1. Exercise Sheet in
Ordered Banach Spaces and Positive Operators

For the exercise classes on April 11 and 12, 2023

with Solutions

Exercise 1 (Cones and wedges in finite dimensions).
(a) Show that $E_+ \coloneqq \{0\} \cup \{(x,y) \mid \ x,y > 0\}$ in $E \coloneqq \mathbb{R}^2$ is a wedge. Is it even a cone? Is it Archimedean?

(b) Give an example of a (non-Archimedean) cone in $\mathbb{R}^2$ that contains a one-dimensional affine subspace.

(c) Give an example of a closed wedge W in $\mathbb{R}^3$ that is not a cone and not a half space. Moreover, give an example a vector $x \neq 0$ in $\mathbb{R}^3$ such that $0 \leq x \leq 0$ with respect to pre-order induced by this wedge W.

(d) Consider the vectors

$$x \coloneqq \begin{pmatrix} -1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad y_n \coloneqq \begin{pmatrix} 1 \\ \frac{1}{n} \end{pmatrix} \quad \text{for each } n \in \mathbb{N}$$

in $\mathbb{R}^2$ and set $S \coloneqq \{x\} \cup \{y_n \mid \ n \in \mathbb{N}\}$.
Determine the smallest wedge W in $\mathbb{R}^2$ that contains S. Is W a cone? Is the closure of W a cone?

(e) Consider the set $E_+ \coloneqq \{0\} \cup \{x \in E \mid \ \langle x', x\rangle > 0\}$, where E is a non-zero real Banach space and $0 \neq x' \in E'$ is a fixed continuous linear functional.
Show that E_+ is a wedge. When is it a cone? When is it Archimedean?

Solution:

(a) Let (x_1, y_1), $(x_2, y_2) \in E_+$ and $\alpha \geq 0$. Then we have the two cases

$$\alpha \cdot (x_1, y_1) = (\alpha x_1, \alpha y_1) = \begin{cases} (0,0), & \text{if } (x_1 = 0 \wedge x_2 = 0) \vee \alpha = 0 \\ (>0, >0), & \text{if } (x_1 > 0 \wedge y_1 > 0) \wedge \alpha > 0 \end{cases}$$

Similarly

$$\begin{aligned}(x_1, y_1) + (x_2, y_2) &= (x_1 + x_2, y_1 + y_2) \\ &= \begin{cases} (0,0), & \text{if } x_1, x_2, y_1, y_2 = 0 \\ (>0, >0), & \text{if } (x_1 > 0 \wedge y_1 > 0) \vee (x_2 > 0 \wedge y_2 > 0) \end{cases}\end{aligned}$$

So $\alpha(x_1, y_1) \in E_+$ and $(x_1, y_1) + (x_2, y_2) \in E_+$. It follwos that E_+ is a wedge. Moreover, $E_+ \cap -E_+ = \{0\}$, so E_+ is also a cone.

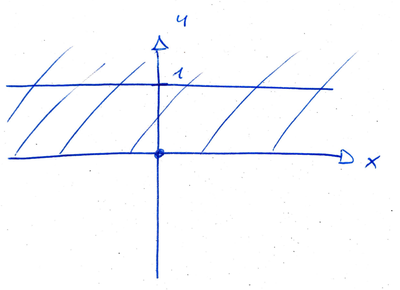

Figure 1: The cone from Exercise 1 (b).

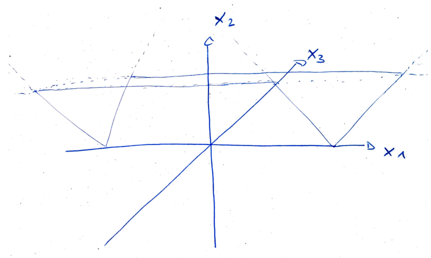

Figure 2: The wedge defined in Exercise 1 (c).

Notice that E_+ is not Archimedean, since

$$\begin{pmatrix}-1\\0\end{pmatrix} \not\leq 0 \qquad \text{and} \qquad \begin{pmatrix}-1\\0\end{pmatrix} \leq \frac{1}{n}\begin{pmatrix}1\\0\end{pmatrix}$$

for all $n \in \mathbb{N}$. (The non-Archimedean property can also be see from Proposition 2.1.2 (ii) "⇒" (i), since E_+ is not closed.)

(b) Consider the cone $E_+ := \{0\} \cup \{(x, y) \mid x \in \mathbb{R},\ y > 0\}$ in $E := \mathbb{R}^2$, see Figure 1. Then E_+ contains the one-dimensional affine subspace $A := \{(x, 1) \mid x \in \mathbb{R}\}$.

Notice also that E_+ is non-Archimedean, since

$$\begin{pmatrix}-1\\0\end{pmatrix} \not\leq 0 \qquad \text{and} \qquad \begin{pmatrix}-1\\0\end{pmatrix} \leq \frac{1}{n}\begin{pmatrix}1\\0\end{pmatrix}$$

for all $n \in \mathbb{N}$. (Notice that the property "non-Archimedean" is in parenthesis, since there exists no Archimedean cone that contains a one-dimensional affine subspace, see Proposition 1.5.2 (i) "⇒" (iv).)

(c) Consider the set $W := \{(x_1, x_2, x_3) \mid x_1 \in \mathbb{R},\ x_2 \geq 0,\ x_2 \geq |x_3|\}$, see Figure 2.

Notice that $\alpha W + \beta W \subseteq W$ for all $\alpha, \beta \geq 0$. So W is indeed a wedge. Moreover, $W \cap -W = \{(x_1, 0, 0) \mid x_1 \in \mathbb{R}\}$. From this it follows that

$$(1, 0, 0) \in W \cap -W, \qquad \text{and thus,} \qquad 0 \leq (1, 0, 0) \leq 0.$$

(d) *Claim.* The wedge W generated by the set S is the rotated lexicographical cone

$$W = \{(x_1, x_2) \mid x \in \mathbb{R},\ x_2 > 0\} \cup \{(x_1, 0) \mid x_1 \leq 0\}.$$

See Figure 3 for an image in of the points in S.

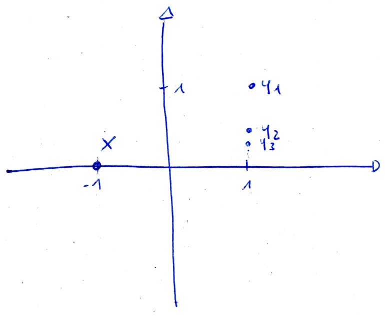

Figure 3: The points in S.

Proof. Clearly, $S \subseteq W$ and W is a wedge. Conversely, let $(x_1, x_2) \in \mathbb{R}^2$ with $x_2 > 0$. Then choose $n \in \mathbb{N}$ large enough such that $n \cdot x_2 - x_1 \geq 0$. Set $\beta := n \cdot x_2$ and $\alpha := \beta - x_1$. Thus, $\alpha \cdot (-1, 0) + \beta \cdot (1, \frac{1}{n}) = (x_1, x_2)$. Now let $(x_1, 0) \in \mathbb{R}^2$ with $x_1 \leq 0$. Then the choice $\alpha = -x_1 \geq 0$ yields $(x_1, 0) = \alpha \cdot (-1, 0) \in E_+$. Hence, W is the wedge generated by S.

Moreover, W is indeed a cone, since $W \cap -W = \{0\}$.

Notice that the closure $\overline{W}$ of W is the half space

$$\overline{W} = \{(x_1, x_2) \mid x_1 \in \mathbb{R},\ x_2 \geq 0\},$$

which satisfies

$$\overline{W} \cap -\overline{W} = \{(x_1, 0) \mid x_1 \in \mathbb{R}\}.$$

Thus, $\overline{W}$ is no cone.

(e) See Figure 4 for an illustration of a cone E_+ for $E = \mathbb{R}^2$.

Let $x_1, x_2 \in E_+$ and $\alpha \geq 0$.

We show that $\alpha \cdot x_1 \in E_+$. If $x_1 = 0$ or $\alpha = 0$, then $\alpha \cdot x_1 = 0$, and thus, $\alpha \cdot x_1 \in E_+$. If $x_1 \neq 0$ and $\alpha > 0$, then $\langle x', \alpha \cdot x_1 \rangle = \alpha \langle x', x_1 \rangle > 0$. So $\alpha \cdot x_1 \in E_+$.

We show that $x_1 + x_2 \in E_+$. If $x_1 = 0$ and $x_2 = 0$, then also $x_1 + x_2 = 0$. If w.l.o.g. $x_1 \neq 0$, then $\langle x', x_1 + x_2 \rangle = \langle x', x_1 \rangle + \langle x', x_2 \rangle > 0$.

It follows $\alpha E_+ + \beta E_+ \subseteq E_+$ for all $\alpha, \beta \geq 0$, and thus, E_+ is a wedge.

Clearly,

$$E_+ \cap -E_+ = \{0\} \cup \{x \in E \mid \langle x', x \rangle > 0,\ \langle x', -x \rangle > 0\} = \{0\},$$

so E_+ is a cone.

Claim. The cone E_+ is non-Archimedean if and only if $\dim(E) \geq 2$.

Proof. "$\Rightarrow$": Let $\dim(E) \leq 1$. Then, since $E \neq \{0\}$, $\dim(E) = 1$. It follows that $E \cong \mathbb{R}$, and thus,

$$E_+ = \{x \mid x \geq 0\} \qquad \text{or} \qquad E_+ = \{x \mid x \leq 0\}.$$

Thus, E_+ is Archimedean.

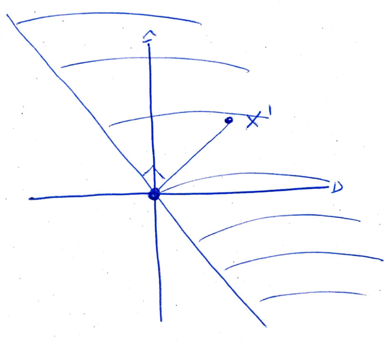

Figure 4: The cone in Exercise 1 (e) in $E = \mathbb{R}^2$.

"$\Leftarrow$": Conversely, let $\dim(E) \geq 2$. Then choose $x \in \ker(x')$ with $\|x\| = 1$ and $y \in E$ with $\langle x', y\rangle > 0$. This is possible, since x' has codimension at least 1. Then

$$x \not\leq 0 \qquad \text{and} \qquad x \leq \tfrac{1}{n} y$$

for all $n \in \mathbb{N}$, since $\langle x', \frac{1}{n}y - x\rangle > 0$. It follows that E_+ is not Archimedean.

Exercise 2 (An ℓ^2-ice cream cone in c_0). Let c_0 denote the space of real-valued sequences (indexed over $\mathbb{N} := \{1, 2, \dots\}$) that converge to 0. Show that

$$(c_0)_+ := \{x \in c_0 \mid \; x_1 \geq 0 \text{ and } x_1^2 \geq \sum_{n=2}^{\infty} x_n^2\}$$

is a cone in c_0. Is $(c_0)_+$ generating? Is the set $(c_0)_+$ closed in c_0 (with respect to the sup norm)?

Solution:

We show that c_0 is a cone. Let $x = (x_n)_{n\in\mathbb{N}}, y = (y_n)_{n\in\mathbb{N}} \in c_0$ and $\alpha, \beta \geq 0$. Define the projection $P : c_0 \to c_0$ by $Px := (0, x_2, x_3, \dots)$ and notice that

$$(c_0)_+ = \{x \in c_0 \mid x_1 \geq \|Px\|_2\}$$

Then

$$\alpha \cdot x_1 + \beta \cdot y_1 \geq \alpha\|Px\|_2 + \beta\|Py\|_2 \geq \|\alpha \cdot Px + \beta \cdot Py\|_2 = \|P(\alpha \cdot x + \beta \cdot y)\|_2.$$

So $\alpha \cdot x + \beta \cdot y \in E_+$. Moreover,

$$\begin{aligned}(c_0)_+ \cap -(c_0)_+ &= \{x \in c_0 \mid 0 \leq x_1 \leq 0,\, x_1^2 \geq \sum_{n=2}^{\infty} x_n^2\} \\ &= \{x \in c_0 \mid 0 \leq x_1 \leq 0,\, 0 \geq \sum_{n=2}^{\infty} x_n^2\} = \{0\}.\end{aligned}$$

So E_+ is indeed a cone.
Notice further that $(c_0)_+$ is a subset of ℓ^2. Thus, by the triangle inequality, it follows that

$$(c_0)_+ - (c_0)_+ \subseteq \ell^2.$$

(It is even the case that equality holds.) Since $\ell^2 \subsetneq c_0$ (for example $(\frac{1}{\sqrt{n}})_{n\in\mathbb{N}}$ is in c_0 but not in ℓ^2), it follows that $(c_0)_+$ is not generating.[1]
To see that $(c_0)_+$ is closed let $(x^{(k)})_{k\in\mathbb{N}}$ be a sequence in $(c_0)_+$ that converges to $x \in c_0$. Then clearly $x_1 \geq 0$, since $x_1^{(k)} \geq 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} x_1^{(k)} = x_1$.
Define

$$f_{(k)}(n) := \left(x_n^{(k)}\right)^2 \qquad \text{and} \qquad f(n) := x_n^2$$

for all $n \in \mathbb{N}$. Notice that $f_{(k)}, f \geq 0$. Then by Fatou's lemma it follows that

$$\begin{aligned}\sum_{n=2}^{\infty} x^2 = \sum_{n=2}^{\infty} f(n) = \sum_{n=2}^{\infty} \liminf_{k\to\infty} f_{(k)}(n) \\ \leq \liminf_{k\to\infty} \sum_{n=2}^{\infty} f_{(k)}(n) \leq \liminf_{k\to\infty} \left(x_1^{(k)}\right)^2 = x_1^2.\end{aligned}$$

Hence, it follows that $x \in (c_0)_+$ and that the cone E_+ is closed.

[1]It is an easy exercise to show that $\overline{(c_0)_+ - (c_0)_+} = c_0$. A cone with this property is called *total*, see Definition 4.3.1.