Free objects in algebraic and analytic categories

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Background and introduction

'Categorical' questions asked at workshop on ordered Banach algebras in Leiden in 2014

- Is there a sensible notion of a free Banach lattice algebra over a set?
- Is there a unitisation of a vector lattice algebra?
- Nobody knew. . .
- Included in a list of problems published by Wickstead in 2017

Since then

- Took this up in 2018: algebraic and analytic part
- Analytic part: joint with Walt van Amstel (Pretoria)
- Today's message: *many such problems are easy to answer*
- Key ingredient: *universal algebra* 'which we should have known'
- Free Banach lattice algebra over a non-empty set: does not exist
- Unitisation of vector lattice algebra: exists

Outline

- Vector lattice algebras and Banach lattice algebras
- Categories and free objects
- Examples of (non-)existing free objects
- Universal algebra: Part I
- Familiar (ordered) structures are abstract algebras with relations
- Universal algebra: Part II
- Existence of free objects in several (ordered) algebraic categories
- Examples of 'concrete models' of free objects in algebraic categories
- Free objects in analytic (ordered) categories 'with bounds'; inverse limits

Guiding examples in the algebraic part: free unital vector lattice algebra over a set and over a vector space where 'familiar constructions will not help you'

Vector lattice algebras and Banach lattice algebras

Vector lattice algebra (Riesz algebra): definition and examples

- An associative algebra A (over $\mathbb{R}$) that is a vector lattice, such that $a, b \geq 0 \Rightarrow ab \geq 0$
- Vector lattice algebras of functions with pointwise operations and ordering
- The regular operators on order complete vector lattices

Banach lattice algebra: definition and examples

- A Banach algebra A that is also a Banach lattice, such that $a, b \geq 0 \Rightarrow ab \geq 0$.
- The bounded continuous functions on a topological space
- The regular operators on an order complete Banach lattice, when supplied with the regular norm $T \mapsto |||T|||$

Some algebraic and analytic categories

Algebraic categories

- Set: the sets with set maps
- VS: the (real) vector spaces with linear maps
- VL: the vector lattices with vector lattice homomorphisms
- VLA: the vector lattice algebras with vector lattice algebra homomorphisms (algebra homomorphism and vector lattice homomorphism)
- VLA^1: the unital vector lattice algebras with the unital vector lattice algebra homomorphisms

Analytic categories

- BS: the Banach spaces with bounded linear maps
- BA: the Banach algebras with bounded algebra homomorphisms
- BLA: the Banach lattice algebras with the vector lattice algebra homomorphisms (automatically bounded)

Free objects

Definition

Suppose that Cat_1 and Cat_2 are categories, and that $U\colon \mathsf{Cat}_2 \mapsto \mathsf{Cat}_1$ is a faithful functor.[a] Take an object O_1 of Cat_1. Then a *free object over O_1 of Cat_2 with respect to U* is a pair $(j, \mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1])$, where $\mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]$ is an object of Cat_2 and $j : O_1 \to \mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]$ is a morphism of Cat_1, with the property that, for every object O_2 of Cat_2 and every morphism $\varphi : O_1 \to O_2$ of Cat_1, there exists a unique morphism $\overline{\varphi} : \mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1] \to O_2$ of Cat_2 such that the diagram

$$
\begin{array}{ccc}
O_1 & \xrightarrow{\ j\ } & U\left(\mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1]\right) \\
 & \searrow^{\varphi} & \downarrow U(\overline{\varphi}) \\
 & & U(O_2)
\end{array}
$$

in Cat_1 is commutative.

[a] Recall that U is *faithful* when the associated map $U\colon \mathrm{Hom}_{\mathsf{Cat}_2}(O_2, O_2') \to \mathrm{Hom}_{\mathsf{Cat}_1}(U(O_2), U(O_2'))$ is injective for all objects O_2, O_2' of Cat_2

Comments

- A pair $(j, \mathsf{F}_{\mathsf{Cat}_1}^{\mathsf{Cat}_2}[O_1])$ as in the definition need not exist.
- A free object over O_1 of Cat_2 with respect to U, if it exists, is determined up to an isomorphism of Cat_2
- So: speak of 'the' free object over O_1 (the morphism j being understood)

Simplification in our case

- The categories are always categories of sets (Set, VLA, BA, . . .).
- The functor U is always the obvious forgetful functor (faithful)
- *Will omit U from the notation*

Example: unitisation is a free object

Take $\mathsf{Cat}_1 = \mathsf{VLA}$ and $\mathsf{Cat}_2 = \mathsf{VLA}^1$ with forgetful functor U

Let A be an object of VLA. A free object over A of VLA^1 is a pair $(j, \mathsf{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[A])$, where $\mathsf{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[A]$ is an object of VLA^1 and $j : A \to \mathsf{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[(] A)$ is a vector lattice algebra homomorphism, with the property that, for every object A^1 of VLA^1 and every vector lattice algebra homomorphisms $\varphi : A \to A^1$, there exists a unique unital vector lattice algebra homomorphism $\overline{\varphi} : \mathsf{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[A] \to A^1$ such that the diagram is commutative.

$$\begin{array}{ccc} A & \xrightarrow{\;j\;} & \mathsf{F}_{\mathsf{VLA}}^{\mathsf{VLA}^1}[A] \\ & \underset{\varphi}{\searrow} & \downarrow \overline{\varphi} \\ & & A^1 \end{array}$$

Comments

- $\mathbb{R} \oplus A$ is a unital vector lattice algebra, but this does not work!
- Still: a unitisation exists (see later)

Example: no free Banach algebras over non-empty sets

Example: take $\mathsf{Cat}_1 = \mathsf{Set}$ and $\mathsf{Cat}_2 = \mathsf{BA}$

Let $S \neq \emptyset$. There exist *no* Banach algebra $\mathrm{F}_{\mathsf{Set}}^{\mathsf{BA}}[S]$ and map $j : S \to \mathrm{F}_{\mathsf{Set}}^{\mathsf{BA}}[S]$ such that, for every Banach algebra A and every map $\varphi : S \to A$, there exists a unique continuous algebra homomorphism $\overline{\varphi}$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\;j\;} & \mathrm{F}_{\mathsf{Set}}^{\mathsf{BA}}[S] \\ & \searrow^{\varphi} & \downarrow{\scriptstyle\overline{\varphi}} \\ & & A \end{array}$$

is commutative. Reason: take $A = \mathbb{R}$. Fix some $s \in S$ and, for every x in $\mathbb{R}$, take a map $\varphi_x : S \to \mathbb{R}$ such that $\varphi_x(s) = x$. The fact that $\overline{\varphi}_x$ is an algebra homomorphism implies that

$$|x|^n \leq \|\overline{\varphi}_x\| \, \|j(s)\|^n$$

for every $n \geq 0$. If $x \neq 0$ then $\overline{\varphi_x} \neq 0$, so letting $n \to \infty$ shows that $\|j(s)\| \geq |x|$ for all $x \neq 0$. This is impossible.

Bounds can be needed

Comments

- No free Banach lattice algebras over non-empty sets or over non-zero Banach spaces (same argument)
- No free Banach spaces over infinite sets (other argument)

Way out (well, not really)

- Impose bounds: consider only $\varphi : S \to A$ such that $\|\varphi(s)\| \leq M(s)$ for some fixed $M : S \to \mathbb{R}_{\geq 0}$.
- Then there *is* a 'solution' for such a fixed M (see later)
- *Not* a free object in our sense because not *all* φ are allowed

Way out (see later)

- Enlarge Cat_2 to a category of inverse limits, and use the 'solutions' for various M as building blocks for an honest free object in the *larger* Cat_2
- For $\mathsf{Cat}_2 = \mathsf{BA}$: larger category is the category of complete locally convex topological algebras

Free objects in algebraic categories

Take $\mathsf{Cat}_1 = \mathsf{Set}$ and $\mathsf{Cat}_2 = \mathsf{VS}$: free vector space over a set

Let $S \neq \emptyset$. Is there a pair $(j, \mathrm{F}_{\mathsf{Set}}^{\mathsf{VS}}[S])$, where $\mathrm{F}_{\mathsf{Set}}^{\mathsf{VS}}[S]$ is a vector space and $j : S \to \mathrm{F}_{\mathsf{Set}}^{\mathsf{VS}}[S]$ is a map, with the property that, for every vector space V and map $\varphi : S \to V$, there exists a unique linear map $\overline{\varphi} : \mathrm{F}_{\mathsf{Set}}^{\mathsf{VS}}[S] \to V$ such that the following diagram is commutative?

$$\begin{array}{ccc} S & \xrightarrow{\ j\ } & \mathrm{F}_{\mathsf{Set}}^{\mathsf{VS}}[S] \\ & {\scriptstyle\varphi}\searrow & \big\downarrow {\scriptstyle\overline{\varphi}} \\ & & V \end{array}$$

Two solutions

- A vector space with basis indexed by S would solve the problem. There is one: the functions on S with finite support. Clever!
- ‘Formal combinations $\sum_{s \in S} \lambda_s s$ with the obvious operations’
- The second solution is actually in the vein of the solution for a ‘free abstract algebra in an equational class over a set’: consists of equivalence classes of strings

Abstract algebras of a given type

Suppose that $\mathcal{F} \neq \emptyset$ and that $\rho : \mathcal{F} \to \mathbb{N}_0$. Then the pair $(\mathcal{F}, \rho)$ is called a *type*. Let $A \neq \emptyset$ and suppose that, for each $f \in \mathcal{F}$, the following is given:

1. when $\rho(f) = 0$: an element f^A of A;
2. when $\rho(f) \geq 1$: a map $f^A : A^{\rho(f)} \to A$.

We set $\mathcal{F}^A := \{ f^A : f \in \mathcal{F} \}$. The pair $\langle A, \mathcal{F}^A \rangle$ is then called an *abstract algebra of type* $(\mathcal{F}, \rho)$. The elements of $\mathcal{F}$ are called *operation symbols*.
If $\rho(f) = 0$, then f^A is called a *constant of* A.
If $\rho(f) \geq 1$, then f^A is called an *operation on* A (taking $\rho(f)$ arguments).

Think of $(\mathcal{F}, \rho)$ as informing us how many distinguished elements (constants) there are in A, and what the numbers of variables are that the operations on A take.

Associating abstract algebras to familiar structures (in simplified notation)

- Example: vector space V
- Have naturally associated abstract algebra $\langle V, \{0, \oplus, \ominus, \{ m_\lambda : \lambda \in \mathbb{R} \}\}\rangle$.
- $\mathcal{F}$ is left unspecified.
- To the elements of $\mathcal{F}$ correspond a constant 0 of V, an obvious binary operation $\oplus$, a unary operation $\ominus$ that sends $x \in V$ to $-x$, and, for every $\lambda \in \mathbb{R}$, a unary operation m_λ that sends $x \in V$ to λx
- $\rho : \mathcal{F} \to \mathbb{N}_0$ is left unspecified
- $\rho(\mathcal{F}) = \{0, 1, 2\}$.

Not every abstract algebra $\langle V, \{0, \oplus, \ominus, \{ m_\lambda : \lambda \in \mathbb{R} \}\}\rangle$ with $\{0, \oplus, \ominus, \{ m_\lambda : \lambda \in \mathbb{R} \}$ as above becomes a vector space when attempting to introduce vector space operations in the obvious way. One has to have $m_{\lambda_1\lambda_2}(x) = m_{\lambda_1}(m_{\lambda_2}(x))$, and more.

Abstract algebra homomorphisms

Let $\langle A, \mathcal{F}^A \rangle$ and $\langle B, \mathcal{F}^B \rangle$ be abstract algebras of the *same* type $(\mathcal{F}, \rho)$. Then $h : A \to B$ is an *abstract algebra homomorphism* when:

1. $h(f^A) = f^B$ for all $f \in \mathcal{F}$ such that $\rho(f) = 0$;
2. $h\big(f^A(a_1, \ldots, a_{\rho(f)})\big) = f^B\big(h(a_1), \ldots, h(a_{\rho(f)})\big)$ for all $f \in \mathcal{F}$ such that $\rho(f) \geq 1$.

Categorical aspects

- Clear: abstract algebras of a given type form and their abstract algebra homomorphisms form a category $\mathsf{AbsAlg}_{(\mathcal{F},\rho)}$
- Will show: if $S \neq \emptyset$, then $\mathsf{F}_{\mathsf{Set}}^{\mathsf{AbsAlg}_{(\mathcal{F},\rho)}}[S]$ exists
- $\mathsf{F}_{\mathsf{Set}}^{\mathsf{AbsAlg}_{(\mathcal{F},\rho)}}[S]$ consists of strings
- It captures the idea of repeatedly applying operations, labelled by $\mathcal{F}$, to their appropriate number of variables labelled by S, thus building operations of ever increasing complexity

Definition of an abstract algebra that contains a set and consists of strings

Let $(\mathcal{F},\rho)$ be a type. Let $S \neq \emptyset$ be disjoint from $\mathcal{F}$. Set

$$T_0(S) := \{\, s : s \in S \,\} \cup \{f \in \mathcal{F} : \rho(f) = 0\}\}.$$

For $n \geq 1$, set

$$T_{n+1}(S) := T_n(S) \cup \{\, f t_1 \dots t_{\rho(f)} : f \in \mathcal{F},\, \rho(f) \geq 1,\, t_1, \dots, t_{\rho(f)} \in T_n(S) \,\}.$$

Define $T_{(\mathcal{F},\rho)}(S) := \bigcup_{n \geq 0} T_n(S)$. The elements of $T_{(\mathcal{F},\rho)}(S)$ are called *terms of type* $(\mathcal{F},\rho)$ *over* S.

If $\rho(f) = 0$, set

$$f^{T_{(\mathcal{F},\rho)}(S)} := f,$$

If $\rho(f) \geq 1$, set

$$f^{T_{(\mathcal{F},\rho)}(S)}(t_1, \dots, t_{\rho(f)}) := f t_1 \dots t_{\rho(f)}$$

for $t_1, \dots, t_{\rho(f)} \in T_{(\mathcal{F},\rho)}(S)$.

This makes the *term algebra* $T_{(\mathcal{F},\rho)}(S)$ into an abstract algebra of type $(\mathcal{F},\rho)$.

Theorem (from the literature)

Let $(\mathcal{F}, \rho)$ be a type, and let $S \neq \emptyset$ be disjoint from $\mathcal{F}$. For every abstract algebra A of type $(\mathcal{F}, \rho)$ and every map $h : S \to A$, there is a unique abstract algebra homomorphism $\overline{h} : T_{(\mathcal{F},\rho)}(S) \to A$ such that $\overline{h}(s) = h(s)$ for all $s \in S$:

$$\begin{array}{ccc} S & \subset & T_{(\mathcal{F},\rho)}(S) \\ & \searrow^{h} & \downarrow \overline{h} \\ & & A \end{array}$$

That is, $\mathrm{F}_{\mathrm{Set}}^{\mathrm{AbsAlg}_{(\mathcal{F},\rho)}}[S]$ exists and is equal to $T_{(\mathcal{F},\rho)}(S)$.

Proof

Every element of $T_{(\mathcal{F},\rho)}(S)$ has an interpretation as a combination of the operation symbols of $\mathcal{F}$, 'applied to the elements of S'. Replace these operation symbols by the corresponding actual operations in A, and every $s \in S$ with $h(s)$. This replacement procedure gives the map $\overline{h}$.

Vector lattice algebras as abstract algebras *with relations*

Vector spaces as abstract algebras 'with relations'

- Take an abstract algebra $\langle V, \{0, \oplus, \ominus, \{ m_\lambda : \lambda \in \mathbb{R} \}\}\rangle$ of the same type as you get from a vector space V
- If the operations happen to be such that $m_{\lambda_1}(m_{\lambda_2}(x)) = m_{\lambda_1\lambda_2}(x)$, $(x \oplus y) \oplus z = x \oplus (y \oplus z)$, $x \oplus \ominus(x) = 0$, etc., then it is clear how to make V into a vector space.
- The vector spaces are the abstract algebras of a fixed type *in which certain relations between the operations hold that are expressed by equalities*

How about structures that are *lattices*?

- Can we do something similar for vector lattice algebras (and for other structures that are lattices)?
- Not so clear: there are *inequalities* in the definition of a vector lattice, and *existence* of suprema 'is a property, not a relation'.

Partially ordered lattices and algebraic lattices

Two kinds of lattices (in ad hoc terminology) for a set $S \neq \emptyset$

1. Suppose that $\leq$ is a partial ordering on S. Then the partially ordered set $(S, \leq)$ is a *partially ordered lattice* if, for all $x, y \in S$, the supremum $x \vee y$ and the infimum $x \wedge y$ exist in S.
2. Suppose that the abstract algebra $(S, \{\owedge, \ovee\})$ has two binary operations. Then $(S, \{\owedge, \ovee\})$ is an *algebraic lattice* if

$$
\begin{aligned}
x \owedge (y \owedge z) &= (x \owedge y) \owedge z, & x \ovee (y \ovee z) &= (x \ovee y) \ovee z, \\
x \owedge x &= x, & x \ovee x &= x, \\
x \owedge y &= y \owedge x, & x \ovee y &= y \ovee x, \\
x \owedge (x \ovee y) &= x, \quad \text{and} & x \ovee (x \owedge y) &= x.
\end{aligned}
$$

Key observation

- There is a natural correspondence between these two types of structures

Partially ordered lattices and algebraic lattices

Lemma (from the literature)

Let $S \neq \emptyset$.

1. *Suppose that $(S, \leq)$ is a partially ordered lattice. For $x, y \in S$, set $x \owedge y := x \wedge y$ and $x \ovee y := x \vee y$. Then the abstract algebra $(S, \{\ovee, \owedge\})$ is an algebraic lattice.*
2. *Suppose that the abstract algebra $(S, \{\owedge, \ovee\})$ is an algebraic lattice. Say that $x \leq y$ if $x \owedge y = x$. Then $\leq$ is a partially ordering on S, and $(S, \leq)$ is a partially ordered lattice. Moreover, for $x, y \in S$, we have $x \wedge y = x \owedge y$ and $x \vee y = x \ovee y$, where $x \wedge y$ and $x \wedge y$ refer to the infimum and the supremum in the partial ordering $\leq$.*
3. *The above transitions from $(S, \leq)$ to $(S, \{\owedge, \ovee\})$ and vice versa are mutually inverse, with lattice homomorphisms corresponding to abstract algebra homomorphisms*

Now: use this result to see that, for example, the unital vector lattice algebras are also precisely the abstract algebras with the obvious operations *satisfying certain relations*.

Unital VLAs as abstract algebras satisfying relations

Proposition (MdJ)

Let A be an abstract algebra with (not necessarily different) constants 0 and 1, a binary map $\oplus$, a unary map $\ominus$, a unary map m_λ for every $\lambda \in \mathbb{R}$, a binary map $\odot$, and binary maps $\owedge$ and $\ovee$. Suppose that all of the following are satisfied:

1. $(x \oplus y) \oplus z = x \oplus (y \oplus z)$ *for all* $x, y, z \in A$
2. $x \oplus 0 = x$ *for all* $x \in A$
3. $x \oplus (\ominus x) = 0$ *for all* $x \in A$
4. $x \oplus y = y \oplus x$ *for all* $x, y \in A$
5. $m_\lambda(x \oplus y) = m_\lambda(x) \oplus m_\lambda(y)$ *for all* $\lambda \in \mathbb{R}$ *and* $x, y \in A$
6. $m_{\lambda+\mu}(x) = m_\lambda(x) \oplus m_\mu(x)$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $x \in A$
7. $m_{\lambda\mu}(x)) = m_\lambda(m_\mu(x))$ *for all* $\lambda, \mu \in \mathbb{R}$ *and* $x \in A$
8. $m_1(x) = x$ *for all* $x \in A$

Unital VLA's as abstract algebras satisfying relations

Proposition (continued)

9. $(x \odot y) \odot z = x \odot (y \odot z)$ *for all* $x, y, z \in A$;
10. $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$ *for all* $x, y, z \in A$
11. $(x \oplus y) \odot z = (x \odot z) \oplus (y \odot z)$ *for all* $x, y, z \in A$
12. $m_\lambda(x \odot y) = m_\lambda(x) \odot y = x \odot m_\lambda(y)$ *for all* $\lambda \in \mathbb{R}$ *and* $x, y \in A$
13. $1 \odot x = x \odot 1 = x$ *for all* $x \in A$
14. $x \owedge (y \owedge z) = (x \owedge y) \owedge z$ *and* $x \ovee (y \ovee z) = (x \ovee y) \ovee z$ *for all* $x, y, z \in A$
15. $x \owedge x = x$ *and* $x \ovee x = x$ *for all* $x \in A$
16. $x \owedge y = y \owedge x$ *and* $x \ovee y = y \ovee x$ *for all* $x, y \in A$
17. $x \owedge (x \ovee y) = x$ *and* $x \ovee (x \owedge y) = x$ *for all* $x, y \in A$
18. $x \oplus (y \owedge z) = (x \oplus y) \owedge (x \oplus z)$ *for all* $x, y, z \in A$
19. $m_\lambda(0 \owedge x) = 0 \owedge (m_\lambda(x))$ *for all* $\lambda \in \mathbb{R}_{\geq 0}$ *and* $x \in A$
20. $0 \owedge ((x \owedge (\ominus x)) \odot (y \owedge (\ominus y))) = 0$ *for all* $x, y \in A$.

Unital VLA's as abstract algebras satisfying relations

Proposition (continued)

Define

(a) $x + y := x \oplus y$ *for* $x, y \in A$

(b) $\lambda \cdot x := m_\lambda(x)$ *for* $\lambda \in \mathbb{R}$ *and* $x \in A$

(c) $xy := x \odot y$ *for* $x, y \in A$

(d) $x \leq y$ *when* $x \owedge y = x$

Then A *is a unital vector lattice algebra with zero element 0 and identity element 1.*

Conversely, every unital vector lattice algebra gives rise, in the obvious way, to an abstract algebra with operations as above satisfying these 20 relations. These two passages are mutually inverse, with unital vector lattice algebra homomorphisms corresponding to abstract algebra homomorphisms.

Where do we stand?

- Have free abstract algebra of a given type $(\mathcal{F}, \rho)$ over a set S: $T_{(\mathcal{F},\rho)}(S)$
- For example: the free abstract algebra of the *type* of unital vector lattice algebras over a set S exists
- This is not yet what we want: we need a free algebra over a set in *'the category of abstract algebras of that type where these 20 relations between the operations are satisfied'*
- How can one, in fact, formalise the concept of *'relations between operations being satisfied'*?

Expressing that relations between operations are satisfied

- Suppose $(\mathcal{F}, \rho)$ is a type, that $f_2, f_3 \in \mathcal{F}$, that $\rho(f_2) = 2$, and that $\rho(f_3) = 3$
- Suppose that A is an abstract algebra of type $(\mathcal{F}, \rho)$, such that

$$f_3^A(f_2^A(x, y), f_2^A(y, z), f_2^A(z, x)) = f_2^A(x, y) \text{ for all } x, y, z \in A$$

- Take a set $S_{\aleph_0}$ with at least three elements s_1, s_2, s_3
- Take the terms $t_1 := f_3(f_2(s_1, s_2), f_2(s_2, s_3), f_2(s_3, s_1))$ and $t_2 := f_2(s_1, s_2)$ in $T_{(\mathcal{F},\rho)}(S_{\aleph_0})$
- *Then the validity of the above relation between the operations in A is equivalent to the fact that $h(t_1) = h(t_2)$ for all abstract algebra homomorphisms $h : T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \to A$*

This leads to the following definition.

Definition

Let $(\mathcal{F}, \rho)$ be a type. Let $S_{\aleph_0}$ be a countably infinite set. Take two terms $t_1, t_2 \in T_{(\mathcal{F},\rho)}(S_{\aleph_0})$. Let A be an abstract algebra of type $(\mathcal{F}, \rho)$. Then A *satisfies* $t_1 \approx t_2$ when $h(t_1) = h(t_2)$ for every abstract algebra homomorphism $h : T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \to A$. For a subset Σ of $T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \times T_{(\mathcal{F},\rho)}(S_{\aleph_0})$, A *satisfies* Σ when A satisfies $t_1 \approx t_2$ for every pair $(t_1, t_2) \in \Sigma$.

For $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \times T_{(\mathcal{F},\rho)}(S_{\aleph_0})$, the class of all abstract algebras of type $(\mathcal{F}, \rho)$ satisfying Σ is called the *equational class defined by* Σ. Together with the abstract algebra homomorphism between them, it forms the subcategory $\mathsf{AbsAlg}_{(\mathcal{F},\rho);\Sigma}$ of $\mathsf{AbsAlg}_{(\mathcal{F},\rho)}$.

Universal algebra: Part II

Theorem

Let $(\mathcal{F}, \rho)$ be a type. Take a countably infinite set $S_{\aleph_0}$ and $\Sigma \subseteq T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \times T_{(\mathcal{F},\rho)}(S_{\aleph_0})$. Let $S \neq \emptyset$. There exists an equivalence relation θ on $T_{(\mathcal{F},\rho)}(S)$ such that $T_{(\mathcal{F},\rho)}(S)/\theta$ is an abstract algebra of type $(\mathcal{F}, \rho)$ satisfying Σ, and with the following property: for every abstract algebra of type $(\mathcal{F}, \rho)$ satisfying Σ and every map $h : S \to A$, there exists a unique abstract algebra homomorphism $\overline{h} : T_{(\mathcal{F},\rho)}(S)/\theta \to A$ such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{q_\theta|_S} & T_{(\mathcal{F},\rho)}(S)/\theta \\ & \searrow^{h} & \downarrow \overline{h} \\ & & A \end{array}$$

is commutative; here $q_\theta : T_{(\mathcal{F},\rho)}(S) \to T_{(\mathcal{F},\rho)}(S)/\theta$ denotes the quotient abstract algebra homomorphism.

That is, $\mathsf{F}_{\mathsf{Set}}^{\mathsf{AbsAlg}(\mathcal{F},\rho);\Sigma}[S]$ exists and is equal to $T_{(\mathcal{F},\rho)}(S)/\theta$.

Comments

- The equivalence relation θ on $T_{(\mathcal{F},\rho)}(S)$ is the smallest so-called congruence relation on $T_{(\mathcal{F},\rho)}(S)$ containing the pairs $(h'(t_1), h'(t_2))$ for all $(t_1, t_2) \in \Sigma$ and all abstract algebra homomorphisms $h' : T_{(\mathcal{F},\rho)}(S_{\aleph_0}) \to T_{(\mathcal{F},\rho)}(S)$.
- These free objects consist of equivalence classes of strings (recall the free vector space over a set!), such that (for example) $\overline{(x \oplus y) \oplus z} = \overline{x \oplus (y \oplus z)}$ and $\overline{m_{\lambda_1 \lambda_2}(x)} = \overline{m_{\lambda_1}(m_{\lambda_2}(x))}$

Free unital vector lattice algebras

Consequence

- Free unital vector lattice algebras over a set exist: the unital vector lattice algebras form an equational class
- Many other free objects in well known categories over sets also exist

Not just over sets

- Construct free unital vector lattice algebra over a vector space V:
- Take $(j, \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)])$ where $j : V \to \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]$
- Let I be the order-and-algebra ideal of $\mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]$ generated by all $j(x+y) - j(x) - j(y)$ and all $j(\lambda x) - \lambda j(x)$
- Then $\mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]\,/I$ is a unital vector lattice algebra
- With $q : \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)] \to \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]\,/I$, the pair $(q \circ j, \mathsf{F}_{\mathsf{Set}}^{\mathsf{VLA}^1}[\mathrm{Set}(V)]\,/I)$ is the free unital vector lattice algebra over V

Theorem (MdJ)

For a non-empty set S, a vector space V, a vector lattice E, a vector lattice algebra A, and a unital vector lattice algebra A^1, the 15 free objects on the following slide all exist, with inclusions as indicated. The surjective unital vector lattice algebra homomorphisms in the rightmost column are the quotient maps corresponding to dividing out the order-and-algebra-ideal that is generated by $(|1| - 1)$.

Existence of free objects in some algebraic categories

$$S \subset \mathrm{F}_{\mathrm{Set}}^{\mathrm{VS}}[S] \subset \mathrm{F}_{\mathrm{Set}}^{\mathrm{VL}}[S] \subset \mathrm{F}_{\mathrm{Set}}^{\mathrm{VLA}}[S] \begin{array}{l} \subset \\ \subset \end{array} \begin{array}{c} \mathrm{F}_{\mathrm{Set}}^{\mathrm{VLA}^1}[S] \\ \twoheaddownarrow \\ \mathrm{F}_{\mathrm{Set}}^{\mathrm{VLA}^{1+}}[S] \end{array}$$

$$V \subset \mathrm{F}_{\mathrm{VS}}^{\mathrm{VL}}[V] \subset \mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}}[V] \begin{array}{l} \subset \\ \subset \end{array} \begin{array}{c} \mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}^1}[V] \\ \twoheaddownarrow \\ \mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}^{1+}}[V] \end{array}$$

$$E \subset \mathrm{F}_{\mathrm{VL}}^{\mathrm{VLA}}[E] \begin{array}{l} \subset \\ \subset \end{array} \begin{array}{c} \mathrm{F}_{\mathrm{VL}}^{\mathrm{VLA}^1}[E] \\ \twoheaddownarrow \\ \mathrm{F}_{\mathrm{VL}}^{\mathrm{VLA}^{1+}}[E] \end{array}$$

$$A \begin{array}{l} \subset \\ \subset \end{array} \begin{array}{c} \mathrm{F}_{\mathrm{VLA}}^{\mathrm{VLA}^1}[A] \\ \twoheaddownarrow \\ \mathrm{F}_{\mathrm{VLA}}^{\mathrm{VLA}^{1+}}[A] \end{array}$$

$$\mathrm{F}_{\mathrm{VLA}^1}^{\mathrm{VLA}^{1+}}\left[A^1\right]$$

What do the abstractly constructed free objects 'look like'?

- Free vector space over a set: 'space of functions with the set as basis'
- Free associative algebra over a vector space: tensor algebra
- Free vector lattice over a set: is a lattice of functions (Bleier)

Theorem (MdJ)

Let V be a vector space. Take a linear subspace $L^\sharp$ of the algebraic dual $V^\sharp$ that separates the points of V. Define the linear map $\Psi : V \to \mathrm{Fun}(L^\sharp, \mathbb{R})$ by setting

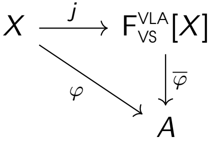
$$[\Psi(v)](l^\sharp) := l^\sharp(v) \quad (v \in V,\ l^\sharp \in L^\sharp).$$

Let $\mathcal{F}$ be the vector sublattice of $\mathrm{Fun}(L^\sharp, \mathbb{R})$ that is generated by its linear subspace $\Psi(V)$. Then $(\Psi, \mathcal{F})$ is a free vector lattice over V.

Use free algebraic objects as starting point

Example: construct the free Banach lattice algebra over a Banach space X. Impossible, but we can still do something.

Start with $\mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}}[X]$. Fix $M > 0$. Take a BLA A and $\varphi : X \to A$ linear such that $\|\varphi\| \leq M$. Have commutative diagram with VLA homomorphism $\overline{\varphi}$



For $a \in \mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}}[X]$, set

$$\rho(a) := \sup\{\|\overline{\varphi}(a)\| : A \text{ is a BLA, } \varphi : X \to A,\ \|\varphi\| \leq M\}$$

The set of all a such that $\rho(a) < \infty$ is a vector lattice subalgebra of $\mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}}[X]$ containing $j(X)$, so it is all of $\mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}}[X]$. Hence ρ is a vector lattice algebra seminorm, so that $\ker \rho$ is an order-algebra ideal.

Free objects in analytic categories

$\mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}}[X] \,/\, \ker \rho$ is then a normed vector lattice algebra.
Let $\mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M$ be its completion.
Have canonical linear map $j'_M : X \to \mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}}[X] \to \mathrm{F}_{\mathrm{VS}}^{\mathrm{VLA}}[X] \,/\, \ker \rho \subseteq \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M$.

Have $\|j'_M\| \leq M$. Furthermore, if A is a BLA and $\varphi : X \to A$ is linear with $\|\varphi\| \leq M$, then there exists a unique Banach lattice algebra homomorphism $\overline{\varphi} : \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M \to A$ such that

$$\begin{array}{ccc} X & \xrightarrow{\;j'_M\;} & \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M \\ & \overset{\varphi}{\searrow} & \Big\downarrow \overline{\varphi} \\ & & A \end{array}$$

is commutative. This $\overline{\varphi}$ is contractive.
So $(j'_M, \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M)$ is not a free Banach lattice algebra over X (which does not exist) but it still solves a universal problem 'with given bound'.

Free objects in analytic categories

Can go one step further to obtain truly free objects *in larger categories*.
Take $M_2 \geq M_1$. Then $\|j'_{M_1}\| \leq M_1 \leq M_2$, so there is a contractive BLA homomorphism $\overline{j'_{M_1}} : \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_{M_2} \to \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_{M_1}$ such that

$$\begin{array}{ccc} X & \xrightarrow{\;j'_{M_2}\;} & \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_{M_2} \\ & {\scriptstyle j'_{M_1}}\searrow & \downarrow{\scriptstyle \overline{j'_{M_1}}} \\ & & \mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_{M_1} \end{array}$$

Follows from this: the $\mathrm{F}_{\mathrm{BS}}^{\mathrm{BLA}}[X]_M$ for $M > 0$ form an inverse system.
More work shows: can take the inverse limit of this system in the category of complete locally convex-solid topological algebras, and this inverse limit is a free object in that category over the Banach space X.

Comments

- Nothing special here in the final two steps: introducing seminorm ρ, passing to quotient, completing, finding inverse system, taking inverse limit in larger category works in much greater generality
- 'General recipe': solve the free algebraic problem first (universal algebra), then apply this 'standard' analytic procedure
- Example: free complete locally convex topological algebras over sets exist
- For a one point set (and over $\mathbb{C}$) this is the algebra of entire functions in the topology of uniform convergence on compact subsets

Current work on free Banach lattices over Banach spaces

- Model for free vector lattice over a vector space + standard analytic procedure predict the following:
- Let X be Banach space. Take a linear subspace $L^\sharp$ of $X^\sharp$ that separates the points of X. Then the free Banach lattice over X is the completion of a quotient of a vector lattice of functions on $L^\sharp$
- Since X^* separates the points of X, this gives intuition for the non-trivial fact (Avilés, Rodríguez, Tradacete; JFA 2018) that it is a Banach lattice of functions on X^*

Literature

References

- M. de Jeu, *Free vector lattices and free vector lattice algebras*; pp.103-139 in "Positivity and Its Applications" (E. Kikianty, M. Mabula, M. Messerschmidt, J.H. van der Walt, and M. Wortel, Eds.); proceedings of the Positivity X conference, July 8-12, 2019, Pretoria; Birkhäuser/Springer, 2021.
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