Direct limits in the category of Banach lattices and almost interval preserving contractions

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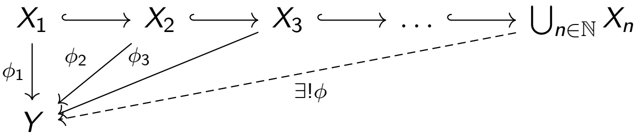
Workshop on Ordered Vector Spaces and Positive Operators
Wuppertal, March 31, 2023

Intuition on direct limits

Example

Suppose (X_n) is an increasing sequence of subspaces of a linear space X, then the union $\bigcup_{n\in\mathbb{N}} X_n$ has a universal property: for any linear space Y and a set of linear maps ϕ_n that form a commutative diagram, there exists a unique linear map ϕ that can fill in the dashed arrow.



Denfinition of direct limits

Direct system
$((O_i), (\phi_{ji})_{j \geq i})$

$$\begin{array}{ccc} O_i & \xrightarrow{\phi_{ji}} & O_j \\ & \underset{\phi_{ki}}{\searrow} & \downarrow \phi_{kj} \\ & & O_k \end{array}$$

Target
$(O, (\phi_i))$

$$\begin{array}{ccc} O_i & \xrightarrow{\phi_{ji}} & O_j \\ & \underset{\phi_i}{\searrow} & \downarrow \phi_j \\ & & O \end{array}$$

Direct limit
$(O^u, (\phi_i^u))$

$$\begin{array}{ccc} O_i & \xrightarrow{\phi_i} & O \\ & \underset{\phi_i^u}{\searrow} & \downarrow \exists! \phi \\ & & O^u \end{array}$$

Standard construction in the category of Banach spaces and contractions

Let $((O_i), (\phi_{ji})_{j \geq i})$ be a direct system in the category of Banach spaces and contractions, then

$$\phi_i^u(x) = (\phi_{ji}(x))/\sim \quad \in \prod_i O_i / \bigoplus_i O_i$$

$$O^u = \overline{\bigcup_i \phi_i^u(O_i)}$$

is a direct limit, where

$$\prod_i O_i := \{(x_i) : x_i \in O_i, \sup_i \|x_i\| < \infty\}$$

$$\bigoplus_i O_i := \{(x_i) : x_i \in O_i, \|x_i\| \to 0 \text{ as } i \to \infty\}$$

and we take the convention that $\phi_{ji} = 0$ if $j \ngeq i$

Category of Banach lattices and almost interval preserving contractions

Category $\mathbf{AIP_1}$
obejects: Banach lattices
morphisms: almost interval preserving contractions

Definition

A positive linear map $\phi : E \to F$ between Banach lattices is *almost interval preserving* if

$$\overline{\phi([0,x])} = [0, \phi(x)] \quad (x \geq 0)$$

Example

1. a lattice homomorphism with a dense image
2. $\mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, y) \mapsto \frac{x+y}{2}$

Dual relation between lattice homomorphisms and almost interval preserving maps

Theorem

Let $T : E \to F$ be a coutinuous linear operator between Banach lattices, then
(a) T is almost interval preserving iff the self-adjoint operator $T^* : F^* \to E^*$ between spaces of continuous functionals is a lattice homomorphism.
(b) T is a lattice homomorphism iff T^* is (almost) interval preserving.

Standard construction may not work

1. If every morphism in a direct system in $\mathbf{AIP_1}$ is a lattice homomorphism, then the standard construction of a direct limit in the Banach spaces category still work in $\mathbf{AIP_1}$.
2. Let $E_i = \ell^p \quad (i \in \mathbb{N})$,

$$\phi_{i+1,i} : E_i \to E_{i+1},$$
$$(x_1, x_2, \dots) \mapsto (x_1, \dots, x_{i-1}, \frac{x_i + x_{i+1}}{2}, x_{i+2}, x_{i+3}, \dots),$$

and ϕ_{ji} be the composition of $\phi_{j,j-1}, \phi_{j-1,j-2}, \dots, \phi_{i+1,i}$. Then $((E_i), (\phi_{ji})_{j \geq i})$ is a direct system in $\mathbf{AIP_1}$. The standard construction of a direct limit does not work when $p = \infty$.
Main reason: the pair (f, g) of almost interval preserving maps f and g may fail to be almost interval preserving.

Order continuity

A Banach lattice is said to be *order continuous* if each order convergent net in the Banach lattice is norm convergent.

Theorem

A Banach lattice is order continuous iff the canonical embedding to its second dual is (almost) interval preserving.

Invariant property of direct limits

Theorem

Let $(E, (\phi_i))$ be a direct limit of a direct system $((E_i), (\phi_{ji})_{j \geq i})$ in $\mathbf{AIP_1}$. *If each E_i is order continuous, so is E.*

Key points of proof.

1. A Banach lattice is order continuous iff the canonical embedding to its second dual is (almost) interval preserving.
2. A linaer map between Banach spaces is almost interval preserving iff its doulble adjoint is.
3.

$$\begin{array}{ccc} E_i & \xrightarrow{\ \phi_i\ } & E \\ {\scriptstyle \iota_i}\downarrow & & \downarrow{\scriptstyle \iota} \\ E_i^{**} & \xrightarrow{\ \phi_i^{**}\ } & E^{**} \end{array}$$

□

Application

A *Banach function space* over a measure space is an order ideal of the space of all equivalence classes of all measurable functions supplied with a Banach lattice norm.

Corollary

Let X be a metric space and μ a measure on all Borel sets of X. If E is a Banach function space over (X, μ) and $C_c(X)$ is continuously included in and dense E, then E is order continuous.

Proof.

$C_c(X)$ is dense in E
$\Rightarrow C(K)$ is dense in $E_K = \{f\chi_K : f \in E\}$ for any compact subset K of X
$\Rightarrow E_K$ is separable and thus order continuous
$\Rightarrow E = \bigcup_{K \text{ compact}} E_K$, being a direct limit of (E_K), is order continuous. □

Thank you!

Summary

1. The existence of a direct limit in the category $\mathbf{AIP_1}$ *of Banach lattices and almost interval preserving contractions remains unclear.*
2. A direct limit in $\mathbf{AIP_1}$ *preserves the order continuity property.*

Paper available on arXiv:
Direct limits in categories of normed vector lattices and Banach lattices