Positive Dilation in order integrals

Xingni Jiang
Sichuan University

Workshop on Ordered Vector Spaces and Positive Operators
Wuppertal 31 March 2023

Joint work with Rui Liu

Algebraic dilations

Let $\mathcal{A}$ be a unital algebra with identity e and X, Y be vector spaces. For any linear operator $T\colon \mathcal{A} \to \mathrm{L}(X, Y)$, there exist a vector space Z, a unital algebra homomorphism $\pi\colon \mathcal{A} \to \mathrm{L}(Z)$, and linear operators $\tau \in \mathrm{L}(X, Z)$, $\alpha \in \mathrm{L}(Z, Y)$ such that

$$T(a) = \alpha\pi(a)\tau \ \forall a \in \mathcal{A}.$$

Algebraic dilations

Let $\mathcal{A}$ be a unital algebra with identity e and X, Y be vector spaces. For any linear operator $T\colon \mathcal{A} \to \mathrm{L}(X, Y)$, there exist a vector space Z, a unital algebra homomorphism $\pi\colon \mathcal{A} \to \mathrm{L}(Z)$, and linear operators $\tau \in \mathrm{L}(X, Z)$, $\alpha \in \mathrm{L}(Z, Y)$ such that

$$T(a) = \alpha\pi(a)\tau \ \forall a \in \mathcal{A}.$$

- $Z = \mathrm{L}(\mathcal{A}, Y)$,
- $\tau\colon x \to T_x$ for $x \in X$, where $T_x = T(\ )x$,
- $\alpha\colon \psi \to \psi(e)$ for $\psi \in \mathrm{L}(\mathcal{A}, Y)$
- $\pi(a)\psi = \psi R_a$ for $a \in \mathcal{A}$, $\psi \in \mathrm{L}(\mathcal{A}, Y)$, where R is the right regular representation of $\mathcal{A}$ (i.e. $R_a(b) = ba$).

Dilation of operators

Dilation in order context

Let $\mathcal{A}$ be a unital Riesz algebra with positive identity e and X, Y be Riesz spaces. For any regular operator $T\colon \mathcal{A} \to \mathrm{L_r}(X, Y)$, there exist a Riesz space Z, a unital algebra homomorphism $\pi\colon \mathcal{A} \to \mathrm{L_r}(Z)$, and regular operators $\tau \in \mathrm{L}(X, Z)$, $\alpha \in \mathrm{L_r}(Z, Y)$ such that

$$T(a) = \alpha\pi(a)\tau \ \forall a \in \mathcal{A}.$$

- $Z = \mathrm{L_r}(\mathcal{A}, Y)$,
- α is positive,
- π is positive,
- if T is positive, then τ is positive.

Executive summary

Dilation theory
- Dilation of operators
- Dilation of operator valued measures

Order integrals
- Measures taking values in patially ordered vector space
- Integrals of such measures (regular operators)
- Representation theorems

Executive summary

Dilation theory
- Dilation of operators
- Dilation of operator valued measures

Order integrals
- Measures taking values in patially ordered vector space
- Integrals of such measures (regular operators)
- Representation theorems

Question:

1. What is the relation between dilation of measures and dilation of operators in the order context ?

Executive summary

Dilation theory
- Dilation of operators
- Dilation of operator valued measures

Order integrals
- Measures taking values in patially ordered vector space
- Integrals of such measures (regular operators)
- Representation theorems

Question:

1. What is the relation between dilation of measures and dilation of operators in the order context ?
2. In the framework of order integrals, will there be more properties to say about a dilation system?

Outline

- Measures taking values in partially ordered vector space
- Dilation of measures taking values in Riesz space
- Order integrals
- Relation between dilation of measures and dilation of order integrals

Measures taking values in P.O.V. (positive case)

Let E be a σ-monotone complete partially ordered vector space, i.e. if every increasing net in E that is bounded from above in E has a supremum in E. We can extend E by adjoin one point ∞ to E such that $x \leq \infty$ for all $x \in E$. Let $\overline{E^+} = E^+ \cup \{\infty\}$.

Definition (J.D.M. Wright 1969)

Let (X, Ω) be a measurable space, and let E be a σ-monotone complete partially ordered vector space. An $\overline{E^+}$*-valued measure* is a map $\mu : \Omega \to \overline{E^+}$ such that:

- $\mu(\emptyset) = 0$;
- whenever $\{A_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence in Ω, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{N=1}^{\infty} \sum_{n=1}^{N} \mu(A_n) \text{ in } \overline{E^+}.$$

We say that μ is finite if $\mu(\Omega) \subseteq E^+$.

Measures taking values in P.O.V. (positive case)

$\mathrm{M}(X,\Omega,\overline{E^+}) = \{\mu\colon \mu$ is an $\overline{E^+}$-valued measure on $(X,\Omega)\}$
$\mathrm{M}(\Omega,E^+) = \{\mu\colon \mu$ is an E^+-valued measure on $(X,\Omega)\}$
A natural partial ordering on $\mathrm{M}(X,\Omega,\overline{E^+})$:

$$\mu_1 \le \mu_2 \Longleftrightarrow \mu_1(A) \le \mu_2(A)\ \forall A \in \Omega.$$

Proposition (M.de Jeu, X. Jiang)

- $\mathrm{M}(\Omega,E^+) \subseteq \mathrm{M}(X,\Omega,\overline{E^+})$.
- $\mathrm{M}(X,\Omega,\overline{E^+})$ *and* $\mathrm{M}(\Omega,E^+)$ *are convex cones.*
- *If E is a Dedekind complete Riesz space, then $\mathrm{M}(\Omega,E^+)$ is a lattice cone, where for all $A \in \Omega$*

$$(\mu_1 \vee \mu_2)(A) = \bigvee \{\mu_1(B) + \mu_2(A\setminus B)\colon B \subseteq A,\ B \in \Omega\},$$

$$(\mu_1 \wedge \mu_2)(A) = \bigwedge \{\mu_1(B) + \mu_2(A\setminus B)\colon B \subseteq A,\ B \in \Omega\}.$$

Measures taking values in P.O.V. (general case)

Definition (M. de Jeu, X.Jiang)

Let (X, Ω) be a measurable space, and let E be a σ-Dedekind complete Riesz space. An *E-valued signed measure* is a map $\mu : \Omega \to E$ such that:

- $\mu(\emptyset) = 0$;
- whenever $\{A_n\}_{n=1}^{\infty}$ is a pairwise disjoint sequence in Ω, then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = o - \lim_{N\to\infty} \sum_{n=1}^{N} \mu(A_n)$$

  in E.

$\mathrm{M}_0(X, \Omega, E) = \{\mu \colon \mu \text{ is an } E\text{-valued signed measure on } (X, \Omega)\}$

Measures taking values in P.O.V. (general case)

Properties of E-valued signed measures

- $\mathrm{M}(\Omega, E^+) \subseteq \mathrm{M}_0(X, \Omega, E)$.
- If E is Dedekind complete, then for any $\mu \in \mathrm{M}_0(X, \Omega, E)$, there exist μ^+ and μ^- in $\mathrm{M}(X, \Omega, \overline{E^+})$ such that $\mu^+ = \mu + \mu^-$.

Measures taking values in P.O.V. (general case)

Properties of E-valued signed measures

- $\mathrm{M}(\Omega, E^+) \subseteq \mathrm{M}_0(X, \Omega, E)$.
- If E is Dedekind complete, then for any $\mu \in \mathrm{M}_0(X, \Omega, E)$, there exist μ^+ and μ^- in $\mathrm{M}(X, \Omega, \overline{E^+})$ such that $\mu^+ = \mu + \mu^-$. Moreover, if one of $\mu^\pm$ is finite, then the other one is finite and

$$\mu = \mu^+ - \mu^-, \ \mu^+ \wedge \mu^- = 0.$$

Measures taking values in P.O.V. (general case)

Properties of E-valued signed measures

- $\mathrm{M}(\Omega, E^+) \subseteq \mathrm{M}_0(X, \Omega, E)$.
- If E is Dedekind complete, then for any $\mu \in \mathrm{M}_0(X, \Omega, E)$, there exist μ^+ and μ^- in $\mathrm{M}(X, \Omega, \overline{E^+})$ such that $\mu^+ = \mu + \mu^-$. Moreover, if one of $\mu^\pm$ is finite, then the other one is finite and

$$\mu = \mu^+ - \mu^-,\ \mu^+ \wedge \mu^- = 0.$$

Let $\mathrm{M}(X, \Omega, E) = \{\mu \in \mathrm{M}_0(X, \Omega, E)$: at least one of $\mu^\pm$ is finite$\}$.

Proposition (M. de Jeu, X. Jiang)

If E is Dedekind complete, then $\mathrm{M}(X, \Omega, E)$ is a Dedekind complete Riesz space with positive cone $\mathrm{M}(\Omega, E^+)$.

Operator valued measures

Let E be a Dedekind complete Riesz space, and we consider an $\mathrm{L_r}(E)$-valued signed measure μ.

1. μ is called a probability measure if $\mu(X) = \mathbf{I}_E$
2. μ is called a spectral measure if $\mu(A_1 \cap A_2) = \mu(A_1)\mu(A_2)$

Let E be a Dedekind complete Riesz space, and we consider an $\mathrm{L_r}(E)$-valued signed measure μ.

1. μ is called a probability measure if $\mu(X) = \mathbf{I}_E$
2. μ is called a spectral measure if $\mu(A_1 \cap A_2) = \mu(A_1)\mu(A_2)$

Question

Given a $\mathrm{L_r}(F, E)$-valued signed measure μ, can we transform it into a spectral measure?

Operator valued measures

Let E be a Dedekind complete Riesz space, and we consider an $\mathrm{L_r}(E)$-valued signed measure μ.

1. μ is called a probability measure if $\mu(X) = \mathbf{I}_E$
2. μ is called a spectral measure if $\mu(A_1 \cap A_2) = \mu(A_1)\mu(A_2)$

Question

Given a $\mathrm{L_r}(F, E)$-valued signed measure μ, can we transform it into a spectral measure? I.e. do there exist a Riesz space Z, a spectral measure $P\colon \Omega \to \mathrm{L_r}(Z)$, $\tau \in \mathrm{L_r}(F, Z)$ and $\alpha \in \mathrm{L_r}(Z, E)$ such that for any $A \in \Omega$,

$$\mu(A) = \alpha P(A)\tau.$$

If it exist, then (P, Z, τ, α) is called a *dilation system of μ*.

Let E be a Dedekind complete Riesz space, and we consider an $\mathrm{L_r}(E)$-valued signed measure μ.

1. μ is called a probability measure if $\mu(X) = \mathbf{I}_E$
2. μ is called a spectral measure if $\mu(A_1 \cap A_2) = \mu(A_1)\mu(A_2)$

Question

Given a $\mathrm{L_r}(F, E)$-valued signed measure μ, can we transform it into a spectral measure? I.e. do there exist a Riesz space Z, a spectral measure $P\colon \Omega \to \mathrm{L_r}(Z)$, $\tau \in \mathrm{L_r}(F, Z)$ and $\alpha \in \mathrm{L_r}(Z, E)$ such that for any $A \in \Omega$,

$$\mu(A) = \alpha P(A)\tau.$$

If it exist, then (P, Z, τ, α) is called a *dilation system of* μ.

In the Banach space case, the answer is affirmative. (σ-additivity is defined by the strong operator topology)

Theorem (X.Jiang, R.Liu)

Let F and E be Riesz spaces. If E is Dedekind complete, then every $\mu \in M(X, \Omega, \mathrm{L_r}(F, E))$ has a dilation system (P, Z, τ, α) where

1. $Z = \mathrm{M}(X, \Omega, E)$,
2. $\tau : x \mapsto \mu_x$, *for every* $x \in F$, *where* $\mu_x = \mu(\ \ )x$,
3. $\alpha : \nu \mapsto \nu(X)$, *for every* $\nu \in \mathrm{M}(X, \Omega, E)$,
4. $P(A)\nu = \nu^A$, *for every* $\nu \in \mathrm{M}(X, \Omega, E)$, *where* $\nu^A(B) = \nu(A \cap B)$ *for every* $B \in \Omega$.

In this case, α is positive and P is a positive probability measure. Moreover:

1. *if μ is positive, then τ is positive,*
2. *if $\mu(\Omega) \subseteq Hom(F, E)$, then τ is a Riesz homomorphism.*

Dilation of operator valued measures

$$\begin{array}{ccc} F & \xrightarrow{\mu(A)} & E \\ \tau \downarrow & & \uparrow \alpha \\ \mathrm{M}(X,\Omega,E) & \xrightarrow{P(A)} & \mathrm{M}(X,\Omega,E) \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\mu(A)} & \mu(A)x \\ \tau \downarrow & & \uparrow \alpha \\ \mu_x & \xrightarrow{P(A)} & \mu_x^A \end{array}$$

$$P(A\cap B)\nu = \nu^{A\cap B} = P(A)\nu^B = P(A)P(B)\nu. \qquad \mu_x(B) = \mu(B)x$$

$$\alpha(\nu) = \nu(X) \qquad \mu_x^A(B) = \mu(A\cap B)x$$

Remark

P is positive and $P(X) = \mathbf{I} \Longrightarrow$ each idempotent $P(A) \leq \mathbf{I}$.

Dilation of operator valued measures

Something more about the dilation spectral measure

The dilated spectral measure $P\colon \Omega \to \mathrm{L_r}(\mathrm{M}(X, \Omega, E))$ satisfies

$$P(A \cap B) = P(A)P(B) = P(A) \wedge P(B),$$

$$P(A \cup B) = P(A) \vee P(B).$$

The integral operator generated by such a spectral measure is a Riesz homomorphism as well.

Definition of order integral w.r.t. a positive measure μ:

- If $\varphi = \sum_{i=1}^{n} r_i \chi_{A_i}$ is an elementary function, where the A_i are pairwise disjoint, then we define its order integral, which is an element of $\overline{E^+}$, by

$$\int_X^{\mathrm{o}} \varphi \,\mathrm{d}\mu := \sum_{i=1}^{n} r_i \mu(A_i)$$

- If $f : X \to \mathbb{R}^+ \cup \{\infty\}$ is measurable, then there exists a sequence $\{\varphi_n\}_{n=1}^{\infty}$ of elementary functions such that $\varphi_n \uparrow f$ pointwise in $\mathbb{R}^+ \cup \{\infty\}$
- We define the (order) integral of f, which is an element of $\overline{E^+}$, by

$$\int_X^{\mathrm{o}} f \,\mathrm{d}\mu := \bigvee_{n=1}^{\infty} \int_X^{\mathrm{o}} \varphi_n \,\mathrm{d}\mu.$$

- This is well defined

Integration

Definition of order integral w.r.t. a positive measure μ:

- If $f : X \to \mathbb{R}$ is measurable, write $f = f^+ - f^-$
- If $\int_X^{\mathrm{o}} f^+ \,\mathrm{d}\mu \in E^+$ and $\int_X^{\mathrm{o}} f^- \,\mathrm{d}\mu \in E^+$, define

$$\int_X^{\mathrm{o}} f \,\mathrm{d}\mu := \int_X^{\mathrm{o}} f^+ \,\mathrm{d}\mu - \int_X^{\mathrm{o}} f^- \,\mathrm{d}\mu$$

- $\mathcal{L}^1(\mu)$

Integration

Definition of order integral w.r.t. a positive measure μ:

- If $f : X \to \mathbb{R}$ is measurable, write $f = f^+ - f^-$
- If $\int_X^{\mathrm{o}} f^+ \,\mathrm{d}\mu \in E^+$ and $\int_X^{\mathrm{o}} f^- \,\mathrm{d}\mu \in E^+$, define

$$\int_X^{\mathrm{o}} f \,\mathrm{d}\mu := \int_X^{\mathrm{o}} f^+ \,\mathrm{d}\mu - \int_X^{\mathrm{o}} f^- \,\mathrm{d}\mu$$

- $\mathcal{L}^1(\mu)$

Definition of order integral w.r.t. a signed measure $\mu \in \mathrm{M}(X, \Omega, E)$:

- $\mu = \mu^+ - \mu^-$
- $\mathcal{L}^1(\mu) = \mathcal{L}^1(X, \Omega, \mu^+; \mathbb{R}) \cap \mathcal{L}^1(X, \Omega, \mu^-; \mathbb{R})$
- $\int_X^{\mathrm{o}} f \,\mathrm{d}\mu := \int_X^{\mathrm{o}} f \,\mathrm{d}\mu^+ - \int_X^{\mathrm{o}} f \,\mathrm{d}\mu^-$

Theorem (M.de Jeu, X. Jiang)

Let E be a Dedekind complete Riesz space, (X,Ω) be a measurable space. Then for any $\mu \in \mathrm{M}(X,\Omega,E)$, $\mathcal{L}^1(\mu)$ is a σ-Dedekind complete Riesz space and the order integral $\int_X^{\mathrm{o}} \cdot \, \mathrm{d}\mu \colon \mathcal{L}^1(\mu) \to E$ is a σ-order continuous operator.

- $\mathcal{B}(X) \subseteq \mathcal{L}^1(\mu)$ for any $\mu \in \mathrm{M}(X,\Omega,E)$.
- If μ is positive, then the order integral w.r.t. μ is a positive operator.
- If $\mu(A \cap B) = \mu(A) \wedge \mu(B)$ for all $A, B \in \Omega$, then the order integral w.r.t to μ is a Riesz homomorphism.
- If $P \colon \Omega \to \mathrm{L}_{\sigma c}(E)$ is a spectral measure, then the order integral w.r.t. P is an algebra homomorphism.

- Let $\mathcal{I}_\mu(f) = \int_X^{\mathrm{o}} f \,\mathrm{d}\mu$ for every $f \in \mathcal{B}(X)$
- Define $\mathcal{I}\colon \mathrm{M}(X, \Omega, E) \to \mathrm{L}_{\sigma\mathrm{c}}(\mathcal{B}(X), E)$ by $\mathcal{I}(\mu) = \mathcal{I}_\mu$ for each $\mu \in \mathrm{M}(X, \Omega, E)$.
- $\mathcal{I}$ is positive and linear.

Theorem (M. de Jeu, X. Jiang)

Let E be a Dedekind complete Riesz space and (X, Ω) a measurable space. Then for any $T \in \mathrm{L}_{\sigma\mathrm{c}}(\mathcal{B}(X),\ E)$, there exists a unique $\mu \in \mathrm{M}(X, \Omega, E)$ such that $T = \mathcal{I}_\mu$. Furthermore, if T is positive, so is μ.

Corollary (M. de Jeu, X. Jiang)

$\mathcal{I}\colon \mathrm{M}(X, \Omega, E) \to \mathrm{L}_{\sigma\mathrm{c}}(\mathcal{B}(X), E)$ *is a Riesz isomorphism.*

Dilation of T

We are interested in the dilation of regular operator

$$T : \mathcal{B}(X) \to \mathrm{L_r}(F, E)$$

where F and E are Riesz spaces.

Dilation of T

We are interested in the dilation of regular operator

$$T\colon \mathcal{B}(X) \to \mathrm{L_r}(F, E)$$

where F and E are Riesz spaces.

- Dilation system of T: $(\mathrm{L_r}(\mathcal{B}(X), E), \pi, \tau_T, \alpha_T)$

Dilation of T

We are interested in the dilation of regular operator

$$T\colon \mathcal{B}(X) \to \mathrm{L_r}(F, E)$$

where F and E are Riesz spaces.

- Dilation system of T: $(\mathrm{L_r}(\mathcal{B}(X), E), \pi, \tau_T, \alpha_T)$
- E is Dedekind complete, T is σ-order continuous, there exists $\mu \in \mathrm{M}(X, \Omega, E)$ such that $T = \mathcal{I}_\mu$.

Dilation of T

We are interested in the dilation of regular operator

$$T\colon \mathcal{B}(X) \to \mathrm{L_r}(F, E)$$

where F and E are Riesz spaces.

- Dilation system of T: $(\mathrm{L_r}(\mathcal{B}(X), E), \pi, \tau_T, \alpha_T)$
- E is Dedekind complete, T is σ-order continuous, there exists $\mu \in \mathrm{M}(X, \Omega, E)$ such that $T = \mathcal{I}_\mu$.
- Dilation system of μ: $(\mathrm{M}(X, \Omega, E), P, \tau_\mu, \alpha_\mu)$

Dilation of T

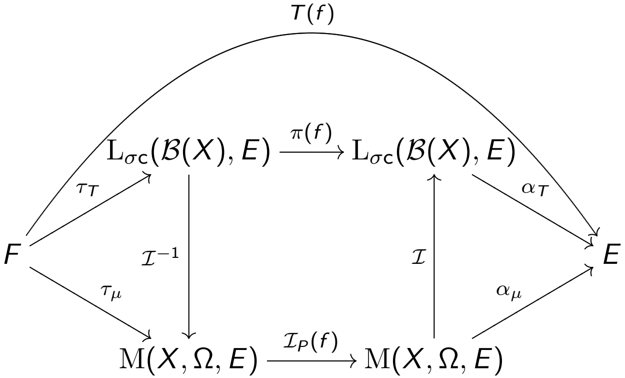
We are interested in the dilation of regular operator

$$T\colon \mathcal{B}(X) \to \mathrm{L_r}(F, E)$$

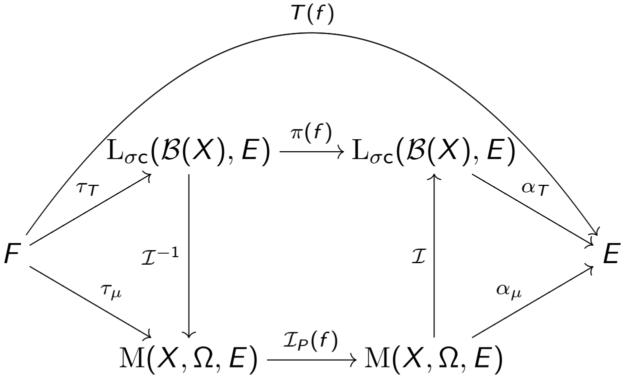
where F and E are Riesz spaces.

- Dilation system of T: $(\mathrm{L_r}(\mathcal{B}(X), E), \pi, \tau_T, \alpha_T)$
- E is Dedekind complete, T is σ-order continuous, there exists $\mu \in \mathrm{M}(X, \Omega, E)$ such that $T = \mathcal{I}_\mu$.
- Dilation system of μ: $(\mathrm{M}(X, \Omega, E), P, \tau_\mu, \alpha_\mu)$
- How are they related? Something more about the dilated operator π?

If $T\colon \mathcal{B}(X) \to \mathrm{L}_{\sigma c}(F, E)$ is a σ-order continuous operator, then the following digram commutes.



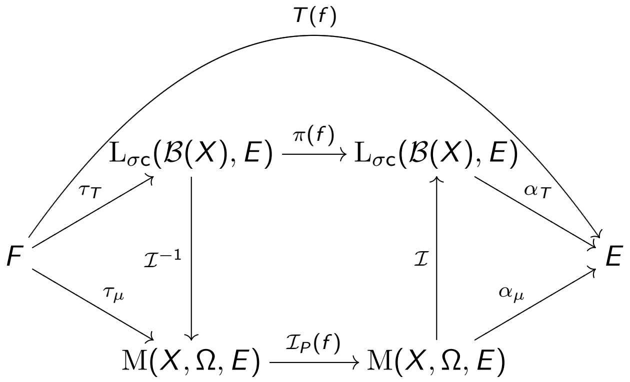
If $T: \mathcal{B}(X) \to \mathrm{L}_{\sigma c}(F, E)$ is a σ-order continuous operator, then the following digram commutes.



- $\mathcal{I}_P(f)\nu = fd\nu: A \to \mathcal{I}_\nu(f\chi_A)$, for any $A \in \Omega$, $f \in \mathcal{B}(X)$, $\nu \in \mathrm{M}(X, \Omega, E)$.

Relation between dilation systems

If $T\colon \mathcal{B}(X) \to \mathrm{L}_{\sigma c}(F, E)$ is a σ-order continuous operator, then the following digram commutes.



- $\mathcal{I}_P(f)\nu = fd\nu\colon A \to \mathcal{I}_\nu(f\chi_A)$, for any $A \in \Omega$, $f \in \mathcal{B}(X)$, $\nu \in \mathrm{M}(X, \Omega, E)$.
- $\mathcal{I}_P$ is Riesz homomorphism $\Longrightarrow$ π is Riesz homomorphism.

Questions

- $\mathrm{C}(K)$, $\mathrm{C}_0(X)$
- Minimal dilation system (order basis, positive frame,...)
- Lattice embeddings (an operator τ that is a lattice homomorphism)
- Banach space, Banach lattice,

Thank you

References:

1. J.D.M. Wright. Stone-algebra-valued measures and integrals. Proc. London Math. Soc. 19, 107-122. (1969)
2. M. de Jeu, X. Jiang. Order integrals. Positivity 26, no. 2, Paper No. 32 (2022)
3. M.de Jeu, X. Jiang. Signed measures and full Riesz representation theorems (In preparation)
4. M.de Jeu, X. Jiang. Riesz representation theorems for positive algebra homomorphisms. arXiv:2109.10690. (2022)
5. B. Liu, D. Han, D. R. Larson and R. Lui. Operator-valued measures, Dilations, and the Theory of Frames, Memoirs of AMS, volume 229 (2014)