that the conditions in Proposition 1.4.6 (iii) are satisfied. Clearly $[0, \infty)\, e^{(k)}$ is a cone. Moreover, let $r \in [0, \infty)\, e^{(k)}$ and $v \in [0, r]$. If $r = 0$, then $v = 0 = 0 \cdot r$. So we may assume that $r \neq 0$. Then for every $l \in \mathbb{N} \setminus \{k\}$ we have $0 \leq v_l \leq r_l = 0$. Hence, $v \in [0, \infty)\, e^{(k)}$ and there exists $\lambda \in [0, \infty)$ such that $v = \lambda \cdot r$. Now Proposition 1.4.6 (iii) $\Rightarrow$ (i) implies that $[0, \infty)\, e^{(k)}$ is an extreme ray.

Conversely, let $R \subseteq (\ell^p)_+$ be an extreme ray. By definition there exists $0 \neq a \in (\ell^p)_+$ such that $R = [0, \infty)\, a$. If there exist distinct $k_1, k_2 \in \mathbb{N}$ such that $a_{k_1}, a_{k_2} > 0$, then $a_{k_1} e^{(k_1)}$ and $a_{k_2} e^{(k_2)}$ are both in $[0, a]$. Thus, by Proposition 1.4.6 (i) $\Rightarrow$ (iii), there exist $\lambda_1, \lambda_2 \in [0, \infty)$ such that

$$\lambda_1 \cdot a_{k_1} \cdot e^{(k_1)} = a = \lambda_2 \cdot a_{k_2} \cdot e^{(k_2)}.$$

This can only happen, when $\lambda_1 = \lambda_2 = 0$, implying that $a = 0$. This is a contradiction and shows that $a \in (0, \infty)\, e^{(k)}$ for some $k \in \mathbb{N}$. Hence, $R = [0, \infty)\, e^{(k)}$. ☐

(b) *Claim.* The cone $(L^p)_+$ does not have any extreme rays.

Proof. To see this, suppose $R = [0, \infty)\, a$ is an extreme ray with $0 \neq a \in (L^p)_+$. Since the map $\gamma(t) := \int_0^t a(x)\, \mathrm{d}x$ is continuous by the monotone convergence theorem and satisfies $\gamma(0) = 0$ and $\gamma(1) > 0$. There exists, by the intermediate value theorem, a $t \in [0, 1]$ such that $\gamma(t) = \frac{1}{2}\gamma(1)$. Then the functions $a_1 := 2a \cdot \mathbb{1}_{[0,t]}$ and $a_2 := 2a \cdot \mathbb{1}_{[t,1]}$ satisfy $a = \frac{1}{2}a_1 + \frac{1}{2}a_2$. Since R is a face, it follows that $a_1, a_2 \in R$. As a_1 and a_2 are linearly independent, it follows that R spans a subspace of at least dimension 2. This contradicts Proposition 1.4.6 (i) $\Rightarrow$ (ii). ☐

Exercise 2 (Cones in $\mathbb{R}^d$).

(a) Let E_+ be a closed and generating cone in $E := \mathbb{R}^2$. Show that the ordered vector space (E, E_+) is isomorphic to $\mathbb{R}^2$ with the standard cone.

(b) Give an example of a closed and generating cone E_+ in $E := \mathbb{R}^3$ such that the ordered vector space (E, E_+) is not isomorphic to $\mathbb{R}^3$ with the standard cone.

(c) Endow $\mathbb{R}^2$ with the standard cone and let $x \in \mathbb{R}^2_+$. Does $[0,1]\,x = [0,x]$ hold?

Solution:

(a) *Claim.* There exist elements a and b that are linearly independent and generate the cone E_+.

It the claim holds then it follows immediately that defines a mapping

$$i : \mathbb{R}^2 \to E, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

is an order isomorphism between $\mathbb{R}^2$ with the standard cone $\mathbb{R}^2_+$ and (E, E_+). Clearly, i is bijective and positive, since $x_1 E_+ + x_2 E_+ \subseteq E_+$. That i^{-1} is positive follows from the fact that a, b generate E_+.

Proof of the Claim. Since the cone E_+ is generating, by Proposition 2.1.1, it has non-empty interior. So $\partial E_+ \neq \emptyset, \{0\}$. Thus, we are able to pick $0 \neq a \in \partial E_+$. Since E_+ is closed, it follows that $a \in E_+$, and thus, $[0,\infty)\,a \subseteq E_+$. Notice that, since E_+ has non-empty interior, $\partial E_+ \setminus [0,\infty)\,a$ is non-empty. If every $b \in \partial E_+ \setminus [0,\infty)\,a$ is linearly dependent on a, then E_+ contains the linear subspace spanned by a. This contradicts the fact that E_+ is a cone. So it follows that a and b are linearly independent and in ∂E_+.

It remains to show that a and b generate E_+. To see this let $z \in E_+$. Since a and b are a basis for E, there exist $x_1, x_2 \in \mathbb{R}$ such that $z = x_1 a + x_2 b$. If $x_1, x_2 < 0$, then $-z \in E_+$, implying that $z = 0$. By linear independence $x_1 = x_2 = 0$. This is a contradiction. If w.l.o.g. $x_1 < 0$ and $x_2 \geq 0$. We show that b is not in the boundary of E_+. Clearly for every $\epsilon \in (0,1)$ the set $(1-\epsilon, 1+\epsilon)b$ is contained in E_+. Moreover, since $x_1 < 0$ we find $0 < \delta < -x_1$ such that $(-\delta,\delta)a + (1-\epsilon, 1+\epsilon)b \subseteq E_+$. This shows that b is in the interior of E_+. This is a contradiction. Thus, we have shown that $x_1, x_2 \geq 0$. It follows that every $z \in E_+$ can be represented as $z = x_1 a + x_2 b$ with $x_1, x_2 \geq 0$. So a and b generate the cone E_+.

(b) *Claim.* We claim that (E, E_+), where E_+ is the ice-cream cone, is not isomorphic to $\mathbb{R}^3$ endowed with the standard cone.

Proof. Notice that it follows from Example 1.4.7 (b) that the ice-cream cone has infinitely many extremal rays. From part (a) of the example it follows that the standard cone in $\mathbb{R}^3$ only has three extremal rays. So it suffices to show that an order isomorphism maps extremal rays to extremal rays.

Clearly an order isomorphism $i : E \to F$ maps $i([0,\infty)\,a) = [0,\infty)\,i(a)$ for each $0 \neq a \in E_+$, so it suffices to show that i maps faces to faces. Let A be a face in E. Let further $x, y \in F$ and suppose there exists $\lambda \in (0,1)$ such that $\lambda x + (1-\lambda)y \in i(A)$. Then $\lambda i^{-1}(x) - (1-\lambda)i^{-1}(y) \in A$, and thus, $i^{-1}(x), i^{-1}(y) \in A$. This implies that $x, y \in i(A)$. So $i(A)$ in indeed a face.

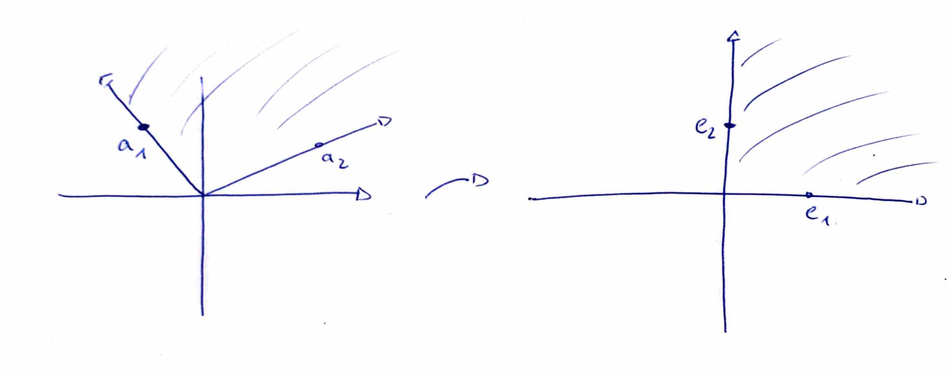

Figure 1: The transform described in the solution of Exercise 2 (a).

(c) No.

Counterexample. Pick $x = (1, 1)$. Then $[0, x] = \{(y_1, y_2) \in \mathbb{R}^2 \mid 0 \leq y_1, y_2 \leq 1\}$, whereas $[0, 1]\, x = \{(y, y) \in \mathbb{R}^2 \mid 0 \leq y \leq 1\}$. So, in particular, the element $(0, 1) \in [0, x]$ but $(0, 1) \notin [0, 1]\, x$.

Exercise 3 (Masquerade of cones). Show that the following ordered vector spaces are isomorphic:

(1) The space $\mathbb{R}^3$ with the ice cream cone.

(2) The space of all symmetric real 2×2-matrices with the Loewner order.

(3) The span of the three real-valued functions $\mathbb{1}$, Re, Im on $\mathbb{T}$ with the pointwise order. Here, $\mathbb{T} := \{z \in \mathbb{C} \mid \ |z| = 1\}$ denotes the complex unit circle.

(4) The span of the functions $\mathbb{1}$, cos, sin on $[0, 2\pi]$ with the pointwise order.

(5) The space of all polynomial functions $\mathbb{R} \to \mathbb{R}$ of degree at most 2 with the pointwise order.

Solution:

"(1)" $\cong$ "(2)" : Denote by $\mathbb{R}^{2\times 2}_{\text{sym}}$ the space of symmetric matrices and by $\left(\mathbb{R}^{2\times 2}_{\text{sym}}\right)_+$ the Loewner cone. Notice that a symmetric matrix is positive semidefinite if and only if all its eigenvalues are non-negative. For a matrix $A \in \mathbb{R}^{2\times 2}_{\text{sym}}$ this is equivalent to $\det A \geq 0$ and $\operatorname{tr} A \geq 0$. We claim that the mapping defined by

$$i : \mathbb{R}^3 \to \mathbb{R}^{2\times 2}_{\text{sym}}, \quad x = (x_1, x_2, x_3) \mapsto \begin{pmatrix} x_1 + x_2 & x_3 \\ x_3 & x_1 - x_2 \end{pmatrix}$$

is an order isomorphism.

Bijectivity is straightforward: clearly, i is injective and its domain and codomain both have dimension 3. To show bipositivity, let $x \in \mathbb{R}^3$. Then

$$\begin{aligned}
& i(x) \geq 0 & & \\
\Leftrightarrow \quad & \operatorname{tr} i(x) \geq 0 & \wedge \quad & \det i(x) \geq 0 \\
\Leftrightarrow \quad & 2x_1 \geq 0 & \wedge \quad & (x_1 + x_2)(x_1 - x_2) - x_3^2 \geq 0 \\
\Leftrightarrow \quad & x_1 \geq 0 & \wedge \quad & x_1^2 \geq x_2^2 + x_3^2 \\
\Leftrightarrow \quad & x \geq 0. & &
\end{aligned}$$

"(3)" $\cong$ "(4)" : It is easily checked that the linear extension of the mapping

$$\mathbb{1} \mapsto \mathbb{1}, \quad \operatorname{Re} \mapsto \cos, \quad \operatorname{Im} \mapsto \sin$$

is the restriction of the bijective and bi-positive mapping

$$T : L^1(\mathbb{T}) \to L^1([0, 2\pi]), \quad Tf(x) := f(\mathrm{e}^{ix})$$

to the three-dimensional space spanned by $\{\mathbb{1}, \cos, \sin\}$.

"(1)" $\cong$ "(4)" : Define the mapping

$$i : \mathbb{R}^3 \to \operatorname{span}\{\mathbb{1}, \cos, \sin\}, \quad (x_1, x_2, x_3) \mapsto x_1 \mathbb{1} + x_2 \cos + x_3 \sin.$$

Then i is clearly bijective.

If $(x_1, x_2, x_3) \in \mathbb{R}^3$ is in the ice-cream cone, then by Cauchy-Schwarz

$$|x_2 \cdot \cos(t) + x_3 \sin(t)| \leq \|(x_2, x_3)\|_2 \|(\cos(t), \sin(t))\|_2 \leq x_1$$

for all $t \in [0, 2\pi]$. Thus, $x_1 \mathbb{1} + x_2 \cos + x_3 \sin \geq 0$.

Conversely, let $x_1 \mathbb{1} + x_2 \cos + x_3 \sin \geq 0$. If $x_2 = x_3 = 0$, then $x \in \mathbb{R}^3_+$. Otherwise, choose $t \in [0, 2\pi]$ such that

$$(\cos(t), \sin(t)) = \frac{1}{\|(x_2, x_3)\|_2}(x_2, x_3).$$

Then $x_1 - \|(x_2, x_3)\|_2 = x_1 - \|(x_2, x_3)\|_2 \cos(t)^2 - \|(x_2, x_3)\|_2 \sin(t)^2 \geq 0$. Hence, (x_1, x_2, x_3) is in the ice-cream cone.

"(2)" $\cong$ "(5)" : Denote by $\mathbb{R}_2[X]$ the space of all polynomial of degree at most 2 with real coefficients. Note that a polynomial $a_1 x^2 + a_2 x + a_3$ takes only non-negative values if and only if either

(i) $a_1 > 0$ and $a_2^2 - 4a_1 a_3 \leq 0$, or

(ii) $a_1 = a_2 = 0$ and $a_3 \geq 0$.

The mapping

$$i : \mathbb{R}^{2\times 2}_{\text{sym}} \to \mathbb{R}_2[X], \quad A = \begin{pmatrix} a & c \\ c & b \end{pmatrix} \mapsto ax^2 + 2cx + b$$

is clearly bijective and we show now that it is an order isomorphism.

Let A be positive. Then $a + b = \operatorname{trace}(A) \geq 0$ and $ab - c^2 = \det(A) \geq 0$. Thus, if $a = 0$, then $-c^2 \geq 0$, and thus, $2c = 0$. It follows that $i(A)$ only takes non-negative values by (ii). We now assume that $a > 0$. Then $(2c)^2 - 4ab = 4(c^2 - ab) \leq 0$. Hence, $i(A)$ only takes non-negative values by (i).

Conversely, let $p := a_1 x^2 + a_2 x + a_3$ satisfy (i). Then

$$\det(i^{-1}(p)) = a_1 a_3 - \tfrac{1}{4} a_2^2 = \tfrac{1}{4}(4a_1 a_3 - a_2^2) \geq 0.$$

And since $a_1 > 0$ and $a_2^2 \geq 0$ we conclude that $a_3 \geq 0$. It follows that $\operatorname{trace}(i^{-1}(p)) = a_1 + a_3 \geq 0$. If p satisfies (ii), then

$$\det(i^{-1}(p)) = 0 \quad \text{and} \quad \operatorname{trace}(i^{-1}(p)) = a_3 \geq 0.$$

In every case, $i^{-1}(p) \in \left(\mathbb{R}^{2\times 2}_{\text{sym}}\right)_+$.

Exercise 4 (Closed faces of the cone in function spaces).

(a) Determine all closed faces of the standard cone in $L^p(\mathbb{R})$ for $p \in [1, \infty)$.

(b) Determine all closed faces of the standard cone in $C([0,1])$.

Solution:

(a) *Claim.* The closed faces of $(L^p)_+$ are exactly the sets

$$I_A := \{f \in \left(L^p(\mathbb{R})\right)_+ \mid f \cdot \mathbb{1}_A = 0\},$$

where A is a measurable set.

Proof. Fix a measurable set A. To show that I_A is a face, let $f, g \in I_A$. Then every element $h \in [0, f + g]$ also satisfies $h \cdot \mathbb{1}_A = 0$. So it follows from Proposition 1.4.3 (iii) $\Rightarrow$ (i) that I_A is indeed a face. Moreover, I_A is clearly closed.

Converse, let Let G be a closed face of $(L^p)_+$. As $L^p(\mathbb{R})$ is a separable metric space, so is G. So we can choose a sequence $(f_n)_{n\in\mathbb{N}}$ in $G \setminus \{0\}$ that is dense in G. Define

$$g := \sum_{n\in\mathbb{N}} 2^{-n} \frac{f_n}{\|f_n\|}$$

Then $g \in L^p$ since the sum is absolutely convergent. Let $\hat{g}$ be a representative of g and set $A := \{x \in \mathbb{R} \mid \hat{g}(x) = 0\}$. We show that $G = I_A$.

"$G \subseteq I_A$": Since for every $n \in \mathbb{N}$ satisfies $f_n \cdot \mathbb{1}_A = 0$ and $(f_n)_{n\in\mathbb{N}}$ is dense in G, it follows that $g \cdot \mathbb{1}_A = 0$ for every $g \in G$.

"$G \supseteq I_A$": Let $h \in I_A$. For $n \in \mathbb{N}$ define $h_n := h \wedge (n \cdot g)$.[1] Then, $h_n \in [0, n \cdot g] \subseteq G$ and $(h_n)_{n\in\mathbb{N}}$ converges almost everywhere[2] as well as monotonously[3] to h. Thus, it follows from the monotone convergence theorem that $(h_n)_{n\in\mathbb{N}}$ converges in norm to h. Since G is closed, $h \in G$.

[1] Here the symbol $\wedge$ denotes the minimum of the two functions. This is defined pointwise on the representatives.

[2] This follows as h vanishes outside of A.

[3] This is clear.

(b) *Claim.* The closed faces of $C([0,1])_+$ are exactly the sets

$$I_A := \{f \in C([0,1])_+ : f|_A = 0\},$$

where A is a closed subset of $[0,1]$.

Proof. Fix A closed and let $f, g \in I_A$. Then every element in $[0, f+g]$ also vanishes on A. So $[0, f+g] \subseteq I_A$. Hence, by Proposition 1.4.3 (iii) $\Rightarrow$ (i), the set I_A is a face. Moreover, I_A is clearly closed.

Conversely, let G be a face of $C([0,1])$.

$$A := \{x \in [0,1] \mid \forall f \in G : f(x) = 0\}$$

Then clearly $G \subseteq I_A$.

To show the converse inclusion, notice that $C([0,1])$ is separable.[4] Thus, we find a sequence $(f_n)_{n\in\mathbb{N}}$ in $G \setminus \{0\}$ that is dense in G. Then the function

$$g := \sum_{n\in\mathbb{N}} 2^{-n} \frac{f_n}{\|f_n\|}$$

is again in $C([0,1])$. Let $h \in I_A$. Then $h_n := h \wedge (n \cdot g)$ defines a monotone increasing sequence that converges pointwisely to h, since h vanishes on A. Since h is continuous, it follows from Dini's theorem that this convergence is even uniform. As $h_n \in [0, n \cdot g] \subseteq G$ it follows from the closedness of G that $h \in G$.

[4] Recall that, if K is a compact Hausdorff space, the space $C(K)$ is separable if and only if K is metrizable. For the space $C([0,1])$ this follows more easily, since by the Weierstraß approximation theorem, the polynomials are a countable and dense set in $C([0,1])$.