1. Exercise Sheet in **Ordered Banach Spaces and Positive Operators**

For the exercise classes on April 11 and 12, 2023

Exercise 1 (Cones and wedges in finite dimensions).

(a) Show that $E_+ \coloneqq \{0\} \cup \{(x,y) \mid \; x,y > 0\}$ in $E \coloneqq \mathbb{R}^2$ is a wedge. Is it even a cone? Is it Archimedean?

(b) Give an example of a (non-Archimedean) cone in $\mathbb{R}^2$ that contains a one-dimensional affine subspace.

(c) Give an example of a closed wedge W in $\mathbb{R}^3$ that is not a cone and not a half space. Moreover, give an example a vector $x \neq 0$ in $\mathbb{R}^3$ such that $0 \leq x \leq 0$ with respect to pre-order induced by this wedge W.

(d) Consider the vectors

$$x \coloneqq \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad y_n \coloneqq \begin{pmatrix} 1 \\ \frac{1}{n} \end{pmatrix} \quad \text{for each } n \in \mathbb{N}$$

in $\mathbb{R}^2$ and set $S \coloneqq \{x\} \cup \{y_n \mid \; n \in \mathbb{N}\}$.
Determine the smallest wedge W in $\mathbb{R}^2$ that contains S. Is W a cone? Is the closure of W a cone?

(e) Consider the set $E_+ \coloneqq \{0\} \cup \{x \in E \mid \; \langle x', x\rangle > 0\}$, where E is a non-zero real Banach space and $0 \neq x' \in E'$ is a fixed continuous linear functional.
Show that E_+ is a wedge. When is it a cone? When is it Archimedean?

Exercise 2 (An ℓ^2-ice cream cone in c_0). Let c_0 denote the space of real-valued sequences (indexed over $\mathbb{N} \coloneqq \{1, 2, \dots\}$) that converge to 0. Show that

$$(c_0)_+ \coloneqq \big\{x \in c_0 \mid \; x_1 \geq 0 \text{ and } x_1^2 \geq \sum_{n=2}^{\infty} x_n^2\big\}$$

is a cone in c_0. Is $(c_0)_+$ generating? Is the set $(c_0)_+$ closed in c_0 (with respect to the sup norm)?