WOVSPO 2023: Compactification of Symmetric Cones under the Hilbert Metric

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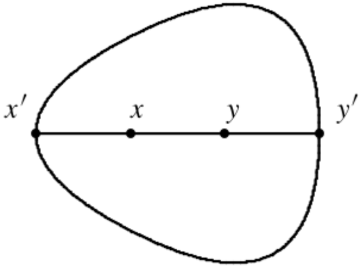
The Hilbert Metric

Consider a bounded convex set:



$$d_H(x,y) = \log\left(\frac{|y'-x|}{|y'-y|}\cdot\frac{|x'-y|}{|x'-x|}\right)$$

Models of Hyperbolic Space

The Lorentz Cone:

$$\Lambda_{n+1} := \{x \in \mathbb{R}^{n+1} : x_1 \geq 0, \text{ and } x_1^2 - \sum_{i=2}^{n+1} x_i^2 \geq 0\}$$

Quadratic Form:

$$\mathbb{R} \times \mathbb{R}^n \ni (\lambda, x) \mapsto Q((\lambda, x)) = \lambda^2 - \langle x, x \rangle \,,$$

so we can actually define the Lorentz cone as:

$$\Lambda_{n+1} := \{(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n : \lambda \geq 0, \text{ and } Q(\lambda, x) \geq 0\}$$

Hyperboloid:

$$\mathbb{H}^n := \{(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n : \lambda > 0, \text{ and } Q(\lambda, x) = 1\}.$$

Models of Hyperbolic Space

Using the associated bilinear form $B : (\mathbb{R} \times \mathbb{R}^n)^2 \to \mathbb{R}$,

$$B((\lambda_1, x_2), (\lambda_2, x_2)) = \lambda_1 \lambda_2 - \langle x_1, x_2 \rangle$$

we can define the length of piecewise C^1 paths:

$$L(\gamma) = \int_a^b \sqrt{-B(\gamma'(t), \gamma'(t))} dt$$

Riemannian Metric:

$$d_{\text{hyp}}(x, y) = \inf_{\gamma} L(\gamma)$$

Relation with Matrices

If we let $\mathrm{Sym}_2(\mathbb{R})$ be the space of positive definite 2×2 matrices, we have a linear bijection between $\mathrm{Sym}_2(\mathbb{R})$ and Λ_3, given by

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \mapsto \left(\frac{a+c}{2}, \frac{a-c}{2}, b \right)$$

We can calculate

$$Q\left(\frac{a+c}{2}, \frac{a-c}{2}, b \right) = \det \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

and

$$\left(\frac{a+c}{2}, \frac{a-c}{2}, b \right) \in B_2^1 \iff \mathrm{tr} \begin{pmatrix} a & b \\ b & c \end{pmatrix} = 2.$$

The Hilbert Metric and M functions

Let (V, V_+, u) be a finite-dimensional order unit space. For $x \in V$ and $y \in V_+$ we say that y *dominates* x if there exists $\alpha, \beta \in \mathbb{R}$ such that $\alpha y \leq x \leq \beta y$. In this case we can define the functions:

$$M(x/y) := \inf\{\beta \in \mathbb{R} : x \leq \beta y\},$$

and

$$m(x/y) := \sup\{\alpha \in \mathbb{R} : \alpha y \leq x\}.$$

If $y \in \mathrm{int}(V_+)$ then y dominates all elements of V. If y dominates x, and x dominates y, we write $y \sim x$, and this is an equivalence relation on V_+. The parts of the equivalence relation correspond with the relative interiors of the faces of V_+.

By Hahn-Banach, we know that $x \leq y$ if and only if $\varphi(x) \leq \varphi(y)$ for all $\varphi \in S(V)$. We can thus write

$$M(x/y) = \sup_{\varphi \in S(V)} \frac{\varphi(x)}{\varphi(y)},$$

and

$$m(x/y) = \inf_{\varphi \in S(V)} \frac{\varphi(x)}{\varphi(y)}.$$

It is also useful to know that for $\alpha, \beta > 0$

$$\log M(\alpha x/\beta y) = \log M(x/y) + \log(\alpha) - \log(\beta),$$

and for $x, y \in \operatorname{int}(V^+)$,

$$\log M(x/y) = \log M(y^{-1}/x^{-1}).$$

We can thus define *Birkhoff's version of the Hilbert distance* on $\mathrm{int}(V_+)$ by

$$d_H(x,y) = \log\left(\frac{M(x/y)}{m(x/y)}\right).$$

We are interested in studying the compactification of *symmetric* Hilbert geometries, where V is an Euclidean Jordan Algebra, and the ordering cone is the cone of squares, and we restrict our attention to the level set of a state, so Birkhoff's version of the Hilbert distance is a proper metric. A motivating example is the space (Ω_V, d_H), where $V = \mathrm{Herm}_n(\mathbb{C})$, and

$$\Omega_V := \{A \in \mathrm{Herm}_n(\mathbb{C}) : \mathrm{tr}(A) = n, \text{ and } A \text{ is positive definite}\}.$$

The Horoboundary Compactification

They are generally attributed to Gromov in his 1978 work on none-positively curved spaces, and were generally involved in the study of geometric group theory.

They have since found application in diverse areas of mathematics, including dynamical systems, Teichmüller theory, complex analysis, noncommutative geometry, and ergodic theory.

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Let (X, d) be a metric space.

The Horoboundary Compactification

Pick a base-point $b \in X$.

$\mathrm{Lip}_b^1(X)$ is a closed and compact subset of $\mathbb{R}^X$ when equipped with the topology of pointwise convergence.

Define, for each $x \in X$, an *internal metric functional* $h_x : X \to \mathbb{R}$ by

$$h_x(y) = d(y,x) - d(b,x).$$

Via the triangle inequality, $h_x \in \mathrm{Lip}_b^1(X)$.

The Horoboundary Compactification

We can identify X within $\mathrm{Lip}_b^1(X)$ via the inclusion map $\iota : X \to \mathrm{Lip}_b^1(X)$, defined by

$$\iota(x) = h_x.$$

ι is a continuous injection.

If X is a proper, geodesic space, then ι is a homeomorphism onto its image.

We call $\bar{X}^h := \overline{\iota(X)}$ the *metric compactification* of X.

We call $\partial\bar{X}^h = \bar{X}^h \backslash \iota(X)$ the *horoboundary of* X, and its elements *horofunctions*.

For proper geodesic spaces this is thus a compactification in the usual topological sense.

Let $X = \ell^1(\mathbb{N})$. Define $(y_n) \subset X$ by $y_n = ne_n$. We can then calculate, for any $x \in X$:

$$h_{y_n}(x) = \sum_{i \neq n} |x_i| + |x_n - n| - n$$
$$\xrightarrow{n} \|x\|_1 = h_0(x)$$

Computing the Horofunctions

In general it's quite difficult to compute the horoboundary, but some things are known:

If X is proper and geodesic, we know that each horofunction is the pointwise limit of a sequence of internal metric functionals, (h_{x_n}), with $d(b, x_n) \to \infty$. For general spaces this is true if we replace sequences with nets.

The horoboundary of certain spaces has been calculated with varying degrees of explicitness:

1. Certain finite and infinite dimensional normed spaces: Gutiérrez, Karlsson, Noskov, Metz, Walsh, Schilling...
2. Certain Teichmüller spaces equipped with the Teichmüller and Thurston metrics: Greenfield, Ji, Walsh...
3. Cones with the projective Hilbert or Thompson metric: Lins, Lemmens, Nussbaum, Walsh, Wortel...
4. Symmetric spaces of non-compact type: Chu, Cueto-Avellaneda, Haettel, Lemmens, Schilling, Walsh, Wienhard...
5. Real infinite dimensional hyperbolic space: Claassen.
6. Schatten p-metrics $(1 < p < \infty)$ on the symmetric cone of Hermitian matrices: Freitas and Friedland.

The Global Topology and Busemann Points

Even when the horoboundary has been identified in some sense, the global topology and geometry is often not well understood.

Rieffel introduced the weaker notion of an *almost-geodesic*: For all $\varepsilon > 0$ and large enough $t \geq s$

$$|d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t| < \varepsilon.$$

Every unbounded almost-geodesic in a proper metric space gives rise to a horofunction. We call such horofunctions *Busemann Points*.

Parts of the Boundary

Akian, Gaubert, and Walsh discovered that we can equip the Busemann points with a possibly infinite valued metric, called the *detour metric*, usually denoted by δ.

This partitions the boundary into parts, where two Busemann points are in the same part if the detour distance between them is finite.

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Question (Kapovich and Leeb)

When does there exist a homeomorphism from the horofunction boundary of a finite-dimensional normed space onto the closed dual unit ball, which maps parts of the boundary onto the relative interior of the faces of the ball?

Finsler Metrics

Finsler manifolds are generalisations of Riemannian manifolds. They consist of a differentiable manifold M, together with a metric $g : TM \to \mathbb{R}^+$, where $g(p, \cdot) = \| \cdot \|_p$ for all $p \in M$.

We can turn such a M into a metric space, by first defining the length L of a C^1 path $\gamma[0, 1] \to M$ by

$$L(\gamma) = \int_0^1 \|\gamma'(t)\|_{\gamma(t)} dt,$$

and then defining a metric $d : M \times M \to \mathbb{R}^+$ by

$$d(x, y) = \inf\{L(\gamma) : \gamma(0) = x,\ \gamma(1) = y.\}$$

What We Know

1. Symmetric spaces with nonpositive sectional curvature are homeomorphic to the Euclidean ball.
2. Ji and Schilling partly answered the question of Kapovich and Leeb, and showed that this is the case for polyhedral norms.
3. Chu, Cueto-Avellaneda, and Lemmens showed that a duality phenomenon also holds for bounded symmetric domains in $\mathbb{C}^n$ with the Caratheodory/Kobayashi distance.
4. P and Lemmens, in recent work, showed that a duality phenomenon exists for certain classes of metric spaces with a Finsler structure, including finite dimensional JB-algebras.
5. In particular we showed this duality phenomenon holds true for symmetric Hilbert Geometries.

Finsler Metric on (Ω_V, d_H)

For any $A \in \mathrm{int}(V_+)$ we can define a semi-norm, $|\cdot|_A$ on V by defining, for any $X \in V$,

$$|X|_A = M(X/A) - m(X/A).$$

This is a genuine norm on the space $V/\mathbb{R}A$, which is the tangent space to Ω_V at A. Furthermore, as I is the order-unit, for every $A \in \Omega_V$ and $B \in V$ we have

$$|B|_A = M(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}/I) - m(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}/I) = |A^{-\frac{1}{2}}BA^{-\frac{1}{2}}|_I,$$

from which we can deduce that the facial structure of the unit ball in the tangent space is identical at all points.

The Dual Ball of the Tangent Space

It turns out that the norm $|\cdot|_I$ on $V/\mathbb{R}I$ is equal to the quotient norm of $2\|\cdot\|_I$, where $\|\cdot\|_I$ is the order-unit norm, which corresponds to the spectral radius norm. As Ω_V is a Euclidean Jordan algebra with the tr inner-product, we can write

$$\mathbb{R}I^{\perp} = \{A \in V : \mathrm{tr}(AI) = 0\}.$$

We thus have the dual space of the tangent space is

$$(V/\mathbb{R}I, |\cdot|_I)^* = (\mathbb{R}I^{\perp}, \frac{1}{2}\|\cdot\|_I^*).$$

Combining the above we see that the dual unit ball B_1^* is given by

$$B_1^* = 2\,\mathrm{conv}[S(V) \cup -S(V)] \cap \mathbb{R}I^\perp$$

Thanks to a result by Edwards and Rüttiman, we know that the closed boundary faces of B_1^* are precisely the nonempty sets of the form $A_{P,Q}$, where P and Q are orthogonal idempotents, and

$$A_{P,Q} = 2\,\mathrm{conv}[(PVP \cap S(V)) \cup (QVQ \cap -S(V))] \cap \mathbb{R}I^\perp$$

The Horofunctions of (Ω_V, d_H)

Lemmens, Lins, Nussbaum and Wortel showed that the horofunction boundary of (Ω_V, d_H) consists entirely of functions $h : \Omega_V \to \mathbb{R}$, defined by

$$h(A) = \log M(P/A) + \log M(Q/A^{-1}),$$

where $P, Q \in \partial V^+$, $\|P\|_I = \|Q\|_I = 1$, and $\mathrm{tr}(PQ) = 0$.

In fact, each horofunction is a Busemann point, and we know the geodesics converging to each one.

Example

Consider $V = \mathrm{Herm}_3(\mathbb{C})$, and $P, Q \in \partial V^+$

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ and } \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can spectrally decompose into a linear combination of idempotents

$$Q = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = Q_1 + \frac{1}{2}Q_2$$

We can then define a sequence $(A_n) \subset \mathrm{int}(V^+)$ by

$$A_n = P + \frac{1}{\frac{1}{2}n^2}Q_1 + \frac{1}{n^2}Q_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{2}{n^2} & 0 \\ 0 & 0 & \frac{1}{n^2} \end{pmatrix}$$

For large enough n, $\|A_n\|_I = 1$, and

$$A_n^{-1} = P + \frac{n^2}{2}Q_1 + n^2 Q_2,$$

from which we see that $\|A_n^{-1}\|_I = n^2$ for large n, meaning that

$$\frac{A_n^{-1}}{\|A_n^{-1}\|_I} = \frac{1}{n^2} = \frac{1}{n^2}P + \frac{1}{2}Q_1 + Q_2$$

We can then calculate

$$\begin{aligned} h_{A_n}(B) &= \log M(B/A_n) - \log M(I/A_n) + \log M(A_n/B) - \log M(A_n/I) \\ &= \log M((A_n^{-1}/\|A_n^{-1}\|)/B^{-1}) - \log M((A_n^{-1}/\|A_n^{-1}\|)/I) \\ &\quad + \log M(A_n/B) - \log M(A_n/I) \\ &\xrightarrow{n} \log M(Q/B^{-1}) - \log M(Q/I) + \log M(P/B) - \log M(P/I) \\ &= h(B) \end{aligned}$$

Can also use the fact that

$$\log M(A/B) = \max \sigma(\log B^{-1/2}AB^{-1/2}),$$

and

$$\log M(B/A) = -\min \sigma(\log B^{-1/2}AB^{-1/2}).$$

With Lemmens we found the parts of the horoboundary:
Let $h_1, h_2 \in \partial\bar{\Omega}_V^h$ be defined by

$$h_1(A) = \log M(P_1/A) + \log M(Q_1/A^{-1}),$$

$$h_2(A) = \log M(P_2/A) + \log M(Q_2/A^{-1}),$$

h_1 is in the same part as h_2 if and only if $P_1 \sim P_2$, and $Q_1 \sim Q_2$, in which case:

$$\delta(h_1, h_2) = d_H(P_1, P_2) + d_H(Q_1, Q_2).$$

The Homeomorphism

We can now define a map $\varphi_H : \overline{\Omega}_V^h \to B_1^*$ via, if $A \in \Omega_V$

$$\varphi_H(A) = \frac{A}{\mathrm{tr}(A)} - \frac{A^{-1}}{\mathrm{tr}(A^{-1})},$$

and

$$\varphi(h) = \frac{P}{\mathrm{tr}(P)} - \frac{Q}{\mathrm{tr}(Q)}$$

for $h \in \partial\bar{\Omega}_V^h$, where P and Q are the unique defining orthogonal matrices for h.

Parts mapped to faces

Let $h \in \partial\bar{\Omega}_V^h$ be defined by $h(A) = \log M(P/A) + \log M(Q/A^{-1})$. We can take the spectral decomposition of P and Q respectively: $P = \sum_{j \in J} \lambda_j P_j$ and $Q = \sum_{k \in K} \mu_k Q_k$, where all $\lambda_j, \mu_k > 0$. Let us define:

$$p_J = \sum_{j \in J} P_j, \quad q_K = \sum_{k \in K} Q_k$$

As the eigenvalues are strictly positive, $p_J \sim P$ and $q_K \sim Q$. By definition we see that $\varphi_H(h)$ lies in the relative interior of

$$A_{p_J, q_K} = 2\,\mathrm{conv}[(p_J V p_J \cap S(V)) \cup (q_K V q_K \cap -S(V))] \cap \mathbb{R}I^{\perp}.$$

If $\mathcal{P}_h$ is the part of the horoboundary containing h, we can show that φ_H maps any $h' \in \mathcal{P}(h)$ into the relative interior of A_{p_J, q_K}

Let $h'(A) = \log M(P'/A) + \log M(Q'/A^{-1})$. As above we can take the spectral decomposition of the defining orthogonal matrices, and thus define p'_J and q'_K similarly. As $h' \in \mathcal{P}(h)$ it follows that $p_J \sim p'_J$ and $q_K \sim q'_K$. Thus $\mathrm{face}(p_J) = \mathrm{face}(p'_J)$ and $\mathrm{face}(q_K) = \mathrm{face}(q'_K)$.

As p_J and p'_J are idempotents in a JB algebra, we have that $\mathrm{face}(p_J) \cap [0, I] = [0, p_J]$ and $\mathrm{face}(p'_J) \cap [0, I] = [0, p'_J]$, from which we deduce that $p_J = p'_J$, and similarly $q_J = q'_J$

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