Embeddability of positive matrices

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Problem

X suitable Banach space, $T \in \mathcal{L}(X)$

Question

Is there a

$$\left.\begin{array}{ll} 1) & - \\ 2) & \textit{Markov} \\ 3) & \textit{real} \\ 4) & \textit{positive} \end{array}\right\} C_0\text{-semigroup } (T(t))_{t\geq 0} \text{ such that } T(1) = T?$$

Recall

A C_0-semigroup is a family of bounded linear operators $(T(t))_{t\geq 0}$ on X such that

- $T(0) = \mathrm{Id}$, $T(t+s) = T(t)T(s)$ for all $s, t \geq 0$,
- $[0, \infty) \to X, \quad t \mapsto T(t)x \quad$ is continuous for all $x \in X$.

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1) T. Eisner (2009)
2) G. Elfving (1937), M. Baake and J. Sumner (2020)

Now 3) and 4), where $X \in \{\mathbb{C}^n; c_0; \ell^p, 1 \leq p < \infty\}$

Spectrum of real-embeddable operators

Proposition (Eisner 2009)

$T \in \mathcal{L}(X)$ embeddable $\Rightarrow \dim\ker T = 0$ or $\dim\ker T = \infty$

Remark

$\dim X < \infty$, T embeddable $\Rightarrow$ T invertible

Example

$T = 0 \in \mathcal{L}(L^2[0,1])$ is real-embeddable into nilpotent shift semigroup.
Since $L^2[0,1] \cong_{\text{real}} \ell^2$, $0 \in \mathcal{L}(\ell^2)$ is real-embeddable.

Spectrum of real-embeddable operators

Remark

For each compact $\emptyset \neq K \subseteq \mathbb{C}$ with $K = \overline{K}$ there exists a real-embeddable operator $T_K \in \mathcal{L}(\ell^2)$ such that $\sigma(T_K) = K$.

Proposition (sufficient condition for real-embeddability)

$$T \in \mathcal{L}(X) \text{ real operator, } \sigma(T) \subseteq \mathbb{C} \setminus (-\infty, 0]$$

$$\Downarrow$$

$$T \text{ real-embeddable into norm continuous semigroup}$$

Proof.

Goal: Find $A \in \mathcal{L}(X)$ real such that $e^A = T$.

Idea: Take a suitable path γ (symmetric, not intersecting $(-\infty, 0]$) surrounding $\sigma(T)$. Define

$$A := \log(T) = \frac{1}{2\pi\,\mathrm{i}} \int_\gamma R(\lambda, T) \log(\lambda) \mathrm{d}\lambda = \overline{A}$$

$\Rightarrow A \in \mathcal{L}(X)$ real $\Rightarrow \left(e^{tA}\right)_{t \geq 0}$ real. $\square$

Spectrum of real-embeddable operators

Proposition (Culver 1966)

Let $T \in \mathbb{R}^{d \times d}, 0 \notin \sigma(T)$. Equivalent are:

(i) T real-embeddable.

(ii) T has a real root (i. e. $\exists S \in \mathbb{R}^{d \times d}$ such that $S \cdot S = T$).

(iii) The number of Jordan blocks of the same dimension of a *negative eigenvalue* is even.

Consequence

$T \in \mathbb{R}^{d \times d}$ real-embeddable $\Rightarrow \det(T) > 0$

Embeddability of positive operators into positive s.g.

Proposition ($\dim X = 2$)

Let $0 \leq T \in \mathbb{R}^{2\times 2}$. Equivalent are:

(i) T is embeddable into a positive semigroup.

(ii) T is invertible and there exists a positive root.

(iii) $\sigma(T) \subseteq (0,\infty)$.

(iv) $\det(T) > 0$.

(v) T is real-embeddable.

Remark

$T := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \geq 0$ is

- positively embeddable only into $\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)_{t\geq 0}$.
- real-embeddable also into $\left(\begin{pmatrix} \cos(2\pi t) & \sin(2\pi t) \\ -\sin(2\pi t) & \cos(2\pi t) \end{pmatrix}\right)_{t\geq 0}$.

Embeddability of positive operators into positive s.g.

Proposition ($\dim X = 2$)

Let $0 \leq T \in \mathbb{R}^{2\times 2}$. Equivalent are:

(i) T is embeddable into a positive semigroup.

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(iii) $\sigma(T) \subseteq (0, \infty)$.

(iv) $\det(T) > 0$.

(v) T is real-embeddable.

Example

$$T := \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, \; a, b, c \geq 0. \text{ Then}$$

- $c \geq \frac{ab}{2} \Leftrightarrow T$ positively embeddable.
- $c \geq \frac{ab}{4} \Rightarrow T$ has a positive root. $\Rightarrow T$ is real-embeddable.

Necessary condition for embeddability into positive s.g.

1. in $\mathbb{C}^n$:
   - $\det(T) > 0$
   - $\sigma_{per}(T) := \sigma(T) \cap \{z \in \mathbb{C} : |z| = \mathrm{r}(T)\} = \{\mathrm{r}(T)\}$
2. in $\mathbb{C}^n, c_0, \ell^p, 1 \leq p < \infty$:
   - diagonal entries of T positive (> 0)
   - T embeddable into positive analytic semigroup $\Rightarrow$ T either strictly positive (each entry > 0) or reducible

Idea to show that diagonal entries are > 0

Let $T(t) = (a_{ij}(t))$. If $a_{jj}(t) = 0$ for some $t > 0$, then

$$0 = a_{jj}(t) \overset{T(t)=T\left(\frac{t}{2}\right)T\left(\frac{t}{2}\right)}{=} \sum_{k\in\mathbb{N}} a_{jk}\left(\tfrac{t}{2}\right) a_{kj}\left(\tfrac{t}{2}\right) \geq a_{jj}\left(\tfrac{t}{2}\right) a_{jj}\left(\tfrac{t}{2}\right) \geq 0.$$

By induction $a_{jj}\left(\frac{t}{2^n}\right) = 0$ for all $n \in \mathbb{N}$. We obtain the contradiction

$$0 = \lim_{n\to\infty} a_{jj}\left(\tfrac{t}{2^n}\right) = a_{jj}(0) = 1.$$

More questions

- Sufficient condition for embeddability into positive semigroups?
  Remark: T invertible + existence of positive n-th roots of all orders $n \in \mathbb{N}$ is not enough! (see example by Kingman below)

  Example: Shift by 1 on $\ell^2(\mathbb{Q})$
  - is invertible,
  - is positive,
  - has positive n-th roots (namely shift by $1/n$).

  However, the diagonal entries in its matrix representation on $\ell^2 \cong_{\text{pos.}} \ell^2(\mathbb{Q})$ are all 0.
- Conditions for uniqueness of embedding?

References

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