Prime ideals in vector lattices

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General properties

A vector subspace J of a vector lattice E is an **(order) ideal** if $x \in J$ and $0 \leq |y| \leq |x|$ imply $y \in E$.

1. The sum of a given family of ideals is an ideal.
2. The intersection of a given family of ideals is an ideal.
3. If $x \in E$, then the set
$$I_x = \{y \in E : \ |y| \leq \lambda |x| \text{ for some } \lambda \geq 0\}$$
is an ideal in E, called the **principal ideal** generated by x.
4. We have $I_x = I_{|x|}$.
5. If $x_1, \ldots, x_n$ are positive, then
$$I_{x_1 + \cdots + x_n} = I_{x_1} + \cdots + I_{x_n} = I_{x_1 \vee \cdots \vee x_n}.$$
Hence, every finitely generated ideal is principal.

Examples of ideals I

Example

- In $\mathbb{R}^n$ every ideal is of the form

$$J_A = \{x \in \mathbb{R}^n : \ x_j = 0 \text{ for all } j \in A\}$$

  where $A \subseteq \{1, \ldots, n\}$.
- In $c_{00}(\Omega)$ every ideal is of the form

$$J_A = \{x \in c_{00}(\Omega) : \ x_j = 0 \text{ for all } j \in A\}$$

  where $A \subseteq \Omega$.
- Closed ideals in $C(K)$ with K compact and Hausdorff are precisely of the form

$$J_F = \{f \in C(K) : \ f|_F \equiv 0\}$$

  where $F \subseteq K$ is a closed set.

Examples of ideals II

Example

- If μ is σ-finite, then every closed ideal in $L^p(\mu)$ $(1 \leq p < \infty)$ is of the form

$$J_F = \{f \in L^p(\mu) : \; f \equiv 0 \text{ almost everywhere on } F\}$$

  where F is a measurable subset of the ambient measurable space Ω.
- If $A \subseteq \mathbb{N}$, then

$$\{x \in \ell^\infty : \; x_n = 0 \text{ for all } n \in A\}$$

  is a closed ideal in ℓ^∞. There are many more closed ideals in ℓ^∞ as we have an isometric isomorphism of C^*-algebras and Banach lattices:

$$\ell^\infty = C_b(\mathbb{N}) \cong C(\beta\mathbb{N}).$$

Maximal ideals

A proper ideal M of E is called **maximal** whenever for each ideal J satisfying $M \subseteq J \subseteq E$ it follows that either $J = M$ or $J = E$.

1. An ideal I of E is maximal if and only if the co-dimension of I in E is one.
2. The Banach lattice $L^1[0,1]$ does not have any maximal ideals.
3. If a vector lattice contains a strong unit, then every proper ideal is contained in a maximal ideal. (A positive vector e in E is a strong unit if $|x| \leq \lambda_x e$, i.e., $I_e = E$.)
4. If E is a norm dense sublattice of $C(K)$, then maximal ideals of E are precisely of the form

$$M_x^E = \{f \in E : \ f(x) = 0\}.$$

5. In particular, maximal ideals of $C(K)$ are precisely of the form $J_{\{x\}} = M_x^{C(K)}$.

Prime ideals

A proper ideal $P \subset E$ is said to be **prime** whenever $x \wedge y \in P$ implies $x \in P$ or $y \in P$.

1. Every maximal ideal is prime.
2. Every ideal containing a prime ideal P is also prime.
3. The family of all ideals containing a prime ideal P is linearly ordered.
4. Every prime ideal contains a minimal prime ideal.

Theorem

For an ideal P in a vector lattice E the following conditions are equivalent.

(i) *P is prime.*

(ii) *If $x \wedge y = 0$, then $x \in P$ or $y \in P$.*

(iii) *The quotient vector lattice E/P is linearly ordered.*

(iv) *For any ideals I and J satisfying $I \cap J \subseteq P$ we have $I \subseteq P$ or $J \subseteq P$.*

Algebraic and lattice ideals in $C(K)$

- Every order ideal is also an algebraic ideal (for $f \in I$ we have $|fg| \leq \|g\|_\infty f$).
- An algebraic ideal I is not necessarily an order ideal. To see this, take $K = [0,1]$ and consider the ideal of all functions f of the form $f(x) = xg(x)$. Then I is an algebraic ideal which does not contain the function h defined as

$$h(x) = \begin{cases} x \sin x^{-1} & : \quad x \neq 0 \\ 0 & : \quad x = 0 \end{cases}$$

Yet, $|h(x)| \leq |x|$ and the identity mapping $x \mapsto x$ is in I.

Algebraic and lattice ideals in $C(K)$

- An order prime ideal is not necessarily an algebraic prime ideal. Define

$$J = \{f \in C[0,1] : \ |f(x)| \leq \alpha_f x \text{ for all } x \text{ and some } \alpha_f > 0\}.$$

Then J is an algebraic and an order ideal. Obviously, the square root $x \mapsto x^{\frac{1}{2}}$ is not in J. By Zorn's lemma type argument there exists an order prime ideal $P \supset J$ with $x^{\frac{1}{2}} \notin P$. However, $x = x^{\frac{1}{2}} \cdot x^{\frac{1}{2}} \in J \subset P$.

- An algebraic prime ideal is an order prime ideal (for $f \in I$ and $|g| \leq |f|$ we have $g = g|f|^{-\frac{1}{2}} \cdot |f|^{\frac{1}{2}}$).
- In $C(K)$ the classes of maximal algebraic and maximal order ideals coincide. They are of the form $J_{\{x\}}$ for some $x \in K$.

If E is a norm dense sublattice of $C(K)$, then for every prime ideal P of E there exists a unique point x in K such that $P \subseteq M_x^E$.

Proposition

Let Ω be a non-empty set. Then an ideal $I \subseteq c_{00}(\Omega)$ is maximal if and only if I is of the form

$$M_w := \{x \in c_{00}(\Omega) : x(w) = 0\}$$

for some $w \in \Omega$. Moreover, for a proper ideal I in $c_{00}(\Omega)$ the following statements are equivalent.

(i) *I is a minimal prime ideal.*

(ii) *I is a prime ideal.*

(iii) *I is a maximal ideal.*

A vector lattice is **Archimedean** if the following holds: if the inequality $0 \leq nx \leq y$ holds for each $n \in \mathbb{N}$, then $x = 0$.

An ideal I in E is **order dense** if for each $0 \leq y \in E^+$ there exists $x \in I$ such that $0 < x \leq y$.

Theorem

Let I be an ideal in a vector lattice E. Consider the following statements.

1. *I is order dense in E.*
2. *$I^d = \{0\}$.*
3. *For each $0 \leq y \in E$ we have*

$$y = \sup\{x \in I :\ 0 \leq x \leq y\}.$$

Then (1) and (2) are equivalent, and (3) implies (1). If E is Archimedean, then all statements are equivalent.

Theorem

Let E be an Archimedean vector lattice.

(i) *E is atomless if and only if every prime ideal in E is order dense in E.*

(ii) *None of the prime ideals are order dense in E if and only if E is equal to the linear span of its atoms.*

- For each positive vector $x \in E$ consider the principal ideal

$$I_x = \{y \in E : \ |y| \leq \lambda x \text{ for some } \lambda > 0\}.$$

- For $y \in I_x$ define $\|y\|_x := \inf\{\lambda : \ |y| \leq \lambda x\}$.
- Then $\| \cdot \|_x$ is a seminorm on I_x for each $x > 0$.
- $(I_x, \| \cdot \|_x)$ is a normed lattice for each $x > 0$ if and only if E is Archimedean.
- To avoid patologies people usually work on Archimedean vector lattices.

Theorem (Kakutani – Bohnenblust – Krein)

If for some $x > 0$ the normed lattice $(I_x, \|\cdot\|_x)$ is a Banach lattice, then I_x is lattice isometric isomorphic to some $C(K)$ space where $x \mapsto \mathbf{1}$.

- Via this representation, one can "generate" adequate vectors to prove theorems.
- To prove an inequality for finitely many vectors, one needs to prove the same inequality for finitely many continuous functions. Hence, for real numbers.
- The lattice structure of I_x is the same as the lattice structure of $C(K)$.
- Sometimes some structure of E passes down to I_x and so to $C(K)$.
- I_x is Dedekind complete if and only if K, in addition, is extremally disconnected.

How to guarantee that $(I_x, \|\cdot\|_x)$ is a Banach lattice?

- A sequence $(x_n)_{n\in\mathbb{N}}$ is relatively uniformly Cauchy if there exists a positive vector y such that for each $\varepsilon > 0$ there exists a $k \in \mathbb{N}$ such that for all $n, m \geq k$ we have
$$|x_n - x_m| \leq \varepsilon y.$$
- If there exists a vector x such that for each $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that
$$|x_n - x| \leq \varepsilon y,$$
then $(x_n)_{n\in\mathbb{N}}$ converges relatively uniformly (with respect to y).
- A vector lattice E is relatively uniformly complete if every relatively uniformly Cauchy sequence converges relatively uniformly.
- $(x_n)_{n\in\mathbb{N}}$ is relatively uniformly Cauchy with respect to y if and only if $(x_n)_{n\in\mathbb{N}}$ is Cauchy in $(I_y, \|\cdot\|_y)$.

Theorem

If an Archimedean vector lattice E contains finitely many minimal prime ideals, then it is finite-dimensional.

Example

The lexicographically ordered real plane $\mathbb{R}^2$ is not Archimedean as

$$(0,0) \leq n \cdot (0,1) \leq (1,0)$$

holds for each $n \in \mathbb{N}$.

The space of all sequences $\mathbb{R}^{\mathbb{N}}$ ordered lexicographically is infinite-dimensional and totally ordered. Hence, the zero ideal $\{0\}$ is a prime ideal, and so, it is the unique minimal prime ideal of $\mathbb{R}^{\mathbb{N}}$.

Theorem

If an Archimedean vector lattice E contains finitely many minimal prime ideals, then it is finite-dimensional.

Sketch of the proof

- Prove that I_x contains finitely many minimal prime ideals.
- $C(K)$ contains a norm dense sublattice $(I_x, \|\cdot\|_x)$ with finitely many minimal prime ideals.
- $C(K)$ contains finitely many minimal prime ideals.
- Every maximal ideal M_t contains a minimal prime ideal and for each prime ideal P there is a unique point t such that $P \subseteq M_t$.
- There are only finitely maximal ideals in $C(K)$.
- K is finite $\Rightarrow C(K) \cong I_x$ is finite-dimensional.
- $E \cong c_{00}(\Omega)$ and therefore, Ω is finite.

From commutative algebra to vector lattices

Theorem

Cohen: *A commutative ring is Noetherian if and only if every prime ideal is finitely generated.*

Kaplansky: *In a Noetherian commutative ring every ideal is principal if and only if every maximal ideal is principal.*

Cohen – Kaplansky: *If every prime ideal in a commutative ring is principal, then every ideal is principal.*

A vector lattice is Noetherian if it satisfies the **ascending chain condition (acc)** for ideals.

Proposition

An Archimedean vector lattice is Noetherian if and only if it is finite-dimensional.

Cohen-Kaplansky's theorem for vector lattices

Let E be a vector lattice.

(i) E is finite-dimensional.

(ii) Every proper ideal in E is principal (= finitely generated).

(iii) Every prime ideal in E is principal (= finitely generated).

Then (i) implies (ii), (i) implies (iii), and (iii) implies (ii). Moreover, in the case where E is Archimedean, we have that (ii) implies (i), so all statements are equivalent.

- If E is Archimedean, then (i) and (iii) are equivalent. This is Cohen's theorem for vector lattices.
- **Cohen-Kaplansky** for vector lattices is the equivalence between (ii) and (iii).

Let $x = (x_n)_{n\in\mathbb{N}} \in c_0$ be any vector with $x_n > 0$ for each $n \in \mathbb{N}$. Define the vector lattice $E := I_x + \mathbb{R}\mathbf{1}$. Then

- E is not norm complete, and its norm completion is $c \cong C(\mathbb{N} \cup \{\infty\})$.
- Maximal ideals in E are of the form

  $$M_t^E = M_t \cap E = \{y \in E : \ y(t) = 0\}$$

  for some $t \in \mathbb{N} \cup \{\infty\}$.
- Since $x(t) > 0$ for each $t \in \mathbb{N}$, we have $e_t \in E$ and so $\mathbf{1} - e_t$ is the generator for M_t^E if $t \in \mathbb{N}$, otherwise $M_\infty^E = I_x$ is principal by construction.
- It turns out that there are no other principal prime ideals.
- E is infinite-dimensional.

A version of Kaplansky's theorem

Cohen-Kaplansky theorem for uniformly complete vector lattices

The following statements are equivalent for a uniformly complete Archimedean vector lattice E.

(i) E is finite-dimensional.

(ii) Every proper ideal in E is principal.

(iii) E contains maximal ideals, and every maximal ideal in E is principal.

Sketch of the proof

- (iii) $\Rightarrow$ (i): It first follows that $E = I_x$ for some $x > 0$.
- Now $E = I_x \cong C(K)$ and since $I_f = M_t = \{f \in C(K) : \ f(t) = 0\}$ is principal, t is isolated (non-trivial).
- Compact space consisting of isolated points is finite.

A vector lattice E is said to be prime Noetherian if every ascending chain of prime ideals $P_1 \subseteq P_2 \subseteq \dots$ in E is stationary.

Auxiliary results

- Let E be a vector lattice and let F be a vector sublattice. If E is prime Noetherian, then F is prime Noetherian.
- In a prime Noetherian vector lattice every prime ideal is contained in a prime ideal of finite co-dimension.
- Let E be an at least two-dimensional prime Noetherian vector lattice. Then every proper ideal of E is contained in a maximal ideal.

Theorem

A uniformly complete Archimedean vector lattice E is prime Noetherian if and only if it is lattice isomorphic to $c_{00}(\Omega)$ for some set Ω.

Corollary

The following assertions are equivalent for a uniformly complete prime Noetherian Archimedean vector lattice E.

(i) *E has a strong unit.*

(ii) *E is lattice isomorphic to a Banach lattice.*

(iii) *E is finite-dimensional.*

Funny Theorem

Let X be a locally compact Hausdorff space and let M be a maximal ideal in $C_0(X)$. Then there exists an $x \in X$ such that

$$M = \{f \in C_0(X) \colon f(x) = 0\}.$$

If M is closed, there are no problems. Why is M closed?

- For $n \in \mathbb{N}$ let $\mathrm{PPol}^n([a,b])$ be the vector lattice of piecewise polynomials of degree at most n that are continuous on the interval $[a,b]$.
- The space of piecewise polynomials that are continuous on the interval $[a,b]$ without any bound on the degree by $\mathrm{PPol}([a,b])$.
- By the lattice version of the Stone-Weierstrass theorem all these spaces are uniformly dense in $C([a,b])$.

For $t_0 \in (a,b]$ we define

$$L_{t_0} := \left\{ f \in M_{t_0}^E \colon \text{there exists a } \delta > 0 \text{ such that } f(t) = 0 \text{ for } t \in (t_0 - \delta, t_0] \right\}$$

and for $t_0 \in [a,b)$ we define

$$R_{t_0} := \left\{ f \in M_{t_0}^E \colon \text{there exists a } \delta > 0 \text{ such that } f(t) = 0 \text{ for } t \in [t_0, t_0 + \delta) \right\}.$$

Lemma

Let E be either $\mathrm{PPol}^n([a,b])$ or $\mathrm{PPol}([a,b])$. The minimal prime ideals in E are precisely L_{t_0} for $t_0 \in (a,b]$ and R_{t_0} for $t_0 \in [a,b)$.

- If we write $E := \mathrm{PPol}([a,b])$, then we consider

$$L_{t_0}^k := \left\{ f \in M_{t_0}^E \colon f_-^{(k)}(t_0) = f_-^{(k-1)}(t_0) = \ldots = f_-'(t_0) = 0 \right\}$$

  for $t_0 \in (a,b]$ and all $k \in \mathbb{N}$ in this case, and

$$R_{t_0}^k := \left\{ f \in M_{t_0}^E \colon f_+^{(k)}(t_0) = f_+^{(k-1)}(t_0) = \ldots = f_+'(t_0) = 0 \right\}$$

  for $t_0 \in [a,b)$ and all $k \in \mathbb{N}$ in this case.
- Note that $\bigcap_{k=1}^{\infty} L_{t_0}^k = L_{t_0}$ and $\bigcap_{k=1}^{\infty} R_{t_0}^k = R_{t_0}$.

Theorem

The non-maximal and non-minimal prime ideals in $\mathrm{PPol}([a,b])$ *are of the form* $L_{t_0}^k$ *for some* $k \in \mathbb{N}$ *and* $t_0 \in (a,b]$*, or are of the form* $R_{t_0}^k$ *for some* $k \in \mathbb{N}$ *and* $t_0 \in [a,b)$*.*

Corollary

The vector lattice $\mathrm{PPol}([a,b])$ *is prime Noetherian and contains ascending chains of prime ideals of arbitrary finite length.*

Theorem

Let E *be the vector lattice* $\mathrm{PPol}^n([a,b])$ *or* $\mathrm{PPol}([a,b])$*. Then all non-minimal prime ideals in* E *are principal.*