The Huijsmans–de Pagter problem in ordered Banach algebras

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More than 30 years ago, Huijsmans and de Pagter posed the following question:

If T is a positive operator on a complex Banach lattice E with the spectrum $\sigma(T) = \{1\}$, does it follow that T is greater than or equal to the identity operator I?

This question was studied by several authors.

In the finite-dimensional case the answer to HdP question is affirmative. One of the proofs is the following.

Since $T - I$ is nilpotent, we have $\mathrm{tr}((T-I)^2) = 0$. If the matrix of T is $[t_{ij}]_{i,j=1}^n$, then we obtain that

$$\sum_{i=1}^{n} (t_{ii} - 1)^2 + \text{ nonnegative terms } = 0 \ .$$

It follows that $t_{ii} = 1$ for all i, showing that $T \geq I$.

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Theorem (Zhang, 1993)

The answer to HdP question is affirmative if we assume in addition that there exist $\alpha \in (0, \frac{1}{2})$ *and a constant* $c \geq 0$ *such that* $\|T^{-n}\| = O(\exp(cn^{\alpha}))$ *as* $n \to \infty$.

Theorem (Drnovšek, 2007)

A positive operator T *on* E *is greater than or equal to the identity operator* I *provided*

$$\lim_{n\to\infty} n\|(T-I)^n\|^{1/n} = 0 \,.$$

In the proof of this theorem we define the operator valued entire function that is of minimal type with respect to the order 1 and is bounded on the real axis. We then use the Phragmén-Lindelöf theorem to conclude that the function is necessarily constant.

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In general this problem that is important for the spectral theory of positive operators is still open.

30 years ago, Zhang also proved the following theorem.

Theorem (Zhang, 1993)

Let T be a positive operator on a complex Banach lattice E with the spectrum $\sigma(T) = \{1\}$. If $\varepsilon \in (0,1)$, then there exists a positive integer n such that

$$T^n \geq (1-\varepsilon)^n I \, .$$

In 2003, Mouton investigated this question in the context of ordered Banach algebras.

Let $\mathscr{A}$ be a complex Banach algebra with unit e. A nonempty set C is called a *cone* of $\mathscr{A}$ if $C + C \subseteq C$ and $\lambda C \subseteq C$ for all $\lambda \geq 0$. If, in addition, $C \cap (-C) = \{0\}$, then C is said to be a *proper* cone. A cone C of $\mathscr{A}$ is *closed* if it is a closed subset of $\mathscr{A}$.

Any proper cone C induces an ordering $\leq$ in the following way:

$$a \leq b \iff b - a \in C.$$

It is easy to see that this ordering is a partial order (reflexive, antisymmetric, and transitive). Clearly, $C = \{a \in \mathscr{A} : a \geq 0\}$, and so elements of C are called *positive*.

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A cone C of $\mathscr{A}$ is *normal* if there exists a constant $\alpha \geq 1$ such that it follows from $0 \leq a \leq b$ that $\|a\| \leq \alpha \|b\|$. If we can take $\alpha = 1$, then we say that the norm is *monotone*.

A cone C is called an *algebra cone* of $\mathscr{A}$ if $C \cdot C \subseteq C$ and $e \in C$. In this case $\mathscr{A}$ is called an *ordered Banach algebra*.

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Let $\mathscr{B}$ be a complex unital Banach algebra with unit e, and let $\mathscr{A}$ be the algebra $\mathscr{B} \times \mathbb{C}$ endowed with multiplication $(a, \xi) \cdot (b, \eta) = (ab, \xi\eta)$. If we define the norm on the algebra $\mathscr{A}$ by $\|(a, \xi)\| = \max\{\|a\|, |\xi|\}$, then $\mathscr{A}$ becomes a unital complex Banach algebra with the unit $(e, 1)$.

Observe that $\sigma((a, \xi)) = \sigma(a) \cup \{\xi\}$ for all $a \in \mathscr{B}$ and $\xi \in \mathbb{C}$. Furthermore, if $\mathscr{B}$ is a C^*-algebra, then $\mathscr{A}$ is also a C^*-algebra with the involution defined by $(a, \xi)^* = (a^*, \overline{\xi})$.

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Lemma

The Banach algebra $\mathscr{A}$ is an ordered Banach algebra with the algebra cone

$$K = \{(a,\xi) \in \mathscr{A} : \|a\| \leq \xi\}$$

that is proper, closed and normal. Furthermore, if $e \neq a \in \mathscr{B}$, $\|a\| = 1$ and $\sigma(a) = \{1\}$, then $(a,1) \in K$, $\sigma((a,1)) = \{1\}$ and $(a,1) - (e,1) \notin K$.

Proof.

It is easy to see that K is an algebra cone that is proper and closed. To show its normality, assume $0 \leq (a,\xi) \leq (b,\eta)$, so that $\|a\| \leq \xi$ and $\|b - a\| \leq \eta - \xi$. Then $\|a\| \leq \xi \leq \eta$, and so $\|(a,\xi)\| \leq \eta \leq \|(b,\eta)\|$. This shows that K is a normal cone. Assume that $e \neq a \in \mathscr{B}$, $\|a\| = 1$ and $\sigma(a) = \{1\}$. Then $(a,1) \in K$, $\sigma((a,1)) = \sigma(a) \cup \{1\} = \{1\}$ and $(a,1) - (e,1) = (a - e, 0) \notin K$, since $\|a - e\| > 0$. □

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The ordered Banach algebra $\mathscr{A}$ was used in the literature to prove some theorems for elements of $\mathscr{B}$ by working in $\mathscr{A}$.

Theorem (Drnovšek, 2018)

There exist an ordered Banach algebra $\mathscr{A}$ with a closed and normal algebra cone and a positive element $a \in \mathscr{A}$ such that $\sigma(a) = \{1\}$ and a is not greater than or equal to the unit element of $\mathscr{A}$.

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Proof.

Let $\mathscr{B}$ be the Banach algebra of all bounded linear operators on the Hilbert space $L^2[0,1]$, and let $\mathscr{A} = \mathscr{B} \times \mathbb{C}$ be an ordered Banach algebra as defined above.

Let V be the Volterra operator on $L^2[0,1]$, that is, the operator defined by $(Vf)(x) = \int_0^x f(y)\,dy \quad (f \in L^2[0,1],\ x \in [0,1])$.

Since $\sigma(V) = \{0\}$, the operator $T = (I+V)^{-1}$ has the spectrum $\sigma(T) = \{1\}$, and it is not equal to the identity operator I.

Let us prove that $\|T\| = 1$. The inequality $\|Tf\| \leq \|f\|$ for $f \in L^2[0,1]$ is equivalent to the inequality $\|T^{-1}g\| \geq \|g\|$ for $g \in L^2[0,1]$, that is, $\|(I+V)g\| \geq \|g\|$. Now, we have

$$\|(I+V)g\|^2 = \|g\|^2 + \langle (V+V^*)g, g\rangle + \|Vg\|^2 \geq \|g\|^2 \ ,$$

since the operator $V+V^*$ is the projection onto one-dimensional subspace of constant functions.

Then it follows from Lemma that $(T,1) \in K$, $\sigma((T,1)) = \{1\}$ and $(T,1) - (I,1) \notin K$. □

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Infinite-dimensionality of the Hilbert space $L^2[0,1]$ is essential in the proof of Theorem, as we have the following observation.

Proposition

Let $\mathscr{B}$ be the Banach algebra of all linear operators on a finite-dimensional Hilbert space, and let $\mathscr{A} = \mathscr{B} \times \mathbb{C}$ be an ordered Banach algebra as defined above. If $(A, \xi) \in K$ with $\sigma((A,\xi)) = \{1\}$, then (A,ξ) is equal to the unit element $(I, 1)$ of $\mathscr{A}$.

Proof.

Since $\sigma((A,\xi)) = \{1\}$ and $\|A\| \leq \xi$, we have $\sigma(A) = \{1\}$, $\xi = 1$, $\|A\| = 1$, and A is unitarily equivalent to triangular matrix that has only 1's on the diagonal. Its norm can be 1 only in the case when $A = I$. □

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Theorem

Let $\mathscr{A}$ be an ordered Banach algebra with a closed and normal algebra cone C. If $a \in C$ then $r(a) \in \sigma(a)$.

This theorem follows from

Theorem (Raubenheimer-Rode, 1995)

Let $\mathscr{A}$ be an ordered Banach algebra with a closed algebra cone C. Assume that the spectral radius is monotone on C, i.e., if $0 \leq a \leq b$, then $r(a) \leq r(b)$. If $a \in C$ then $r(a) \in \sigma(a)$.

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We may assume that $r(a) = 1$. Suppose that $1 \notin \sigma(a)$. Choose $\alpha \in (0,1)$ such that $\sigma(a) \subseteq \{\lambda \in \mathbb{C} : \mathrm{Re}\,\lambda \leq \alpha\}$. Given $t > 0$, the spectral mapping theorem implies that

$$\sigma(e^{ta}) = e^{t\sigma(a)} \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq e^{t\alpha}\}.$$

Since $a \in C$ and C is closed, we have

$$e^{ta} = 1 + ta + \frac{t^2}{2!}a^2 + \ldots \in C,$$

so that

$$0 \leq \frac{t^n a^n}{n!} \leq e^{ta}$$

for all n and for all $t > 0$. It follows that

$$\frac{t^n}{n!} = r\left(\frac{t^n a^n}{n!}\right) \leq e^{t\alpha} .$$

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$$0 \leq \frac{t^n a^n}{n!} \leq e^{ta}$$

for all n and for all $t > 0$. It follows that

$$\frac{t^n}{n!} = r\left(\frac{t^n a^n}{n!}\right) \leq e^{t\alpha} \ .$$

Proof continued.

Putting $t = \frac{n}{\alpha}$, we obtain that

$$\frac{n^n}{\alpha^n \, n!} \leq e^n,$$

and so

$$\frac{1}{n!}\left(\frac{n}{e}\right)^n \leq \alpha^n.$$

Now, we recall Stirling's formula

$$\lim_{n\to\infty} \frac{\sqrt{2\pi n}}{n!}\left(\frac{n}{e}\right)^n = 1.$$

It follows that

$$1 \leq \lim_{n\to\infty} \sqrt{2\pi n} \cdot \alpha^n = 0 \ .$$

A contradiction. □

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