Order and metric structures on cones

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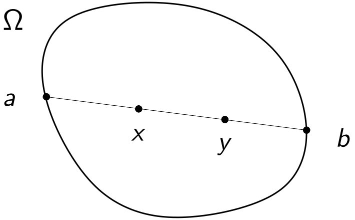
Hilbert's metric

In a letter to Klein in 1894, Hilbert generalised Klein's model of hyperbolic space.



Definition

Hilbert metric

$$d_F(x, y) := \frac{1}{2} \log \frac{|ay||bx|}{|ax||by|}.$$

If Ω is a disk, then Ω is isometric to the hyperbolic plane.

Order unit spaces

Let V be a real vector space.

Definition Cone C in V:

$$
\begin{aligned}
x + y \in C &\qquad \text{for } x, y \in C \\
\lambda x \in C &\qquad \text{for } x \in C, \quad \lambda \geq 0 \\
C \cap -C = \{0\}.
\end{aligned}
$$

Partial order on V: $x \leq y$ if $y - x \in C$.
$(V, \leq)$ is an ordered vector space.

Definition V is Archimedean if

$$(x \in V, \quad y \in C, \quad nx \leq y \text{ for all } n \in \mathbb{N}) \implies x \leq 0.$$

Definition An order unit: $u \in C$ such that, for each $x \in V$ there is a $\lambda > 0$ such that $x \leq \lambda u$.

Definition (V, C, u) is an order-unit space.

The order-unit norm

Definition The order-unit norm:

$$||x||_u := \inf\{\lambda > 0 \mid -\lambda u \leq x \leq \lambda u\}, \qquad \text{for all } x \in V.$$

Give V the topopogy induced by this norm. Then, C is closed with non-empty interior.

Proposition

$$\{\text{order units}\} = \text{int } C$$

Assumption The order-unit space V is complete with respect to the order-unit norm.

Bonsall's $M(\cdot, \cdot)$ function

Definition

$$M(x, y) := \inf\{\lambda > 0 \mid x \leq \lambda y\}, \qquad \text{for } x, y \in \operatorname{int} C.$$

Proposition

$$M(x, y) = \sup_{z \in C^*} \frac{\langle z, x \rangle}{\langle z, y \rangle}$$

Example

For the positive cone $\mathbb{R}^n_+$,

$$M(x, y) = \max_i \frac{x_i}{y_i}.$$

The Hilbert pseudo-metric on the cone

The Hilbert (pseudo-)metric is defined to be, for $x, y \in \operatorname{int} C$,

$$\tilde{d}_H(x, y) := \frac{1}{2} \log M(x, y)M(y, x).$$

Hilbert's metric satisfies

- (positivity) $\tilde{d}_H(x, y) \geq 0$;
- (pseudo-definiteness) $\tilde{d}_H(x, y) = 0$ iff $x = \lambda y$ for some $\lambda > 0$;
- (symmetry) $\tilde{d}_H(x, y) = \tilde{d}_H(y, x)$;
- (triangle inequality) $\tilde{d}_H(x, z) \leq \tilde{d}_H(x, y) + \tilde{d}_H(y, z)$.

Proposition For x and y in a cross-section Ω of the cone,

$$d_H(x, y) = \tilde{d}_H(x, y).$$

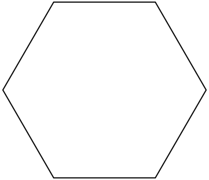
The case of simplices

Proposition (Nussbaum, de la Harpe)

Ω is an n-simplex $\Longrightarrow$ (Ω, d_H) is isometric to a normed space

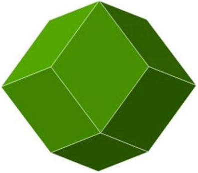
The unit ball of the normed space:

dim $= 2$



hexagon

dim $= 3$



rhombic dodecahedron

Theorem (Foertsch–Karlsson)

Ω is an n-simplex $\Longleftrightarrow$ (Ω, d_H) is isometric to a finite-dimensional normed space

Infinite-dimensional "simplices"

Definition

- $\mathcal{C}(K)$ the continuous functions on a compact Hausdorff space K;
- $\mathcal{C}^+(K)$ the positive continuous real-valued functions on K;
- $\operatorname{cl}\mathcal{C}^+(K)$ the non-negative continuous functions on K;
- u the function on K that is identically 1.

$(\mathcal{C}(K), \operatorname{cl}\mathcal{C}^+(K), u)$ is an order-unit space.

$$d_H(x,y) = \frac{1}{2}\log \sup_{j,k\in K} \frac{x(j)\,y(k)}{y(j)\,x(k)}, \qquad \text{for } x,y \in \mathcal{C}^+(K).$$

Theorem (W)

A Hilbert geometry on a cone C is isometric to a Banach space $\iff$ C is linearly isomorphic to $\operatorname{cl}\mathcal{C}^+(K)$, for some compact Hausdorff space K.

Isometries of the Hilbert metric

Let (X, d) be a metric space.

A map $\phi \colon X \to X$ is an isometry if ϕ is a bijection and

$$d\big(\phi(x), \phi(y)\big) = d(x, y).$$

Order isomorphisms and antimorphisms

Let (V, C, u) be an order-unit space, and let $\phi\colon \operatorname{int} C \to \operatorname{int} C$.

ϕ is an order isomorphism if ϕ is a bijection and

$$x \leq y \iff \phi(x) \leq \phi(y).$$

ϕ is an order antimorphism if ϕ is a bijection and

$$x \leq y \iff \phi(x) \geq \phi(y).$$

ϕ is homogeneous of degree α if

$$\phi(\lambda x) = \lambda^{\alpha}\phi(x), \qquad \text{for all } x \in \operatorname{int} C \text{ and } \lambda > 0.$$

Proposition

- *If ϕ is an order isomorphism and homogeneous of degree 1, then ϕ is an isometry on $P(C)$.*
- *If ϕ is an order antimorphism and homogeneous of degree -1, then ϕ is an isometry on $P(C)$.*

Hilbert isometries and order iso/anti-morphisms

Theorem (W)

In finite dimension, every isometry of the Hilbert metric arises as the projective action of either

- *an order isomorphism that is homogeneous of degree* 1;
- *or, an order antimorphism that is homogeneous of degree* -1.

Theorem (Noll–Schaffer)

If $\phi\colon \operatorname{int} C \to \operatorname{int} C$ *is an order isomorphism and homogeneous of degree* 1*, then* ϕ *is the restiction of a linear map.*

What about order antimorphisms?

Examples

R^n_+ $\qquad (\phi x)_i := 1/x_i, \qquad$ for all $i \in \{1, \ldots, n\}$.

$\mathrm{Pos}(n, \mathbb{C})$ $\qquad \phi A := A^{-1}$.

Symmetric cones

Let C be a closed cone in a finite-dimensional Hilbert space H.

The dual cone is

$$C^* := \{y \in H \mid \langle y, x\rangle \geq 0 \text{ for all } x \in C\}$$

The automorphism group of C is

$$\text{Aut}(C) := \{\phi \in \text{GL}(H) \mid \phi(C) = C\}$$

Definition: C is symmetric if

1. (Homogeneous) For all $x, y \in \text{int}\, C$, there exists $\phi \in \text{Aut}(C)$ such that $\phi(x) = y$
2. (Self dual) $C^* = C$

Classification of symmetric cones in finite dimension

Theorem (Jordan–von Neumann–Wigner)

In finite dimension, every symmetric cone is the direct sum of the simple ones, of which there are five types:

1. $\mathrm{Pos}(n, \mathbb{F})$, *the* $n \times n$ *positive definite Hermitian matrices, over field* $\mathbb{F}$ *equal to* $\mathbb{R}$, $\mathbb{C}$, *or* $\mathbb{H}$, *with* $n \geq 3$;
2. L_n, *the n-dimensional Lorentz cone, with* $n \geq 2$;
$$\left\{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : ||x||_2 \leq t\right\}.$$
3. $\mathrm{Pos}(3, \mathbb{O})$, *where* $\mathbb{O}$ *are the octonians.*

Order antimorphisms in finite dimension

Theorem (W)

*Let (V, C, u) be a finite-dimensional order unit space.
Then, there exists an order-antimorphism $\phi\colon \operatorname{int} C \to \operatorname{int} C$ that is homogeneous of degree -1 if and only if C is a symmetric cone.*

Corollary (W)

Let $\Omega := P(C)$ be a finite-dimensional Hilbert geometry.

- *If C is a symmetric cone and not a Lorentz cone, then $\operatorname{Isom}(\Omega)$ is a subgroup of order two in $\operatorname{Proj}(\Omega)$;*
- *otherwise,*

$$\operatorname{Isom}(\Omega) = \operatorname{Proj}(\Omega).$$

Notation $\operatorname{Proj}(X)$ denotes the projective linear maps such that $\phi(X) = X$.

JB-algebras

A (real) Jordan algebra is a real linear space J with a bilinear product $a \bullet b \in J$ satisfying

1. Commutivity: $a \bullet b = b \bullet a$
2. Jordan identity: $a^2 \bullet (a \bullet b) = a \bullet (a^2 \bullet b)$

for all a and b in J.

A JB-algebra is a real Jordan algebra J with a norm $||\cdot||$ making it a Banach space satisfying

1. $||a \bullet b|| \leq ||a|| \, ||b||$
2. $||a^2|| = ||a||^2$
3. $||a^2 + b^2|| \geq ||a^2||$

Let J be a JB-algebra with algebraic unit e.
Define the positive cone $C := \{a^2 \mid a \in J\}$.
Then, (J, C, e) is an order unit space.

Conjecture

Conjecture (Lemmens–Roelands–van Imhoff)

Let (V, C, u) be a complete order-unit space. Then, there exists an anti-homogeneous order-antimorphism $\phi\colon \operatorname{int} C \to \operatorname{int} C$ if and only if V is a JB-algebra with unit u, positive cone C, and norm $||\cdot||_u$.

Theorem (Lemmens–Roelands–van Imhoff)

Let (V, C, u) be a complete order unit space with a strictly convex cone C. Then, there exists an anti-homogeneous order-antimorphism $\phi\colon \operatorname{int} C \to \operatorname{int} C$ if and only if C is a spin factor.

Another result

Theorem (Roelands–Wortel)

Let C be the cone of a unital JB-algebra.

- *If the JB-algebra is not a spin factor, then* $\mathrm{Isom}(C)$ *is a subgroup of index two in* $\mathrm{Proj}(C)$*;*
- *if the JB-algebra is a spin factor, then* $\mathrm{Isom}(X) = \mathrm{Proj}(C)$.

Is the assumption of anti-homogeneity really necessary?

Theorem (W)

Let (V, C, u) be a finite-dimensional order unit space.
Then, there exists an order-antimorphism $\phi\colon \operatorname{int} C \to \operatorname{int} C$ if and only if C is a symmetric cone.

A sharper version of the Lemmens–Roelands–van Imhoff conjecture:

Conjecture

Let (V, C, u) be a complete order-unit space. Then, there exists an order-antimorphism $\phi\colon \operatorname{int} C \to \operatorname{int} C$ if and only if V is a JB-algebra with unit u, positive cone C, and norm $||\cdot||_u$.