Nonlocal Problems with Integral Conditions for a Loaded Fourth-Order Equation

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Introduction

- Initial and boundary value problems for partial differential equations
- Nonclassical problems for differential equations
- Nonlocal problems

In nonlocal problems, given boundary values are partially or completely replaced by additional conditions on the function connecting the values at the boundary and the inner of the domain.

Introduction

- Nonlocal problems
- Problems with nonlocal integral conditions for a loaded equation

By a loaded equation we mean a differential equation involving also values of a required solution and its derivatives at certain fixed points.

- Second-order equations
- Nonlocal problem with integral conditions for a loaded fourth-order equation

Statement of the problem PK

Introduce some notation

$$\mathcal{L} \equiv A(x,y)\frac{\partial^2}{\partial x^2} + B(x,y)\frac{\partial^2}{\partial y^2} + C(x,y),$$

$$\Omega = (0,a) \times (0,b), \quad \text{where} \;\; a > 0,\; b > 0,$$

$$\widetilde{\mathbf{C}}(\Omega) = \{u : \; u \in \mathbf{C}^2(\bar{\Omega}), \frac{\partial^4 u}{\partial x^2 \partial y^2} \in \mathbf{C}(\Omega)\},$$

and consider the following problem.

Statement of the problem PK (continuation)

Problem PK (Goursat-type nonlocal problem for a fourth-order loaded equation):
Find a function $u(x, y)$ such that $u \in \widetilde{\mathbf{C}}(\Omega)$,

$$\frac{\partial^4 u(x,y)}{\partial x^2 \partial y^2} + \mathcal{L}[u(x,y) - u(x,0) - u(0,y)] + C(x,y)u(0,0) = F(x,y) \quad (1)$$

$$u_y(x,0) - u_y(0,0) = \varphi(x) - \varphi(0), \quad u_x(0,y) - u_x(0,0) = \psi(y) - \psi(0), \quad (2)$$

$$\int_0^a K(x)u(x,y)dx = h(y), \quad \int_0^b L(y)u(x,y)dy = p(x), \quad (3)$$

where $A, B, C, F, \varphi, \psi, K, L, h, p$ are given functions.

Result

Theorem (Goursat-type nonlocal problem PK for a fourth-order loaded equation)

Suppose that

$$A, B, C, F \in \mathbf{C}(\bar{\Omega}),\ K, p \in \mathbf{C}[0, a],\ L, h \in \mathbf{C}[0, b],$$

$$\varphi \in \mathbf{C}^2[0, a],\ \ \psi \in \mathbf{C}^2[0, b],\ \varphi'(0) = \psi'(0),$$

$$\int_0^a K(x)dx \neq 0,\ \ \int_0^b L(y)dy \neq 0,$$

$$\int_0^a K(x)p(x)dx = \int_0^b L(y)h(y)dy.$$

Then there exists a unique solution of the problem PK (1)-(3).

Proof of the theorem PK

Step 1: Transfering the problem for 4th-order equation to problems for 2-order equations
Step 2: Goursat problem for differential-integral equation
Step 3: Nonlocal Goursat problem

First point to be noticed is that (1) can be represent as follows:

$$\left(\frac{\partial^2}{\partial x \partial y} + A\frac{\partial}{\partial x}\int_0^y d\eta + B\frac{\partial}{\partial y}\int_0^x d\xi + C\int_0^y\int_0^x d\xi d\eta\right) u_{xy} = F(x,y) \quad (4)$$

Introduce a new function $v(x,y) = u_{xy}$. If $u(x,y)$ is a solution to (1) and satisfies (2) then $v(x,y)$ satisfies integro-differential equation

$$\frac{\partial^2 v}{\partial x \partial y} + A\int_0^y v_x d\eta + B\int_0^x v_y d\xi + C\int_0^y\int_0^x v d\xi d\eta = F(x,y) \quad (5)$$

and boundary conditions

$$v(x,0) = \varphi'(x), \quad v(0,y) = \psi'(y). \quad (6)$$

As $y = 0$ and $x = 0$ are characteristics of (5) then the problem (5), (6) is none other than a Goursat problem.

Step 1: Transfering (continuation)

If there exists a solution to (5), (6) we can find $u(x, y)$ as a solution of

$$u_{xy} = v. \tag{7}$$

Attaching integral conditions (3) we come to Goursat problem with nonlocal conditions.

Lemma

The problem PK (1) – (2) – (3) is equivalent to a system of problems: problem (5), (6) and problem (7), (3).

Proof.

Let now $v(x, y), u(x, y)$ are the solutions to (5), (6) and (7), (3) respectively. It is clear that (4) holds, therefore (1) also holds. It follows from (6) and $v = u_{xy}$ that $u_{xy}(x, 0) = \varphi'(x), \quad u_{xy}(0, y) = \psi'(y)$. Integrate these equalities by x and y respectively we get (2). □

Step 2: Goursat problem for differential-integral equation

Find a solution of (5):

$$\frac{\partial^2 v}{\partial x \partial y} + A\int_0^y v_x d\eta + B\int_0^x v_y d\xi + C\int_0^y\int_0^x v d\xi d\eta = F(x,y)$$

satisfying the boundary conditions (6):

$$v(x,0) = \varphi'(x), \quad v(0,y) = \psi'(y).$$

Step 2: Goursat problem for d.-i. equation (continuation)

Theorem (Goursat problem for differential-integral equation (5), (6))

Let the following assumptions are valid

$$A, B, C, F \in \mathbf{C}(\bar{\Omega}),$$

$$\varphi \in \mathbf{C}^2[0, a], \quad \psi \in \mathbf{C}^2[0, b], \quad \varphi'(0) = \psi'(0).$$

Then there exists a unique solution $v(x, y)$ *to the problem (5), (6) such that* $v \in \mathbf{C}^1(\bar{\Omega}) \cap \mathbf{C}^2(\Omega), \ v_{xy} \in \mathbf{C}(\Omega)$.

Step 3: Nonlocal Goursat problem

Find a solution of (7):

$$u_{xy} = v(x, y)$$

satisfying the nonlocal conditions (3):

$$\int_0^a K(x)u(x, y)dx = h(y), \quad \int_0^b L(y)u(x, y)dy = p(x).$$

Step 3: Nonlocal Goursat problem (continuation)

Theorem (Nonlocal Goursat problem (7), (3))

Let the following assumptions are valid

$$K \in \mathbf{C}[0, a], \quad L \in \mathbf{C}[0, b],$$

$$\int_0^a K(x)dx \neq 0, \quad \int_0^b L(y)dy \neq 0,$$

$$\int_0^a K(x)p(x)dx = \int_0^b L(y)h(y)dy.$$

Then there exists a unique solution $u(x, y)$ to the problem (7), (3) such that $u \in \widetilde{\mathbf{C}}(\Omega)$.

Conclusion

Now the proof of theorem PK is trivial because of the lemma, the theorem 3, and the theorem 4.
The Goursat-type nonlocal problem PK for a fourth-order loaded equation has a unique solution in

$$\widetilde{\mathbf{C}}(\Omega) = \{u : \; u \in \mathbf{C}^2(\bar{\Omega}), \frac{\partial^4 u}{\partial x^2 \partial y^2} \in \mathbf{C}(\Omega)\}.$$

Thank you very much for your attention.