Relatively uniform spectral theory for operators on vector lattices

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Marjeta Kramar Fijavž, University of Ljubljana, Slovenia

University of Ljubljana
Faculty of Civil and Geodetic Engineering

Institute of Mathematics, Physics and Mechanics

Mathematical models for interacting dynamics on networks

References

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Relatively uniform convergence

Relatively uniform convergence

Let X be an Archemedean vector lattice.

Definition

A net (x_α) *converges relatively uniformly* to $x \in X$,

$$(x_\alpha) \xrightarrow{\text{ru}} x,$$

if there exists $u \in X$ such that for each $\varepsilon > 0$ there exists α_0 such that

$$|x_\alpha - x| \leq \varepsilon \cdot u \quad \text{holds for all } \alpha \geq \alpha_0.$$

We call such $u \in X$ a *regulator* of $(x_\alpha)_\alpha$.

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   (i) $f_\alpha \xrightarrow{\|\cdot\|_\infty} f$ and

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2. $f_\alpha \xrightarrow{\text{ru}} f$ on $\mathrm{C}_c(\Omega) \iff$
   (i) $f_\alpha \xrightarrow{\|\cdot\|_\infty} f$ and
   (ii) there exists a compact set $K \subset \Omega$ and α_0 such that $f_\alpha|_{K^c} = 0$ for all $\alpha \geq \alpha_0$.

Relatively uniform convergence

Definition

$f\colon \mathbb{R}_+ \to X$ is called ru-*continuous* if one can find $u\colon \mathbb{R}_+ \to X$ such that for each $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(h+t) - f(t)| \le \varepsilon \cdot u(t)$$

holds for all $t \ge 0$ and $h \in [-\min\{\delta, t\}, \delta]$.

We write

$$f(h+t) \xrightarrow{\text{ru}} f(t) \ \text{ as } h \to 0 \quad \text{or} \quad \text{ru-}\lim_{h\to 0} f(h+t) = f(t).$$

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ru-derivative & ru-integral

can be defined analogously

Relatively uniform spectrum and spectral radius

Relatively uniform spectrum and spectral radius

Let $T\colon X \to X$ be linear operator.

Relatively uniform resolvent set

$$\rho_{\mathrm{ru}}(T) := \{\lambda \in \mathbb{R} : (\lambda I - T)^{-1} = R(\lambda, T) \text{ exists and is positive}\}$$

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Spectral radius

In a Banach space: $\mathrm{r}(T) = \lim_{n\to\infty} \sqrt[n]{\|T\|^n}$

Relatively uniform spectrum and spectral radius

Inspired by V. Troitsky[1] we define the *relatively uniform spectral radius* of a linear operator T on vector lattice X by

$$\mathrm{r}_{\mathsf{ru}}(T) := \inf\left\{\nu > 0 : \; \frac{T^n x}{\nu^n} \xrightarrow{\mathsf{ru}} 0 \; \forall x \in X\right\}$$

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Lemma

$$\mathrm{r}_{\mathrm{ru}}(T) = \inf\left\{\nu > 0 : \ \left(\frac{T^n x}{\nu^n}\right) \text{ is order bounded for every } x \in X\right\}$$

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Relatively uniform spectrum and spectral radius

Theorem

Let X be a ru*-complete Archimedean vector lattice, $T\colon X \to X$ a linear operator and $\lambda > \mathrm{r}_{\mathsf{ru}}(T)$. Then the Neumann series*

$$\sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}$$

converges pointwise relatively uniform to a linear operator R_λ^0 satisfying $R_\lambda^0(\lambda I - T) = I$.

Moreover, if T is positive, then R_λ^0 is positive, $R_\lambda^0 = R(\lambda, T)$, and $|\sigma_{\mathsf{ru}}(T)| \leq \mathrm{r}_{\mathsf{ru}}(T)$.

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(ii) for every $x \in X$ the mapping $t \mapsto T(t)x \in X$ is ru-continuous, i.e.,

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A semigroup $(T(t))_{t\geq 0}$ is *positive* if each $T(t)$ is a positive operator on X.

Relatively uniformly continuous semigroups

Some properties

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- If $(T(t))_{t\geq 0}$ is a positive ru-continuous semigroup on a vector lattice X then for each $s \geq 0$ and $x \in X$ the set

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- A positive semigroup $(T(t))_{t\geq 0}$ is ru-continuous $\iff$ $T(t)x \xrightarrow{\text{ru}} x$ as $t \downarrow 0$ for $x \in X_+$.

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- If $(T(t))_{t\geq 0}$ is a positive ru-continuous semigroup on a vector lattice X then for each $s \geq 0$ and $x \in X$ the set

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  is order bounded in X.
- A positive semigroup $(T(t))_{t\geq 0}$ is ru-continuous $\iff$ $T(t)x \xrightarrow{\text{ru}} x$ as $t \downarrow 0$ for $x \in X_+$.
- Every positive ru-continuous semigroup on a Banach lattice is strongly continuous.

Theorem

For a positive strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X the following assertions are equivalent.

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(i) $(T(t))_{t\geq 0}$ *is relatively uniformly continuous.*

Relatively uniformly continuous semigroups

Theorem

For a positive strongly continuous semigroup $(T(t))_{t\geq 0}$ *on a Banach lattice* X *the following assertions are equivalent.*

(i) $(T(t))_{t\geq 0}$ *is relatively uniformly continuous.*

(ii) *There exists* $s > 0$ *such that for each* $x \in X$ *the set* $\{|T(t)x| : t \in [0, s]\}$ *is order bounded in* X.

Relatively uniformly continuous semigroups

Theorem

For a positive strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach lattice X the following assertions are equivalent.

(i) $(T(t))_{t\geq 0}$ *is relatively uniformly continuous.*

(ii) *There exists $s > 0$ such that for each $x \in X$ the set $\{|T(t)x| : t \in [0,s]\}$ is order bounded in X.*

(iii) *For each $x \in X$ and $t \geq 0$ we have*

$$T(h+t)x \xrightarrow{o} T(t)x \quad \text{as } h \to 0.$$

The left translation semigroup

$$(T_l(t)f)(x) := f(t+x), \quad x \in \mathbb{R}$$

is relatively uniformly continuous on $\mathrm{Lip}(\mathbb{R})$, $\mathrm{UC}(\mathbb{R})$, $\mathrm{C}_c(\mathbb{R})$, $\mathrm{C}(\mathbb{R})$, $\mathrm{W}^{1,p}(\mathbb{R})$ for $1 \leq p < \infty$.

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Ohrstein-Uhlenbeck semigroup

$$(T_{OU}(t)f)(x) := \int_{\mathbb{R}^n} f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right) d\gamma(y)$$

is relatively uniformly continuous on $\mathrm{L}^p(\gamma)$.

Relatively uniformly continuous semigroups

Definition

The *generator A of a ru-continuous semigroup* $(T(t))_{t\geq 0}$ is defined as

$$Ax := \mathrm{ru} - \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x)$$

$$D(A) := \left\{ x \in X \mid \mathrm{ru} - \lim_{h\downarrow 0} \frac{1}{h}(T(h)x - x) \text{ exists in } X \right\}$$

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The left translation semigroup

The ru-generator of $(T_l(t))_{t\geq 0}$ on $\mathrm{C_c}(\mathbb{R})$ is $A := \frac{\mathrm{d}}{\mathrm{d}x}$ with

$$D(A) = \{f \in \mathrm{C}_c(\mathbb{R}) \mid f \text{ is continuously differentiable}\}.$$

Relatively uniformly continuous semigroups

Definition

$D \subset X$ is ru-*dense* if for each $x \in X$ there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D$ such that $x_n \xrightarrow{\text{ru}} x$ as $n \to \infty$.

We call an operator B on X

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- ru-*densely defined* if its domain $D(B)$ is ru-dense in X,
- ru-*closed* whenever $x_n \xrightarrow{\text{ru}} x$ and $Bx_n \xrightarrow{\text{ru}} y$ imply that $x \in D(B)$ and $Bx = y$.

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Proposition

Every generator of a positive ru-*continuous semigroup is* ru-*closed and* ru-*densely defined.*

Relatively uniformly continuous semigroups

Definition

$(T(t))_{t\geq 0}$ is called *exponentially order bounded* if for each $x \in X$ there exists $u \in X$ such that for all $t \geq 0$ we have

$$|T(t)x| \leq e^{\omega t}u.$$

Let $\omega_{\mathrm{ru}}(T)$ be the infimum of such ω's.

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- is exponentially order bounded on $\mathrm{Lip}(\mathbb{R})$, $\mathrm{UC}(\mathbb{R})$, and $\mathrm{W}^{1,p}(\mathbb{R})$,

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The left translation semigroup

- is exponentially order bounded on $\mathrm{Lip}(\mathbb{R})$, $\mathrm{UC}(\mathbb{R})$, and $\mathrm{W}^{1,p}(\mathbb{R})$,
- is not exponentially order bounded on $\mathrm{C}_c(\mathbb{R})$ and $\mathrm{C}(\mathbb{R})$.

Relatively uniform spectral bound

$$\mathrm{s}_{\mathsf{ru}}(A) := \sup\{\mathrm{Re}\lambda : \lambda \in \sigma_{\mathsf{ru}}(A)\}$$

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Proposition

Let $(T(t))_{t\geq 0}$ be an exponentially order bounded positive ru-*continuous semigroup and A its generator. Then for each $\lambda > \omega_{\mathsf{ru}}(T)$ one has $\lambda \in \rho_{\mathsf{ru}}(A)$ and*

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, \mathrm{d}t.$$

Moreover, $\mathrm{s}_{\mathsf{ru}}(A) \leq \omega_{\mathsf{ru}}(T)$.

Relatively uniformly continuous semigroups

A vector lattice X has *property* (D) if for each net of linear operators $(T_\alpha)_\alpha$ on X the following two assertions imply $T_\alpha x \xrightarrow{\text{ru}} 0$ for each $x \in X$.

Relatively uniformly continuous semigroups

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(a) There exists an ru-dense subset $D \subset X$ such that $T_\alpha y \xrightarrow{ru} 0$ for each $y \in D$.

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(a) There exists an ru-dense subset $D \subset X$ such that $T_\alpha y \xrightarrow{ru} 0$ for each $y \in D$.

(b) For each sequence $(x_n)_{n\in\mathbb{N}} \subset X$ with $x_n \xrightarrow{ru} 0$ there exists $u \in X_+$ such that for each $\varepsilon > 0$ there exist $N_\varepsilon \in \mathbb{N}$ and α_ε such that

$$|T_\alpha x_n| \leq \varepsilon \cdot u$$

holds for all $n \geq N_\varepsilon$ and $\alpha \geq \alpha_\varepsilon$.

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holds for all $n \geq N_\varepsilon$ and $\alpha \geq \alpha_\varepsilon$.

Examples

$\mathrm{Lip}(\mathbb{R})$, $\mathrm{UC}(\mathbb{R})$, $\mathrm{C}_c(\mathbb{R})$, $\mathrm{C}(\mathbb{R})$, $\mathrm{L}^p(\mathbb{R})$ for $0 < p < \infty$

Lemma

Let X have the property (D) and $(T(t))_{t\geq 0}$ be an exponentially order bounded positive semigroup on X. If there exists an ru*-dense set $D \subset X$ such that $T(h)y \xrightarrow{\text{ru}} y$ as $h \downarrow 0$ holds for each $y \in D$, then $(T(t))_{t\geq 0}$ is relatively uniformly continuous on X.*

Relatively uniformly continuous semigroups

Lemma

Let X have the property (D) and $(T(t))_{t\geq 0}$ be an exponentially order bounded positive semigroup on X. If there exists an ru*-dense set $D \subset X$ such that $T(h)y \xrightarrow{\text{ru}} y$ as $h \downarrow 0$ holds for each $y \in D$, then $(T(t))_{t\geq 0}$ is relatively uniformly continuous on X.*

Proposition

Let X be an ru*-complete vector lattice with property (D). Every positive exponentially order bounded* ru*-continuous semigroup on X is uniquely determined by its generator.*

Theorem

Let X be an ru*-complete vector lattice with property (D). Then the following assertions are equivalent.*

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(i) *A generates an exponentially order bounded positive* ru*-continuous semigroup.*

Relatively uniformly continuous semigroups

Theorem

Let X be an ru*-complete vector lattice with property (D). Then the following assertions are equivalent.*

(i) *A generates an exponentially order bounded positive* ru*-continuous semigroup.*

(ii) *A is* ru*-closed,* ru*-densely defined, for every $\lambda > \omega_{\mathrm{ru}}(T) =: \omega$ one has $\lambda \in \rho_{\mathrm{ru}}(A)$ and for each $x \in X$ there exists $u \in X$ such that*

$$|R(\lambda, A)^k x| \leq (\lambda - \omega)^{-k} \cdot u \quad \textit{for all } k \in \mathbb{N}.$$

What next?

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Operator semigroups on vector lattices

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Thank you!