Cesàro Sequence Vector Lattices. Duals and Ideals of Finite Elements

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Workshop on Ordered Vector Spaces and Positive Operators

Wuppertal, March 29 – April 1, 2023

Topics

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1. Cesàro Sequence Vector Spaces

Let C be the Cesàro matrix defined by

$$c_{nm} = \begin{cases} \frac{1}{n}, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}, \quad \text{i.e.}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & . & . & . & . & . \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & . & . & . & . & . \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & . & . & . & . & . \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \\ \frac{1}{n} & \frac{1}{n} & . & . & . & . & . & \frac{1}{n} & 0 & 0 & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . \end{pmatrix}.$$

Then $C \geq 0$, $dom(C) = \mathbb{R}^{\mathbb{N}}$ and $Cx = \left(\frac{1}{n} \sum_{k=1}^{n} x_k \right)_{n \in \mathbb{N}}$ for all sequences $x = (x_k)_{k \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$.

For a subset $\lambda \subset \mathbb{R}^{\mathbb{N}}$ define the set

$$sol\big(C^{-1}(\lambda)\big) = \{x \in \mathbb{R}^{\mathbb{N}} \colon C|x| \in \lambda\}$$

Properties of $sol\big(C^{-1}(\lambda)\big)$ for the case that λ is an order ideal in $\mathbb{R}^{\mathbb{N}}$:

- $sol\big(C^{-1}(\lambda)\big)$ is an order ideal in $\mathbb{R}^{\mathbb{N}}$,
- $sol\big(C^{-1}(\lambda)\big) \subset C^{-1}(\lambda) = \{x \in \mathbb{R}^{\mathbb{N}} \colon Cx \in \lambda\}$,
- $sol\big(C^{-1}(\lambda)\big)$ is the largest solid subset which is contained in $C^{-1}(\lambda)$.

As an ideal $sol\big(C^{-1}(\lambda)\big)$ is a vector lattice with respect to the order inherited from the natural coordinatewise order of $\mathbb{R}^{\mathbb{N}}$.

We study the ideals $sol\big(C^{-1}(\lambda)\big)$ generated by the ideals

$$\lambda = \begin{cases} \ell_p, \quad \text{for } \ 0 < p < \infty & \rightsquigarrow \ ces_p \\ \ell_\infty, & \rightsquigarrow \ ces_\infty \\ c_0, & \rightsquigarrow \ ces_0. \end{cases}$$

Clearly, c has to be omitted since it is not an order ideal in $\mathbb{R}^{\mathbb{N}}$!

The spaces $sol\big(C^{-1}(\lambda)\big)$ share the properties above.

2. The Cesàro Sequence Spaces for $0 < p < \infty$

These vector lattices are defined as

$$\boldsymbol{ces_p := sol\big(C^{-1}(\ell_p)\big) = \{x \in \mathbb{R}^{\mathbb{N}} \colon C|x| \in \ell_p\}.}$$

It turns out that for $0 < p \leq 1$ these spaces are trivial, i.e. $\{\mathbf{0}\}$.

If $1 < p < \infty$ then ℓ_p is a proper solid (normal) subset of ces_p. Fix $m > 2 + \frac{2}{p-1}$. Then the sequence $x = (x_n)_{n\in\mathbb{N}}$ with

$$x_n = \begin{cases} k, & \text{if } n = k^m \\ 0, & \text{if } n \notin \{k^m \colon k \in \mathbb{N}\} \end{cases}. \qquad n \in \mathbb{N}$$

Then $x \notin \ell_p$, however it can be shown that Cx is in ℓ_p: For $p \geq 4$ and $m = 3$ take $x = (1, 0, \ldots 0, 2, 0, \ldots 0, 3, 0, \ldots)$. Then $Cx = (1, \frac{1}{2}, \ldots, \frac{1}{7}, \frac{3}{9}, \ldots, \frac{3}{26}, \frac{6}{27}, \ldots, \frac{6}{63}, \frac{10}{64}, \ldots) \in \ell_p$, and so $x \in ces_p$.

Norm on ces_p: $\|x\|_{ces_p} := \|C\,|x|\|_p = \Big(\sum_{n=1}^{\infty} \big(\frac{1}{n}\sum_{k=1}^{n} |x_k|\,\big)^p\Big)^{\frac{1}{p}}$.

$$x, y, \in \ell_p,\ C \geq 0,\ |x| \leq |y| \Longrightarrow C|x| \leq C|y| \text{ and so}$$

$$\|x\|_{ces_p} = \|C|x|\|_p \leq \|C|y|\|_p = \|y\|_{ces_p} \text{ – Riesz norm.}$$

Remember:
M-norm: $x, y \geq 0 \Longrightarrow \|x \vee y\| = \max\{\|x\|, \|y\|\}$.
L_p-norm: $x, y \geq 0 \Longrightarrow \|x + y\| = \|x\| + \|y\|$.

An element $u \in E_+$, $u \neq 0$ of a vector lattice E is called an atom, whenever $0 < x, y \leq u$ and $x \wedge y = 0$ imply either $x = 0$ or $y = 0$, or equivalent: whenever $0 \leq x \leq u$ implies $x = \lambda u$ for some $\lambda \in \mathbb{R}_+$.
A vector lattice E is said to be atomic if for each $x > 0$, there exists an atom u, such that $0 < u \leq x$.
Examples: c_0, c, ℓ^p $(1 \leq p \leq \infty)$ are atomic vector lattices, but $C([0,1])$ is atomless.

Properties of $\left(ces_p, \|\cdot\|_{ces_p}\right)$:

- Dedekind complete vector lattice (in particular, Archimedean)
- separable Banach lattice with respect to its norm (Shiue, 1970 and Leibowitz, 1971)
- reflexive if and only if $1 < p < \infty$ (Jagers, 1974) and so the norm of ces_p is order continuous, i.e. $x_n \downarrow 0$ implies $\|x_n\| \downarrow 0$ (Aliprantis/Border, 1994)
- atomic vector lattice, where the only atoms are the coordinate sequences $e_n = (\underbrace{0, 0, \ldots, 0}_{n-1}, 1, 0, 0, \ldots)$ for $\forall n \in \mathbb{N}$
- neither an AM-space nor an abstract L_p-space.

3. The Dual of ces_p

The problem of the dual space ces'_p of the Banach lattices $(ces_p, \|\cdot\|)$, formulated first 1971 in a Dutch Journal, was solved by Jagers (1974). We rest here on Bennett's approach (1996), which provides an explicit lattice isomorphism between ces'_p and the new constructed spaces d_q. For $0 < q < \infty$ consider

$$d_q := \Big\{ a = (a_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} \colon \Big(\underbrace{\sup_{k\geq n} |a_k|}_{=:\, \hat{a}_n} \Big)_{n\in\mathbb{N}} \in \ell_q \Big\}.$$

The sequence $\hat{a} = (\hat{a}_n)_{n\in\mathbb{N}}$ is called least decreasing majorant of a. $d_0 = c_0$ and $d_\infty = \ell_\infty$ are not of interest.

d_q is proper subset of ℓ_q: $a = (0, \frac{1}{2^1}, 0, \frac{1}{2^2}, 0, 0, 0, \frac{1}{2^3}, \ldots) \in \ell_1$. But $\hat{a} = (\frac{1}{2^1}, \frac{1}{2^1}, \frac{1}{2^2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^3}, \frac{1}{2^3}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots) \notin \ell_1$, hence $a \notin d_1$.

Norm on d_q: $\|a\|_{d_q} := \|\hat{a}\|_q = \left(\sum_{n=1}^{\infty} \sup_{k\geq n} |a_k|^q \right)^{1/q}$.

Properties of $\left(d_q, \|\cdot\|_{d_q}\right)$:

- $(d_q, \|\cdot\|_{d_q})$ is a Dedekind complete Banach lattice,
- $c_{00} \subsetneqq d_q \subsetneqq \ell_q$,
- the norm in d_q is order continuous for $1 \leq q < \infty$.

Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\langle a, x\rangle := \sum_{n=1}^{\infty} a_n x_n, \quad x \in ces_p,\ a \in d_q$$

is a lattice isomorphism between ces_p' and d_q.

Interesting: the identification is even isometric when ces_p is endowed with an other norm, equivalent to $\|\cdot\|_{ces_p}$ (s. Bennett, Bonet/Ricker for details).

- $ces_0' = d_1$ with equality of norms (Curbera/Ricker),
- $ces_0'' = d_1' = ces_\infty$ with equality of the norms (Alexiewicz),
- Pettis' Theorem implies that d_q is reflexive for $1 < q < \infty$ as ces_p is for $1 < p < \infty$.
- the norm of d_1 is order continuous (see above). However the norm of ces_∞ is not order continuous (GPW).

4. Finite Elements in Vector Lattices

- An element φ of an Archimedean vector lattice E is called **finite** if if there is an element $z \in E$ satisfying the following condition: for any element $x \in E$ there exists a number $c_x > 0$ such that the following inequality holds

$$\boldsymbol{|x| \wedge n|\varphi| \leq c_x\, z \quad \text{for} \quad \forall n \in \mathbb{N}.}$$

The element z is called *E-majorant* of φ.

$\Phi_1(E)$ denotes the ideal of all finite elements of E.

- $\varphi \in E$ is called **totally finite**, if there exists an E-majorant $z \in \Phi_1(E)$.

$\Phi_2(E)$ denotes the ideal of all totally finite elements of E.

- $\varphi \in E$ is called **self-majorizing**, if $|\varphi|$ is an E-majorant of φ, i.e. $\forall x \in E$ there is $c_x > 0$ with

$$|x| \wedge n|\varphi| \leq c_x\, |\varphi| \quad \text{for} \quad \forall n \in \mathbb{N}$$

$S(E)$ denotes the set of all self-majorizing elements of E. $S_+(E) = S(E) \cap E_+$. The set $\Phi_3(E) = S_+(E) - S_+(E)$ is also an ideal and

$$0 \in \Phi_3(E) \subseteq \Phi_2(E) \subseteq \Phi_1(E) \subseteq E.$$

5. The Finite Elements in ces_p and in d_p

Each atom of an Archimedean vector lattice is also a self-majorizing element.

For a normed vector lattice E denote by Γ_E the set of all atoms with the norm equal to 1. $\Gamma_E \subseteq \Phi_3(E)$ and, it consists of pair-wise disjoint linearly independent elements.

Let c_{00} denote the space of all finite sequences, i.e. $x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ is in c_{00} if exists n_x such that $x_n = 0$ for all $n \geq n_x$. It is clear that $\operatorname{span}(\Gamma_{ces_p}) = c_{00} \neq ces_p$.

Theorem 1.

Let be $1 < p < \infty$. *Then*
(1) $\Phi_3(ces_p) = \Phi_2(ces_p) = \Phi_1(ces_p) = c_{00}$,
(2) *the space* ces_p *has no order unit.*

Proof. (1): For a Banach lattice E with order continuous norm by (Thm. Chen/W.,2006) there holds $\Phi_i(E) = c_{00},\ i \in \{1, 2, 3\}$.
(2): If there would be an order unit in ces_p then $\Phi_i(E) = ces_p,\ i \in \{1, 2, 3\}$, and by (1) should be $ces_p = c_{00}$, i.e. a contradiction.

Theorem 2.

Let be $1 \leq q < \infty$. *Then*
(1) $\Phi_3(d_q) = \Phi_2(d_q) = \Phi_1(d_q) = c_{00}$,
(2) *the space* d_q *has no order unit.*

6. The Cesàro Space ces_∞

$$\boldsymbol{ces_\infty := sol\big(C^{-1}(\ell_\infty)\big) = \{x \in \mathbb{R}^{\mathbb{N}} \colon C|x| \in \ell_\infty\}}$$

with the **norm** $\|x\|_{ces_\infty} := \sup_n \left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right) < \infty.$

Properties of $\left(ces_\infty, \|\cdot\|_{ces_\infty}\right)$:

- non-atomic Dedekind complete Banach lattice
- ℓ_∞ is a proper solid subspace of ces_∞.
  For that consider the sequence $x = (x_n)_{n\in\mathbb{N}}$ with
  $$x_n = \begin{cases} k, & \text{if } \; n = k^2 \\ 0, & \text{if } \; n \notin \{k^2 \colon k \in \mathbb{N}\} \end{cases}, \qquad n \in \mathbb{N}$$
  then $x = (1, 0, 0, 2, 0, 0, 0, 0, 3, 0, 0, \ldots) \notin \ell_\infty$,
  but $Cx = (1, \frac{1}{2}, \frac{1}{3}, \frac{3}{4}, \frac{3}{5}, \ldots, \frac{3}{8}, \frac{6}{9}, \frac{6}{10}, \ldots)$, i.e. $Cx \in \ell_\infty$,
  and so $x \in ces_\infty$.
- not an AM-space, and the norm is not σ-order continuous
- vector lattice without an order unit, see also Corollary 1 below.

7. Self-majorizing and Finite Elements in ces_∞

For $x \in \mathbb{R}^{\mathbb{N}}$ define its support by $\operatorname{supp}(x) = \{n \in \mathbb{N} \colon x_n \neq 0\}$.

Theorem 3.

Let φ be an element of ces_∞. Then $\varphi \in \Phi_3(ces_\infty)$ if and only if

$$r_\varphi = \inf \left\{ \frac{|\varphi_n|}{n} \colon n \in \operatorname{supp}(\varphi) \right\} > 0.$$

Example: Existence of elements in $\Phi_3(ces_\infty) \setminus \ell_\infty$. Consider $x = (x_n)_{n \in \mathbb{N}}$ with

$$x_n = \begin{cases} n, & \text{if } n = 2^k \\ 0, & \text{otherwise} \end{cases}, \qquad k \in \{0\} \cup \mathbb{N}.$$

Then $x \notin \ell_\infty$ (all the more $x \notin c_{00}$). Let be $y = Cx$. Then

$$y_{2^k} = (Cx)_{2^k} = \frac{1}{2^k} \sum_{n=1}^{2^k} x_n = \frac{\sum\limits_{n=1}^{2^k} 2^k}{2^k} = \frac{2^{k+1} - 1}{2^k(2-1)} = 2 - \frac{1}{2^k} \leq 2$$

while, for the subscript N satisfying $2^k \leq N < 2^{k+1}$ one has

$$0 < y_N = \frac{1}{N} \sum_{n=1}^{N} x_n = \frac{1}{N} \sum_{n=1}^{2^k} x_n \leq \frac{1}{2^k} \sum_{n=1}^{2^k} x_n = y_{2^k} \leq 2.$$

It is clear that $y = Cx \in \ell_\infty$, i.e. $x \in ces_\infty$.

Moreover, since $\inf\left\{\frac{|x_n|}{n} : n \in \operatorname{supp}(x)\right\} = 1$, Theorem 3 $\Longrightarrow$ $x \in \Phi_3(ces_\infty)$.

The Dedekind completeness of ces_∞ implies that ces_∞ has (ppp). Hence $\Phi_1(ces_\infty) = \Phi_2(ces_\infty)$.

Theorem 4.

Let φ be an element of ces_∞. The following statements are equivalent:

(1) φ *is a finite element of* ces_∞.

(2) $\{\varphi\}^{dd}$ *has an order unit* x *with* $\inf\{\frac{x_n}{n} : n \in \operatorname{supp}(x)\} > 0$

(3) The sequence $z = (z_n)_{n\in\mathbb{N}}$, *where*

$$z_n = \begin{cases} n, & \text{if } n \in \operatorname{supp}(\varphi) \\ 0, & \text{otherwise} \end{cases}, \qquad n \in \mathbb{N}.$$

belongs to ces_∞.

Theorem 5.

Let φ be an element of ces_∞. Let $\operatorname{supp}(\varphi)$ be an infinite set written as an increasing sequence $(k_n)_{n\in\mathbb{N}}$.

(1) If $\limsup \frac{k_{n+1}}{k_n} = 1$, *then* φ *is not a finite element of* ces_∞.

(2) If $\liminf \frac{k_{n+1}}{k_n} > 1$, *then* φ *is a finite element of* ces_∞.

Corollary 1.

(i) ces_∞ *has no order unit.*

(ii) *Not every element of* ℓ_∞ *is a finite element of* ces_∞*, and therefore,* ℓ_∞ *is neither a norm-closed ideal nor a band in* ces_∞*.*

The next characterization of finite elements uses the distribution of their supports within the intervals $[2^n, 2^{n+1})$ along the positive real numbers.

Theorem 6.

The element φ *of* ces_∞ *is finite if and only if*

$$\beta := \sup \left\{ \beta_n : n \in \{0\} \cup \mathbb{N} \right\} < \infty,$$

where β_n *denotes the cardinality of the set* $[2^n, 2^{n+1}) \cap \operatorname{supp}(\varphi)$ *for each* $n \in \{0\} \cup \mathbb{N}$*.*

Idea of the proof: For the finite element φ define the sequence z by $z_n = n$ if $n \in \operatorname{supp}(\varphi)$ and $z_n = 0$ otherwise. Then Theorem 4(3) implies $z \in ces_\infty$, i.e. $\|z\|_{ces_\infty} = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} z_k \right) < \infty$. By using the so-called *blocking technique* this "series form" of the norm is transformed into its "block form".

With the parameter $p = 1$, the blocks

$$\{n \in \mathbb{N} \colon 2^n \leq k < 2^{n+1}\}$$

and the sequence $a = (\frac{1}{n})_{n\in\mathbb{N}}$ from some Theorem (see Grosse-Erdmann, 1998), we obtain the equivalence

$$\sup_n \left(\frac{1}{n}\sum_{k=1}^{n} z_k\right) < \infty \overset{(\equiv)}{\Longleftrightarrow} \varrho_z := \sup_n \left(\frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}-1} z_k\right) < \infty.$$

By taking into consideration that $z_k = k \geq 2^n$ and so, $z_k \neq 0$ only for $k \in \operatorname{supp}(\varphi)$ one has

$$\varrho_z \geq \frac{1}{2^n}\sum_{\substack{k=2^n\\ k\in\operatorname{supp}(\varphi)}}^{2^{n+1}-1} z_k = \frac{1}{2^n}\sum_{\substack{k=2^n\\ k\in\operatorname{supp}(\varphi)}}^{2^{n+1}-1} k \geq \frac{1}{2^n}2^n\beta_n = \beta_n,$$

what implies $\beta < \infty$.

Conversely, if $\beta < \infty$ then, due to $k < 2^{n+1}$, one has

$$\frac{1}{2^n}\sum_{k=2^n}^{2^{n+1}-1} z_k = \frac{1}{2^n}\sum_{\substack{k=2^n\\ k\in\operatorname{supp}(\varphi)}}^{2^{n+1}-1} k < \frac{1}{2^n}2^{n+1}\beta_n = 2\beta_n \leq 2\beta,$$

such that $\varrho_z < \infty$. From $(\equiv)$ we get $z \in ces_\infty$ and, Theorem 4(3) implies then that φ is finite.

8. The Cesàro space ces_0 and its finite elements

$$\boldsymbol{ces_0 := sol\left(C^{-1}(c_0)\right) = \{x \in \mathbb{R}^{\mathbb{N}} \colon C|x| \in c_0\}}$$

with the **norm** $\|x\|_{ces_0} := \|C\,|x|\|_\infty = \sup_n \left(\frac{1}{n}\sum_{k=1}^{n} |x_k|\right)$.

Properties of $\left(ces_0, \|\cdot\|_{ces_0}\right)$:

- it is an atomic Dedekind complete Banach lattice,
- c_0 is a proper solid subspace of ces_0, e.g. the sequence (x_n), where $x_n = 1$ if $n = k^2$ and $x_n = 0$ otherwise, belongs to ces_0 but not to c_0,
- $\ell_\infty \not\subset ces_0$, what is easily seen from $\mathbf{1} \in \ell_\infty$, but $C\mathbf{1} \notin c_0$,
- $ces_0 \not\subset \ell_\infty$, since y belongs to $ces_0 \setminus \ell_\infty$, where $y_n = k$, if $n = k^3$ and $y_n = 0$ otherwise,
- not an AM-space, although c_0 is an AM-space with order continuous norm,
- the norm is order continuous.

The finite elements in ces_0 are characterized next (the same as for ces_p).

Theorem 7.

(1) $\Phi_3(ces_0) = \Phi_2(ces_0) = \Phi_1(ces_0) = c_{00}$,

(2) *the space ces_0 has no order unit.*

9. The Cesàro sums and their duals

Let $\mathfrak{X}$ be a sequence $(X_n, \|\cdot\|_n)_{n\in\mathbb{N}}$ of **Banach spaces** and $p \in \{0\} \cup [1,\infty]$. Define the *$p$-Cesàro sums* and *$d_p$-sums* of $\mathfrak{X} = (X_n)_{n\in\mathbb{N}}$ as follows

$$ces_p(\mathfrak{X}) := \{x = (x_n)_{n\in\mathbb{N}} \colon x_n \in X_n, (\|x_n\|_n)_{n\in\mathbb{N}} \in ces_p\}$$
$$d_p(\mathfrak{X}) := \{x = (x_n)_{n\in\mathbb{N}} \colon x_n \in X_n, (\|x_n\|_n)_{n\in\mathbb{N}} \in d_p\}.$$

Further on we simply write $\|\cdot\|$ instead of $\|\cdot\|_n$ and $\mathbf{0}$ for the zero vector in X_n for each $n \in \mathbb{N}$. Under the coordinatewise algebraic operations and the norms

$$|||x|||_{ces_p(\mathfrak{X})} = \left\|(\|x_n\|)_{n\in\mathbb{N}}\right\|_{ces_p}, \quad |||y|||_{d_p(\mathfrak{X})} = \left\|(\|y_n\|)_{n\in\mathbb{N}}\right\|_{d_p}$$

the spaces $ces_p(\mathfrak{X})$ and $d_p(\mathfrak{X})$ are Banach spaces.

Let now all X_n be **Banach lattices**. Then with coordinatewise order the spaces $ces_p(\mathfrak{X})$ and $d_p(\mathfrak{X})$ are also Banach lattices.

Since $ces_1 = \{\mathbf{0}\}$ also $ces_1(\mathfrak{X})$ is trivial.

For $p \in \{0\} \cup (1,\infty]$ define the map $J_j : X_j \to ces_p(\mathfrak{X})$ by

$$J_j x = (x_n)_{n\in\mathbb{N}} = (\mathbf{0}, \ldots, \mathbf{0}, \underbrace{x}_{j^{th}\ term}, \mathbf{0}, \ldots) = \begin{cases} \mathbf{0}, & n \neq j \\ x, & n = j \end{cases}$$

for $x \in X_j$ and $j \in \mathbb{N}$.

Properties of the map J_j:

- a lattice isomorphism from X_j to $ces_p(\mathfrak{X})$,
- $J_j X_j$ is a projection band in $ces_p(\mathfrak{X})$ such that
- $\Phi_1(J_j X_j) = J_j \Phi_1(X_j)$,
- for the band projection $P_j : ces_p(\mathfrak{X}) \to J_j X_j$ one has

$$P_j\big(\Phi_1\big(ces_p(\mathfrak{X})\big)\big) = \Phi_1(J_j X_j) = J_j \Phi_1(X_j).$$

Theorem 8.

Let $\mathfrak{X} = (X_n)_{n\in\mathbb{N}}$ be a sequence of Banach lattices, $\mathfrak{X}' = (X'_n)_{n\in\mathbb{N}}$ the sequence of their dual spaces and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.
Then the mapping $y' = (y'_n)_{n\in\mathbb{N}} \mapsto f_{y'}$ from $d_q(\mathfrak{X}')$ to $ces'_p(\mathfrak{X})$, defined by

$$f_{y'}(x) := \sum_{n=1}^{\infty} \langle y'_n, x_n \rangle, \quad x = (x_n)_{n\in\mathbb{N}} \in ces_p(\mathfrak{X})$$

is a vector lattice isomorphism from $d_q(\mathfrak{X}')$ onto $ces'_p(\mathfrak{X})$ and satisfies for all $y' \in d_q(\mathfrak{X}')$ the relations

$$\frac{1}{q} |||y'|||_{d_q(\mathfrak{X}')} \le \|f_{y'}\| \le (p-1)^{1/p}\, |||y'|||_{d_q(\mathfrak{X}')}.$$

Similarly, we have $ces'_0(\mathfrak{X}) = d_1(\mathfrak{X}')$ with equality of the norms, i.e. $\|f_{y'}\| = |||y'|||_{d_1(\mathfrak{X}')}$.

10. Finite elements in Cesàro sums

As the characterization of the finite elements in the Banach lattices $\boldsymbol{ces}_p(\mathfrak{X})$ for $p \in \{0\} \cup (1, \infty]$, we get a quite direct generalization of the results for the classical cases $X_n = c_0$, ℓ_p and ℓ_∞.

Theorem 9.

The following statements hold:

(1) *For $p = 0$ and $1 < p < \infty$ the element $\varphi_= (\varphi_n)_{n\in\mathbb{N}}$ is finite in $\boldsymbol{ces}_p(\mathfrak{X})$ if and only if $\varphi_n \in \Phi_1(X_n)$ for all $n \in \mathbb{N}$ and $\varphi_n = \mathbf{0}$ for all but finite many $n \in \mathbb{N}$.*

(2) *The element $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ is finite in $\boldsymbol{ces}_\infty(\mathfrak{X})$ if and only if there exist $w_n \in X_n^+$ such that $(n\,\|w_n\|)_{n\in\mathbb{N}} \in \boldsymbol{ces}_\infty$ and*

$$B_{\{\varphi_n\}^{dd}} \subset [-w_n,\ w_n]\,.$$

In particular, $\varphi_n \in \Phi_1(X_n)$ for all $n \in \mathbb{N}$.

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Grazie per la vostra attenzione Obrigado pela sua atenção

Dinlediğiniz için teşekkür ederim ! Dziękuję za Państwa uwagę

⋆⋆ Thank you for your attention !! ⋆⋆

Спасибо за Ваше внимание ! Danke für Ihre Aufmerksamkeit !

Hvala na pazhnji Дякую за Вашу увагу *ϵυχαϱιστω*