Algebraic structures in pre-Riesz spaces

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Motivation (Lattice ordered algebras)

Let A be an l-algebra, i.e., a vector lattice with associative multiplication $\cdot$ such that, for all $a, b \in A_+$, we have $a \cdot b \in A_+$.

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d-algebra

$$\forall a, b, c \in A_+ :$$
$$c \cdot (a \vee b) = c \cdot a \vee c \cdot b,$$
$$(a \vee b) \cdot c = a \cdot c \vee b \cdot c$$

almost f-algebra

$$\forall a, b \in A_+ :$$
$$a \perp b \Rightarrow a \cdot b = 0$$

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Let K be a non-empty compact Hausdorff space and $A = \mathrm{C}(K)$.

f-algebra[1]
$$\Leftrightarrow (f \cdot g)(t) = w(t)f(t)g(t)$$
$$(w \in \mathrm{C}(K)_+)$$

$\swarrow$ $\searrow$

d-algebra[2]
$$\Leftrightarrow (f \cdot g)(t) = w(t)f(\alpha_1(t))g(\alpha_2(t))$$
$$(w \in \mathrm{C}(K)_+, \alpha_i \colon K \to K)$$

almost f-algebra[3]
$$\Leftrightarrow (f \cdot g)(t) = \int_K fg \, d\mu_t$$
$((\mu_t)_{t\in K}$ positive measures$)$

[1](Conrad, 1974)
[2](Boulabiar, 2004)
[3](Scheffold, 1981)

Generalizations of Riesz homomorphisms

Definition.

Let X, Y be directed povs. A linear map $T\colon X \to Y$ is called

(a) (van Haandel, 1993) a *Riesz* homomorphism* if, for every non-empty finite subset F of X, one has

$$T[F^{\mathrm{ul}}] \subseteq T[F]^{\mathrm{ul}},$$

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(b) (Buskes-van Rooij, 1993) a *Riesz homomorphism* if, for every $x, y \in X$, one has

$$T[\{x,y\}^{\mathrm{u}}]^{\mathrm{l}} = T[\{x,y\}]^{\mathrm{ul}}.$$

Pre-Riesz spaces

Definition/Theorem (van Haandel, 1993).

Let X be a povs. The following statements are equivalent:

(i) X is a pre-Riesz space.

(ii) There exist a Riesz space Y and a bipositive linear map $i\colon X \to Y$ such that $i[X]$ is order dense in Y.

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Moreover, all Riesz spaces Y as in (iii) are isomorphic as Riesz spaces.

We call a pair (Y, i) as in (ii) a *vector lattice cover of* X and as in (iii) the *Riesz completion of* X and denote it by (X^ρ, i).

Extension of Riesz* homomorphisms

Theorem (van Haandel, 1993).

Let X_1 and X_2 be pre-Riesz spaces with Riesz completions (X_1^ρ, i_1) and (X_2^ρ, i_2), respectively. Let $T\colon X_1 \to X_2$ be a linear map. The following statements are equivalent:

(i) T is a Riesz* homomorphism.

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(i) T is a Riesz* homomorphism.

(ii) There exists a Riesz homomorphism $S\colon X_1^\rho \to X_2^\rho$ satisfying $S \circ i_1 = i_2 \circ T$.

Moreover, if (i) is satisfied, then the Riesz homomorphism S in (ii) is unique.

Riesz* homomorphisms on spaces of continuous functions

Theorem (van Imhoff, 2018).

Let P and Q be nonempty compact Hausdorff spaces and let X and Y be order dense subspaces of $\mathrm{C}(P)$ and $\mathrm{C}(Q)$, respectively. Let $T\colon X \to Y$ be linear. Then, under some mild conditions on X, the following statements are equivalent:

(i) T is a Riesz* homomorphism

(ii) There exist $w \in \mathrm{C}(Q)$, $w \geq 0$, and $\alpha\colon Q \to P$ continuous on $\{q \in Q;\ w(q) > 0\}$ such that

$$T(x)(q) = w(q)x(\alpha(q)) \quad (x \in X).$$

Order unit spaces

Definition.

Let X be a povs.

(a) An element $u \in X$ is called *order unit* if, for every $x \in X$, there is $\lambda \in (0, \infty)$ such that $\pm x \leq \lambda u$.

(b) If X is, in addition, Archimedean, then we can define a norm $\|x\|_u := \inf\{\lambda \in (0, \infty); -\lambda u \leq x \leq \lambda u\}$ $(x \in X)$ on X.

(c) If X is an Archimedean povs with order unit, then we call X an order unit space.

Note: Every order unit space is pre-Riesz.

Functional representation

We outline the construction of (Kadison, 1951).
Let X be an order unit space with order unit u. Define the weakly-$*$ compact convex set

$$\Sigma := \{\varphi \in X'; \varphi \text{ positive }, \varphi(u) = 1\}$$

and define Λ as the set of extreme points of Σ. The weak-$*$ closure $\overline{\Lambda}$ of Λ is a compact Hausdorff space (with the weak-$*$ topology) and the map

$$\Phi\colon X \to \mathrm{C}(\overline{\Lambda}), \quad x \mapsto (\varphi \mapsto \varphi(x)),$$

is linear and bipositive.

Functional representation

Theorem (Kalauch, Lemmens, van Gaans, 2014).
Let X be an order unit space. Then $(\mathrm{C}(\overline{\Lambda}), \Phi)$ is a vector lattice cover of X.

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Proposition.

Let X be an order unit space and let
$\varphi \in \Sigma = \{\varphi \in X'; \varphi \text{ positive }, \varphi(u) = 1\}$.

(a) (Hayes, 1966) $\varphi \in \Lambda$ if and only if φ is Riesz homomorphism.

(b) (van Haandel, 1993) $\varphi \in \overline{\Lambda}$ if and only if φ is Riesz* homomorphism.

Riesz* bi-morphisms

Recall: An l-algebra A is a d-algebra if, for all $a, b, c \in A_+$, we have $c \cdot (a \vee b) = c \cdot a \vee c \cdot b$ and $(a \vee b) \cdot c = a \cdot c \vee b \cdot c$.

In other words, for each $c \in A_+$, the maps $x \mapsto c \cdot x$ and $x \mapsto x \cdot c$ are Riesz homomorphisms.

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Definition.
Let X_1, X_2, Y be povs and $T\colon X_1 \times X_2 \to Y$ be bilinear. Then T is called a Riesz* bi-morphism (resp. Riesz bi-morphism) if, for all positive $x_1 \in X_1$, $x_2 \in X_2$, the linear operators $T(\cdot, x_2)$ and $T(x_1, \cdot)$ are Riesz* homomorphisms (resp. Riesz homomorphisms).

Extension of Riesz bi-morphisms

Recall: Riesz* homomorphisms between pre-Riesz spaces are exactly the operators that can be extended to a Riesz homomorphism between the Riesz completions.

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Question.

Do we have a similar extension result for Riesz* bi-morphisms on pre-Riesz spaces?

Theorem (Kalauch-Kusraeva, 2022).

Let X_1, X_2, Y be Archimedean pre-Riesz spaces with Riesz completions $X_1^\rho, X_2^\rho, Y^\rho$, resp., and $T\colon X_1 \times X_2 \to Y$ a Riesz bi-morphism. Then T has a Riesz bi-morphism extension $T^\rho\colon X_1^\rho \times X_2^\rho \to Y^\rho$.

Localizing on principal ideals

Let X be a povs. For $x \in X$, $x > 0$, we denote by I_x the principal ideal generated by x, i.e.,

$$I_x := \{y \in X;\ \exists \lambda \in \mathbb{R}:\ \pm y \le \lambda x\}$$

and let $j_x \colon I_x \to X$, $j_x = \mathrm{id}|_{I_x}$. Note that x is an order unit in I_x.

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We say that X has property (P) if the set

$$P_X := \{x \in X;\ x > 0,\ j_x \text{ is a Riesz* homomorphism}\}$$

is majorizing in X.

Localizing on principal ideals

Observation.
Let X, Y be pre-Riesz spaces and let $T\colon X \to Y$ be a Riesz* homomorphism. Note that, for all $x \in P_X, y \geq T(x)$, the restriction $T|_{I_x}\colon I_x \to I_y$ is a Riesz* homomorphism ($T|_{I_x} = T \circ j_x$). This is, in general, not true for restrictions to any kind of subspace!

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Theorem (S., 2022).

If X has (P) and for all $x \in P_X, y \geq T(x)$, the restriction $T|_{I_x}\colon I_x \to I_y$ is a Riesz* homomorphism, then T is a Riesz* homomorphism.

Localizing on principal ideals

Definition.
A pre-Riesz space X with Riesz completion (X^ρ, i) is called *pervasive* if, for every $y \in X^\rho$, $y > 0$, there exists $x \in X$ such that $0 < i(x) \leq y$.

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Examples.

- Every order unit space with o.u. u satisfies (P) as $u \in P_X$.
- Every Archimedean pervasive pre-Riesz space satisfies (P) ($P_X = X_+ \setminus \{0\}$).
- Every pre-Riesz space with RDP satisfies (P) ($P_X = X_+ \setminus \{0\}$).

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- Every pre-Riesz space with RDP satisfies (P) ($P_X = X_+ \setminus \{0\}$).

There is no known example of a pre-Riesz space which does not satisfy property (P) yet!

Extension of Riesz* bi-morphisms

Theorem (S. 2022).

Let X_1, X_2, Y be Archimedean pre-Riesz spaces with Riesz completions $X_1^\rho, X_2^\rho, Y^\rho$, resp., and $T\colon X_1 \times X_2 \to Y$ a Riesz* bi-morphism.

(1) If $Y = \mathbb{R}$, then T has a Riesz bi-morphism extension $T^\rho\colon X_1^\rho \times X_2^\rho \to \mathbb{R}$.

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$\Downarrow$ (Functional representation)

(2) If X_1, X_2, Y are order unit spaces, then T has a Riesz bi-morphism extension $T^\rho\colon X_1^\rho \times X_2^\rho \to Y^\rho$.

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$\Downarrow$ (Localizing on principal ideals)

(3) If X_1, X_2 satisfy (P), then T has a Riesz bi-morphism extension $T^\rho\colon X_1^\rho \times X_2^\rho \to Y^\rho$.

Lattice ordered algebras

Let A be an l-algebra, i.e., a vector lattice with associative multiplication $\cdot$ such that, for all $a, b \in A_+$, we have $a \cdot b \in A_+$.

f-algebra

$$\forall a, b, c \in A_+ : a \perp b \Rightarrow a \cdot c \perp b \text{ and } c \cdot a \perp b$$

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d-algebra

$$\forall a, b, c \in A_+ :$$
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$\forall a, b \in A_+ :$

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Disjointness in povs

Let X be a vector lattice. Two elements $x_1, x_2 \in X$ are called *disjoint*, denoted by $x_1 \perp x_2$, if $|x_1| \wedge |x_2| = 0$. Moreover,

$$x_1 \perp x_2 \Leftrightarrow |x_1 + x_2| = |x_1 - x_2| \quad (x_1, x_2 \in X).$$

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Definition (Kalauch, van Gaans, 2006).

Let X be a povs. Two elements $x_1, x_2 \in X$ are called *disjoint*, denoted by $x_1 \perp x_2$, if

$$\{x_1 + x_2, -x_1 - x_2\}^u = \{x_1 - x_2, -x_1 + x_2\}^u.$$

Disjointness in povs

Proposition (Kalauch, van Gaans, 2006).

Let X be a pre-Riesz space with Riesz completion (X^ρ, i). For all $x_1, x_2 \in X$, we have

$$x_1 \perp x_2 \text{ in } X \Leftrightarrow i(x_1) \perp i(x_2) \text{ in } X^\rho.$$

Example

The proposition allows us easily characterize disjointness in an order dense subspace X of $\mathrm{C}(K)$. For all $x_1, x_2 \in X$, we have

$$x_1 \perp x_2 \text{ in } X \Leftrightarrow x_1(t)x_2(t) = 0 \quad \forall t \in K.$$

o-algebras

Let X be an ordered algebra (o-algebra), i.e., a partially ordered vector space (povs) with associative multiplication $\cdot$ such that, for all $a, b \in X_+$, we have $a \cdot b \in X_+$.

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o-d-algebra

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o-d^*-algebra

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almost o-f-algebra

$\forall a, b \in A_+ :$

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Note on disjointness

Being an o-f-algebra is a weaker condition in povs than in vector lattices as there might be very few or even no non-trivial pairs of positive disjoint elements.

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Example.

Let $X = \mathrm{P}([0,1])$ be the space of all polynomials on $[0,1]$. Then there are no non-trivial disjoint pairs of elements. Hence, any o-algebra multiplication is already an o-f-algebra multiplication. E.g., consider the multiplication $\odot$ on X given by

$$(p_1 \odot p_2)(t) = \int_0^t p_1(s)p_2(s)\mathrm{d}s$$

for $p_1, p_2 \in X$ and $t \in [0,1]$. Then $\odot$ is an o-f-algebra multiplication by the above, but not an o-d-algebra multiplication.

Definition.
Let K be a non-empty compact Hausdorff space and X a subspace of $\mathrm{C}(K)$. X has the property (SD) ('separation by disjoint elements') if, for every $t_1, t_2 \in K$, $t_1 \neq t_2$, there exist $x_1, x_2 \in X_+$, $x_1 \perp x_2$, such that $x_1(t_1) > 0$ and $x_2(t_2) > 0$.

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Example.

(a) $\mathrm{C}^k([0,1])$ has (SD).

(b) The Namioka space $N := \{x \in \mathrm{C}([-1,1]); 2x(0) = x(-1) + x(1)\}$ does not have (SD).

Representations on subspaces of $\mathrm{C}(K)$

Let K be a non-empty compact Hausdorff space and X an order dense subspace of $\mathrm{C}(K)$.

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$$\overset{\text{(SD)}}{\Leftrightarrow} (f \cdot g)(t) = w(t)f(t)g(t)$$
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d-algebra

$$\Leftrightarrow (f \cdot g)(t) = w(t)f(\alpha_1(t))g(\alpha_2(t))$$
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$$\overset{\text{(SD)}}{\Leftrightarrow} (f \cdot g)(t) = \int_K fg \, d\mu_t$$
$$((\mu_t)_{t \in K} \text{ positive measures})$$

Let X be an o-algebra with multiplication $\cdot$.

o-f-algebra

$\forall a, b, c \in A_+ : a \perp b \Rightarrow a \cdot c \perp b$ and $c \cdot a \perp b$

$\swarrow$ (SD) $\searrow$

o-d-algebra

$\cdot$ is a Riesz bi-morphism

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Markov operators

Let X, Y be o-algebras with units e_X, e_Y, resp. Define

$$\mathcal{M}(X,Y) := \{T\colon X \to Y;\ T \geq 0,\ T(e_X) = e_Y\}.$$

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Theorem (van Putten, 1980).

Let A, B be Archimedean f-algebras with unit elements, and let $T \in \mathcal{M}(A, B)$. Then the following are equivalent:

(i) T is an extreme point in $\mathcal{M}(A, B)$.

(ii) T is an algebra homomorphism.

(iii) T is a Riesz homomorphism.

Theorem (S., 2022).

Let X, Y be order dense subalgebras of $\mathrm{C}(K_1)$ and $\mathrm{C}(K_2)$ equipped with any f-algebra multiplication and units $e_1, e_2 > 0$, respectively, such that $e_1 \in X$ and $e_2 \in Y$ and X satisfies (SD).

Let $T \in \mathcal{M}(X, Y)$. Then the following are equivalent:

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o-d-algebras as order dense subalgebras of d-algebras

Theorem.
If X is

(a) an Archimedean pre-Riesz o-d-algebra, or

(b) an Archimedean pre-Riesz o-d^*-algebra with (P),

then the multiplication in X can be extended to a d-algebra multiplication in the Riesz completion X^ρ of X, i.e., X can be seen as an order dense subalgebra of X^ρ.

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Conversely, every order dense subalgebra of a d-algebra is a pre-Riesz o-d^*-algebra.

Basic results in d-algebras

Let A be an Archimedean d-algebra. Denote the set of nilpotent elements by $N_A := \{a \in A;\ \exists n \in \mathbb{N}\colon a^n = 0\}$. A is called semiprime if $N_A = \{0\}$.

Proposition (cf. Bernau-Huijsmans, 1990).

(a) The set of the nilpotent elements is given by $N_A = \{a \in A; a^3 = 0\}$. N_A is an order ideal and a ring ideal.

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(b) If A has a unit $e > 0$, then e is a weak order unit in A. Moreover, in this case, A is already a semiprime f-algebra.

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(b) If A has a unit $e > 0$, then e is a weak order unit in A. Moreover, in this case, A is already a semiprime f-algebra.

(c) If A is semiprime, then A is an f-algebra.

Basic results in o-d-algebras

Let X be an Archimedean pre-Riesz o-d-algebra with Riesz completion (X^ρ, i).

Proposition (S., 2022).

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(c) If X is, in addition, pervasive, then X is semiprime if and only if X^ρ is semiprime. Hence, in this case, X^ρ becomes a semiprime f-algebra and, therefore X is an o-f-algebra.

o-algebras

Let X be an o-algebra with multiplication $\cdot$.

o-f-algebra

$\forall a, b, c \in A_+ : a \perp b \Rightarrow a \cdot c \perp b$ and $c \cdot a \perp b$

$\swarrow$ (SD) $\qquad\qquad\qquad\qquad \searrow$

o-d-algebra
$\cdot$ is a Riesz bi-morphism

$\downarrow$

o-d^*-algebra
$\cdot$ is a Riesz* bi-morphism

almost o-f-algebra
$\forall a, b \in A_+ :$
$a \perp b \Rightarrow a \cdot b = 0$

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Thank you a lot for your attention.