Duality of Riesz* Homomorphisms and Interval Preserving Operators in Ordered Vector Spaces

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Definition

Let X, Y be ordered vector spaces and $T\colon X \to Y$ a linear operator.

(i) T is called *interval preserving* if T is positive and

$$\forall x \in X_+ : \quad T[0,x] = [0, Tx].$$

(ii) Let $X^\sim$ denote the space of all order bounded linear functionals on X. If T is order bounded, then the linear operator

$$T^\sim\colon Y^\sim \longrightarrow X^\sim, \quad g \longmapsto g \circ T$$

is called the *order adjoint* of T.

Theorem (Kim-Andô, 1975)

Let X, Y be vector lattices and $T : X \to Y$ a positive operator.

$$T \text{ Riesz hom.} \xrightarrow{\qquad\qquad} T^\sim \text{ int. pres.}$$
$$T \text{ Riesz hom.} \xleftarrow[Y^\sim \text{ separating}^1]{\qquad\qquad} T^\sim \text{ int. pres.}$$

$$T \text{ int. pres.} \Longrightarrow T^\sim \text{ Riesz hom.}$$

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Can this be generalized to the setting of ordered vector spaces?
(Joint work with A. Kalauch, O. van Gaans, and J. Stennder.)

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A first generalization

Proof of T int. pres. $\Rightarrow$ $T^{\sim}$ Riesz hom.

Suppose that $T\colon X \to Y$ is interval preserving. For every $g \in Y^{\sim}$ and $x \in X_+$, we have

$$
\begin{aligned}
T^{\sim}\left(g^{+}\right)(x) &= g^{+}(Tx) \\
&= \sup\left\{g(v);\ v \in [0, Tx]\right\} \\
&= \sup\left\{g(v);\ v \in T[0,x]\right\} \\
&= \sup\left\{g(Tu);\ u \in [0,x]\right\} \\
&= (g \circ T)^{+}(x) = T^{\sim}(g)^{+}(x).
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The proof only relies on the Riesz-Kantorovich-formula. $\Rightarrow$ This implication is also true for directed ordered spaces X, Y with the Riesz decomposition property (RDP).

$$T \text{ Riesz hom.} \underset{X,\,Y \text{ v.l., } Y^{\sim} \text{ separating}}{\overset{X,\,Y \text{ v.l.}}{\rightleftarrows}} T^{\sim} \text{ int. pres.}$$

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Definition

Let X, Y be ordered vector spaces. A linear operator $T\colon X \to Y$ is called a

(i) *Riesz* homomorphism* if

$$\forall \varnothing \neq F \subseteq X \text{ finite}: \quad T\left[F^{\mathrm{u}\ell}\right] \subseteq T[F]^{\mathrm{u}\ell}.$$

(ii) *Riesz homomorphism* if

$$\forall x_1, x_2 \in X: \quad T\left[\{x_1, x_2\}^{\mathrm{u}}\right]^{\ell} = \{Tx_1, Tx_2\}^{\mathrm{u}\ell}.$$

(iii) *complete Riesz homomorphism* if

$$\forall \varnothing \neq A \subseteq X: \quad \inf A = 0 \Longrightarrow \inf T[A] = 0.$$

In vector lattices:

(i) Riesz* and Riesz homomorphism coincide with the Riesz homomorphisms of vector lattices.

(ii) Complete Riesz homomorphisms coincide with the order continuous Riesz homomorphisms.

Definition

An ordered vector space X is called a *pre-Riesz space* if there exists a vector lattice Y and a bipositive operator $i\colon X \to Y$ such that $i[X]$ is order dense in Y, i.e.,

$$\forall y \in Y: \quad y = \inf \{i(x); x \in X, i(x) \geq y\}.$$

In this case, such a pair (Y, i) is called a *vector lattice cover* of X. If $i[X]$ generates Y as a vector lattice, then (Y, i) is called the *Riesz completion*.

(i) The Riesz completion is unique up to order isomorphisms.

(ii) Every pre-Riesz space is directed.

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(ii) Namioka space: $\{x \in \mathrm{C}[-1, 1];\ x(-1) + x(1) = 2x(0)\}$

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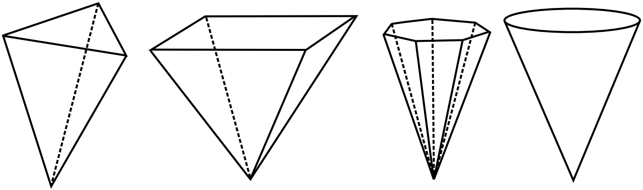
(iii) $\mathrm{L}^{\mathrm{r}}(X,Y)$ with X directed and Y Archimedean

(iv) Finite-dimensional spaces X with closed positive cone X_+ and $\operatorname{int} X_+ \neq \varnothing$.



In pre-Riesz spaces:

complete Riesz homomorphism

$\Downarrow \quad \not\Uparrow$

Riesz homomorphism

$\Downarrow \quad \not\Uparrow$

Riesz* homomorphism

$\Downarrow \quad \not\Uparrow$

positive

The van Haandel Extension

Theorem (Van Haandel, 1993)

Let X, Y be pre-Riesz spaces with respective Riesz completions (X^ρ, i_X), (Y^ρ, i_Y) and $T\colon X \to Y$ a linear operator. T is a Riesz homomorphism if and only if there exists a Riesz homomorphism $T^\rho\colon X^\rho \to Y^\rho$ such that $T^\rho \circ i_X = i_Y \circ T$, i.e.,*

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ X^\rho & \xrightarrow{T^\rho} & Y^\rho \end{array}$$

In this case, the Riesz homomorphism T^ρ is unique and called the van Haandel extension.

Back to the Main Topic

$$T \text{ Riesz hom.} \underset{X,\,Y \text{ v.l., } Y^\sim \text{ separating}}{\overset{X,\,Y \text{ v.l.}}{\rightleftarrows}} T^\sim \text{ int. pres.}$$

$$T \text{ int. pres.} \xRightarrow{X,\,Y \text{ RDP}} T^\sim \text{ Riesz hom.}$$

Two Useful Characterizations

Definition

Let X be an ordered vector space. An element $y \in X_+$ is called *extremal* if

$$\forall x \in [0, y] \, \exists \lambda \in [0, 1] : \quad x = \lambda y.$$

Proposition (Hayes, 1966)

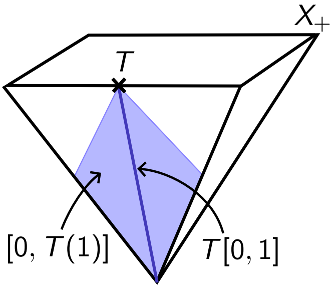
Let X be a directed ordered vector space and $f : X \to \mathbb{R}$ a positive functional. Then

$$f \text{ is a Riesz homomorphism} \iff f \text{ is extremal.}$$

Proposition

Let X be an ordered vector space and $T : \mathbb{R} \to X$ a positive operator. Then

$$T \text{ is interval preserving} \iff T \text{ is extremal.}$$



For a positive operator $T\colon X \to Y$, the question whether

$$\begin{cases} T \text{ is int. pres.} \Leftrightarrow T^\sim \text{ is a Riesz hom.} & \text{if } X = \mathbb{R}, \\ T \text{ is a Riesz hom.} \Leftrightarrow T^\sim \text{ is int. pres.} & \text{if } Y = \mathbb{R} \end{cases}$$

is equivalent to the question whether

$$T \text{ is extremal} \Leftrightarrow T^\sim \text{ is extremal.}$$

Proposition

Let X, Y be finite-dimensional directed Archimedean ordered vector spaces. Then

$$\mathrm{L}^{\mathrm{b}}(X, Y) \longrightarrow \mathrm{L}(Y^{\sim}, X^{\sim}), \quad T \longmapsto T^{\sim}$$

is an order isomorphism. In particular, for every positive operator $T\colon X \to Y$, one has

$$T \text{ is extremal} \iff T^{\sim} \text{ is extremal.}$$

Corollary

Let Y be a finite-dimensional directed Archimedean ordered vector space and $T\colon \mathbb{R} \to Y$ a positive operator.

$$T \text{ is interval preserving} \iff T^{\sim} \text{ is a Riesz homomorphism.}$$

$$T \text{ Riesz hom.} \underset{X,\,Y \text{ v.l., } Y^{\sim} \text{ separating}}{\overset{X,\,Y \text{ v.l.}}{\rightleftarrows}} T^{\sim} \text{ int. pres.}$$

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$$T \text{ int. pres.} \; \underset{X=\mathbb{R},\, Y \text{ Arch., } \dim(Y)<\infty}{\overset{X,Y \text{ RDP}}{\Longleftrightarrow}} \; T^{\sim} \text{ Riesz hom.}$$

Remark

In general, the order dual of a directed ordered vector space is not directed.[1]

Proposition

Let X, Y be directed ordered vector spaces with $Y^\sim$ directed and separating and let $T : X \to Y$ be a positive operator.

$$T \text{ is extremal } \overset{Y=\mathbb{R}}{\underset{}{\rightleftarrows}} T^\sim \text{ is extremal.}$$

[1]See: Otto van Gaans, *An elementary example of an order bounded dual space that is not directed*, Positivity 9(2):265-267, 2005

Corollary

Let Y be a directed ordered vector space with $Y^\sim$ directed and separating and let $T: \mathbb{R} \to Y$ be a positive operator.

$$T^\sim \text{ is a Riesz homomorphism} \implies T \text{ is interval preserving.}$$

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T Riesz hom. $\Longrightarrow$ $T^\sim$ int. pres. (X, Y v.l.)

$T^\sim$ int. pres. $\Longrightarrow$ T Riesz hom. (X, Y v.l., $Y^\sim$ separating)

T int. pres. $\Longrightarrow$ $T^\sim$ Riesz hom. (X, Y RDP)

$T^\sim$ Riesz hom. $\Longrightarrow$ T int. pres. ($X = \mathbb{R}$, Y Arch., $\dim(Y) < \infty$)

$T^\sim$ Riesz hom. $\Longrightarrow$ T int. pres. ($X = \mathbb{R}$, $Y^\sim$ directed, separating)

Proposition

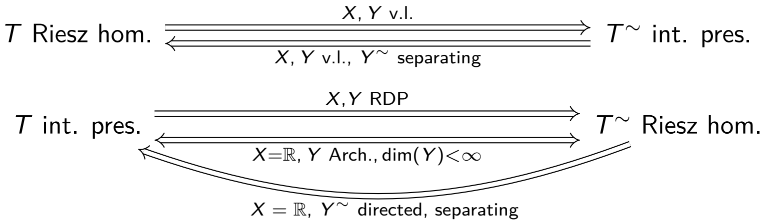
Let X, Y be directed ordered vector spaces with $Y^\sim$ directed and separating and let $T : X \to Y$ be a positive operator.

$$T \text{ is extremal } \underset{\Longleftarrow}{\overset{Y=\mathbb{R}}{\Longrightarrow}} T^\sim \text{ is extremal.}$$

Corollary

Let X be a directed ordered vector space and let $T : X \to \mathbb{R}$ be a positive functional.

$$T \text{ is a Riesz homomorphism} \iff T^{\sim} \text{ is interval preserving.}$$



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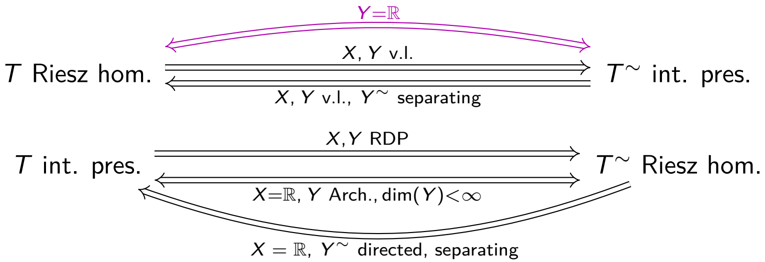


Theorem (B., van Gaans, Kalauch, Stennder, 2023)

Let X, Y be pre-Riesz spaces. Suppose that

(a) every positive functional on Y is a positive-linear combination of Riesz homomorphisms,*

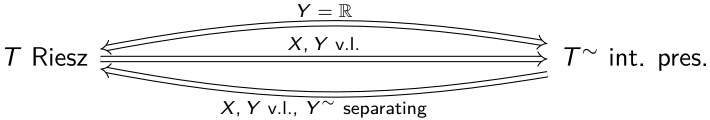
(b) $Y^\sim$ has the Riesz decomposition property,

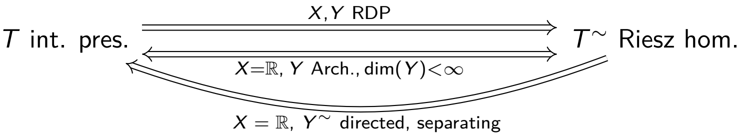
(c) every Riesz homomorphism in X_+^* is a Riesz homomorphism.*

If $T\colon X \to Y$ is a Riesz homomorphism, then $T^\sim$ is interval preserving.*

Example

If $X = \{x \in \mathrm{C}[-1,1];\, x(-1) + x(1) = 2x(0)\}$ is the Namioka space, then there exists a complete Riesz homomorphism $T\colon X \to X$ such that $T^\sim$ is not interval preserving.


T Riesz
$Y = \mathbb{R}$
X, Y v.l.
X, Y v.l., $Y^{\sim}$ separating
$T^{\sim}$ int. pres.


T int. pres.
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$X{=}\mathbb{R}$, Y Arch., $\dim(Y){<}\infty$
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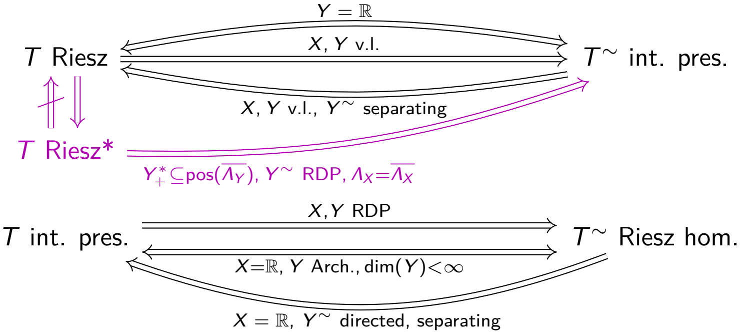
$T^{\sim}$ Riesz hom.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T Riesz*
$Y_+^* \subseteq \mathrm{pos}(\overline{\Lambda_Y})$, $Y^{\sim}$ RDP, $\Lambda_X = \overline{\Lambda_X}$
T int. pres.
X, Y RDP
$X = \mathbb{R}$, Y Arch., $\dim(Y) < \infty$
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An Open Problem

Let X, Y be ordered vector spaces and $T\colon X \to Y$ a linear operator. T is a Riesz* homomorphism if

$$\forall \varnothing \neq F \subseteq X \text{ finite}: \quad T\left[F^{\mathrm{u}\ell}\right] \subseteq T[F]^{\mathrm{u}\ell}.$$

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Question

Is T a Riesz* homomorphism if and only if

$$\forall x_1, x_2 \in X: \quad T\left[\{x_1, x_2\}^{\mathrm{u}\ell}\right] \subseteq T[\{x_1, x_2\}]^{\mathrm{u}\ell}? \qquad (1)$$

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Definition

We call linear operators that satisfy (1) *mild Riesz* homomorphisms*.

A Half-Solution

Theorem (Van Haandel, 1993)

Let X, Y be ordered vector spaces. Suppose that

$$\forall \varnothing \neq F, G \subseteq X \text{ finite}: \quad (F \cup G)^{\mathrm{u}\ell} = \bigcup_{b \in G^{\mathrm{u}\ell}} (F \cup \{b\})^{\mathrm{u}\ell}. \qquad (2)$$

Then a linear operator $T : X \to Y$ is a Riesz homomorphism if and only if T is a mild Riesz* homomorphism.*

The condition (2) is true in vector lattices, but not in general pre-Riesz spaces. We know (2) to be false in $\mathrm{P}^2[-1, 1]$.

Theorem (B., van Gaans, Kalauch, Stennder, 2023)

Let X be a pre-Riesz space, Y a directed ordered vector space such that Y_+^* is total[2], and $T: X \to Y$ a positive operator.

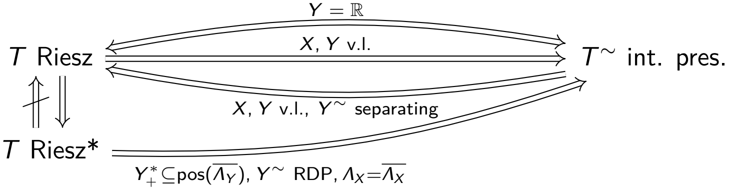
$$T^\sim \text{ is interval preserving} \Longrightarrow T \text{ is a mild Riesz* homomorphism.}$$

Question

(i) Can one show that T even is a Riesz* homomorphism?

(ii) Can this be used to find mild Riesz* homomorphisms that are not Riesz* homomorphisms?

[2] i.e., $\forall y \in Y : y \geq 0 \Leftrightarrow (\forall g \in Y_+^* : g(y) \geq 0)$.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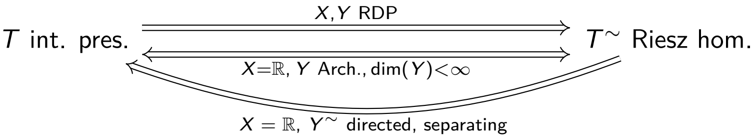
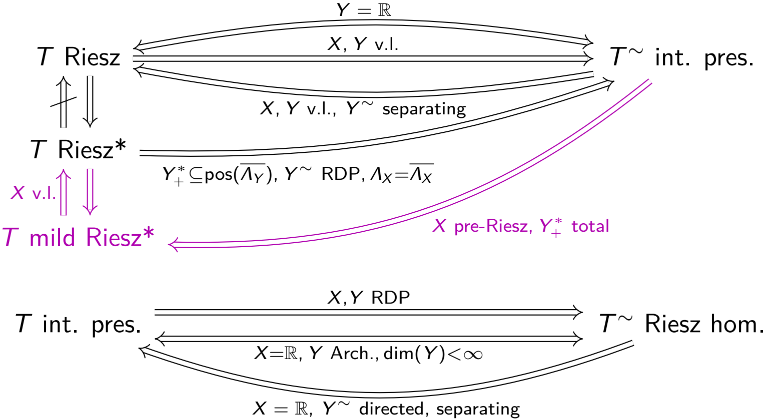
T Riesz*
Y = ℝ
X, Y v.l.
X, Y v.l., Y~ separating
Y*+ ⊆ pos(Λ̄Y), Y~ RDP, ΛX = Λ̄X


T int. pres.
T~ Riesz hom.
X, Y RDP
X=ℝ, Y Arch., dim(Y)<∞
X = ℝ, Y~ directed, separating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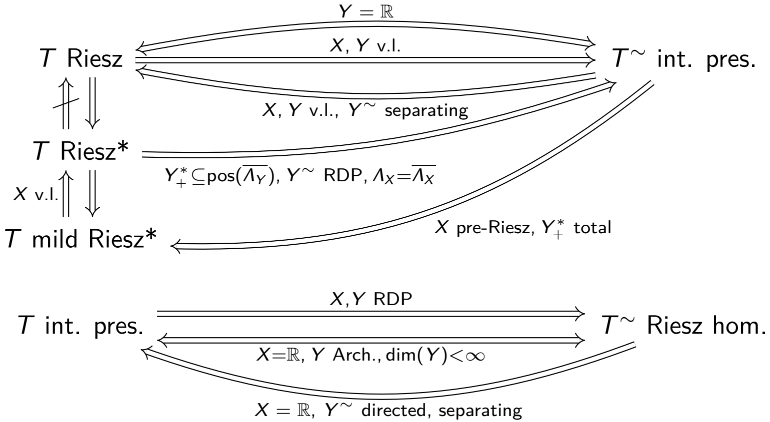
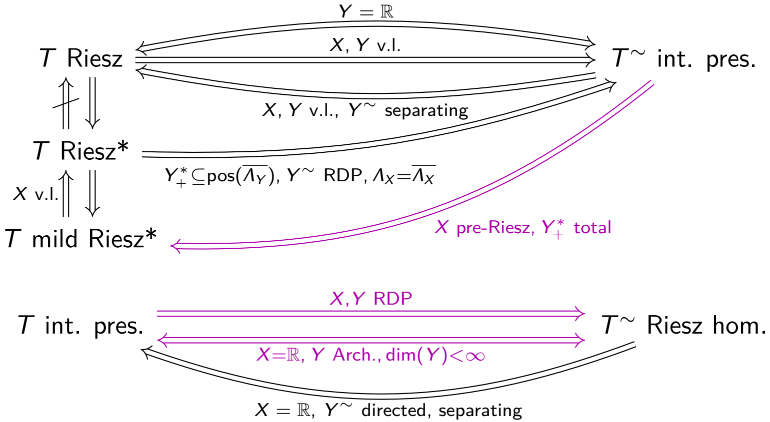
State of the Art



Some Open Questions

Can the presented results be further improved?



Some Open Questions

Definition

Let X be an ordered vector spaces, Y an ordered normed space, and $T\colon X \to Y$ a positive operator. T is called *almost interval preserving* if

$$\forall x \in X_+ : \quad \overline{T[0,x]} = [0, T(x)].$$

Theorem (Andô, 1975)

Let X, Y be normed vector lattices and $T\colon X \to Y$ a continuous linear operator.

$$T \text{ is almost int. pres.} \iff T' \text{ is a Riesz homomorphism.}$$

Can this also be generalized?

References

[1] F. Boisen, O. van Gaans, A. Kalauch, and J. Stennder. Riesz* homomorphisms and dual Maharam operators in pre-Riesz spaces. In preperation. 2023.

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[6] H. P. Lotz. Extensions and liftings of positive linear mappings on Banach lattices. In: *Trans. Am. Math. Soc.* 211 (1975), pp. 85–100.

Thank you :)