FernUniversität
in Hagen

Comparing the spectrum of Schrödinger operators on quantum graphs

Joint work with J. Kerner (Hagen)

WOVSPO 2023
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FernUniversität in Hagen

What is a *metric* graph?

„A metric graph is a collection of intervals glued together at their ends in a certain sense.“

(1) E (finite) edge set, $[0, \ell_e]_{e \in E}$ family of intervals, consider disjoint union

$$\mathcal{E} := \coprod_{e \in E} [0, \ell_e] := \bigcup_{e \in E} [0, \ell_e] \times \{e\}.$$

(2) Put $\mathcal{V} := \coprod_{e \in E} \{0, \ell_e\}$ *vertices*, $\sim$ some equivalence relation on $\mathcal{V}$.

(3) Extend this relation $\sim$ onto $\mathcal{E}$ via

$$(x_1, e_1) \sim (x_2, e_2) \ :\Leftrightarrow\ (x_1, e_1) = (x_2, e_2) \text{ or } (x_1, e_1), (x_2, e_2) \in \mathcal{V} \text{ and } (x_1, e_1) \sim (x_2, e_2).$$

(4) Define $\mathcal{G} := \mathcal{E}/\sim$ *metric graph*, E edge set, $V := \mathcal{V}/\sim$ vertex set.

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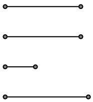

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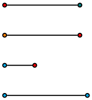

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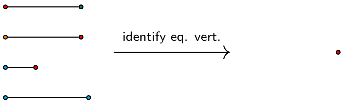

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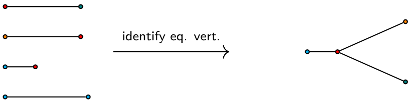

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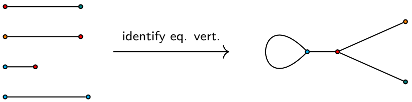
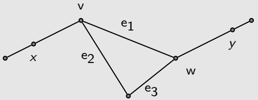


Metric graphs – Why „*metric*“?

$\mathcal{G} = (\mathsf{V}, \mathsf{E})$ metric graph, $x, y \in \mathcal{G}$.

Define **path pseudo-metric**

$$d_{\mathcal{G}}(x, y) := \text{length of shortest path between } x \text{ and } y.$$



- $x \in \mathsf{e} \simeq [0, \ell_{\mathsf{e}}]$ with $\mathsf{e} \in \mathsf{E}$, then

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- $d_{\mathcal{G}}(x, y) = d_{\mathcal{G}}(x, \mathsf{v}) + \text{shortest path between v and w} + d_{\mathcal{G}}(y, \mathsf{w})$

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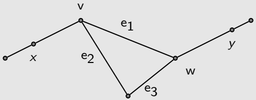

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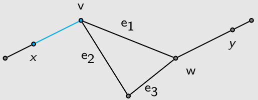

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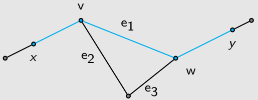

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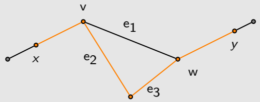

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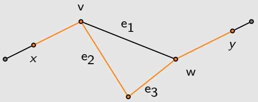

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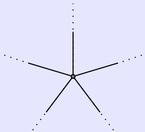
Metric graphs – Relevant spaces

Definition

$\mathcal{G} = (\mathsf{V}, \mathsf{E})$ metric graph. $f = (f_\mathsf{e})_{\mathsf{e}\in\mathsf{E}} : \mathcal{G} \to \mathbb{C}$ is **continuous** if

$$f_\mathsf{e} \in C[0, \ell_\mathsf{e}], \quad f_\mathsf{e}(\mathsf{v}) = f_{\mathsf{e}'}(\mathsf{v}) \ \forall \mathsf{v} \in \mathsf{V}, \forall \mathsf{e}, \mathsf{e}' \in \mathsf{E} : \ \mathsf{e}, \mathsf{e}' \sim \mathsf{v}.$$

Notation: $f \in C(\mathcal{G})$.



- Relevant *Hilbert spaces*

$$L^2(\mathcal{G}) := \bigoplus_{\mathsf{e}\in\mathsf{E}} L^2(0, \ell_\mathsf{e}), \ \ H^1(\mathcal{G}) := \bigoplus_{\mathsf{e}\in\mathsf{E}} H^1(0, \ell_\mathsf{e}) \cap C(\mathcal{G}),$$

- *Differential expression* on the edges; $q_\mathsf{e} \in C^\infty(0, \ell_\mathsf{e}) \cap L^\infty(0, \ell_\mathsf{e})$, $\mathsf{e} \in \mathsf{E}$

$$\Delta_{\mathcal{G};q} f := \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} f_\mathsf{e} + q_\mathsf{e} f_\mathsf{e} \right)_{\mathsf{e}\in\mathsf{E}} \quad \text{for „suitable“ } f = (f_\mathsf{e})_{\mathsf{e}\in\mathsf{E}} \in L^2(\mathcal{G}).$$

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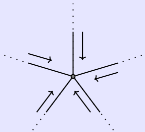

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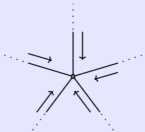

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Suitable operator domains – Vertex conditions

Consider edgewise Schrödinger operator $\Delta_{\mathcal{G};q} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q$:

$$\Delta_{\mathcal{G};q} f := (-f_e'' + q_e f_e)_{e \in E} \quad \text{for suitable } f \in L^2(\mathcal{G}).$$

Question: Which functions do we consider for $\Delta_{\mathcal{G};q}$?.

Define for parameters $\sigma = (\sigma_1, \dots, \sigma_{|V|}) \in \mathbb{R}^{|V|}$

$$\mathcal{D}_{\mathcal{G};\sigma} := \left\{ f \in H^1(\mathcal{G}) \,\middle|\, f_e' \in H^1(0, \ell_e), \; \sum_{e \sim v} \partial_e f(v) = \sigma_v f(v) \right\}$$

δ-**type vertex conditions** (**Kirchhoff-Neumann conditions** if $\sigma = 0$).

- $(\Delta_{\mathcal{G};q}, \mathcal{D}_{\mathcal{G};\sigma})$ self-adj., comp. resolvent, disc. spec. $\rightsquigarrow (\lambda_n^q(\sigma))_{n \in \mathbb{N}} \subset \mathbb{R}$ s.t.

$$\lambda_1^q(\sigma) \le \lambda_2^q(\sigma) \le \dots \le \lambda_n^q(\sigma) \le \dots \to +\infty$$

- If $\sigma, q \ge 0$, then $\Delta_{\mathcal{G};q}$ has positive spectrum; $f \in \mathcal{D}_{\mathcal{G};q}$, then:

$$\langle f, \Delta_{\mathcal{G};q} f \rangle_{L^2(\mathcal{G})} = a_{\mathcal{G};q,\sigma}(f) := \int_{\mathcal{G}} |f'(x)|^2 + q(x)|f(x)|^2 \mathrm{d}x + \sum_{v \in V} \sigma_v |f(v)|^2 \ge 0.$$

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$$\langle f, \Delta_{\mathcal{G};q} f\rangle_{L^2(\mathcal{G})} = a_{\mathcal{G};q,\sigma}(f) := \int_{\mathcal{G}} |f'(x)|^2 + q(x)|f(x)|^2 \mathrm{d}x + \sum_{\mathsf{v}\in\mathsf{V}} \sigma_{\mathsf{v}} |f(\mathsf{v})|^2 \ge 0.$$

Suitable operator domains – Vertex conditions

Consider edgewise Schrödinger operator $\Delta_{\mathcal{G};q} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q$:

$$\Delta_{\mathcal{G};q} f := (-f_{\mathsf{e}}'' + q_{\mathsf{e}} f_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}} \quad \text{for suitable } f \in L^2(\mathcal{G}).$$

Question: Which functions do we consider for $\Delta_{\mathcal{G};q}$?.

Define for parameters $\sigma = (\sigma_1, \dots, \sigma_{|\mathsf{V}|}) \in \mathbb{R}^{|\mathsf{V}|}$

$$\mathcal{D}_{\mathcal{G};\sigma} := \left\{ f \in H^1(\mathcal{G}) \;\middle|\; f_{\mathsf{e}}' \in H^1(0,\ell_{\mathsf{e}}),\ \sum\nolimits_{\mathsf{e}\sim\mathsf{v}} \partial_{\mathsf{e}} f(\mathsf{v}) = \sigma_{\mathsf{v}} f(\mathsf{v}) \right\}$$

δ**-type vertex conditions** (**Kirchhoff-Neumann conditions** if $\sigma = 0$).

- $(\Delta_{\mathcal{G};q}, \mathcal{D}_{\mathcal{G};\sigma})$ self-adj., comp. resolvent, disc. spec. $\rightsquigarrow (\lambda_n^q(\sigma))_{n\in\mathbb{N}} \subset \mathbb{R}$ s.t.

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The main theorem – an asymptotic result

$\mathcal{G} = (\mathsf{V}, \mathsf{E})$ metric graph, $\Delta_{\mathcal{G};q,\sigma}$ Schrödinger op. on $\mathcal{D}_{\mathcal{G};\sigma}$, $\sigma \in \mathbb{R}^{|\mathsf{V}|}$

Study **Robin-KN-gaps**:
$$d_n^q(\sigma) := \lambda_n^q(\sigma) - \lambda_n^q(0), \quad \hat{d}_n^q(\sigma) := \lambda_n^q(\sigma) - \lambda_n^{q=0}(0), \qquad (n \in \mathbb{N})$$

- $|d_n^q(\sigma)|$, $|\hat{d}_n^q(\sigma)|$ uniformly bounded in $n \in \mathbb{N}$.
- **But**: Not strictly positive bounded from below (e.g., 2-star).

Theorem (B., Kerner, 2022)

$\mathcal{G} = (\mathsf{V}, \mathsf{E})$ metric graph, $q \in \bigoplus_{e \in \mathsf{E}} (C^\infty \cap L^\infty)(0, \ell_e)$, $\sigma \in \mathbb{R}^{|\mathsf{V}|}$.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \hat{d}_n^q(\sigma) = \frac{2}{|\mathcal{G}|} \left(\sum_{v \in \mathsf{V}} \frac{\sigma_v}{\deg(v)} + \frac{1}{2} \sum_{e \in \mathsf{E}} \int_0^{\ell_e} q_e(x) \mathrm{d}x \right).$$

In particular: $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n^q(\sigma) = \frac{2}{|\mathcal{G}|} \sum_{v \in \mathsf{V}} \frac{\sigma_v}{\deg(v)}$.

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Sketch of proof. $(\lambda_n(\sigma), f_n^\sigma)$ norm. eigenpairs of $\Delta_{\mathcal{G};q,\sigma}$, $\sigma \in \mathbb{R}^{|\mathsf{V}|}$

① $[0,1] \ni \tau \mapsto \lambda_n(\tau\sigma)$ differentiable a.e. with

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Robin-KN-gap – Some futher notes

- First studied by Riviére and Royer ('20): For certain star-graphs.
- Rudnick, Wigman, Yesha ('21): Planar domains in $\mathbb{R}^2$ and single $\sigma \geq 0$.

$$\Omega \subset \mathbb{R}^2 \text{ planar domain.} \quad \text{Then: } \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n(\sigma) = \frac{2\sigma}{|\Omega|} |\partial\Omega|$$

- Band, Schanz and Sofer ('22): For KN-Laplacian $\Delta_{\mathcal{G}}$ and single $\sigma \geq 0$.
- $\mathcal{C}(\mathcal{G}) := \sum_{v\in V} \frac{1}{\deg(v)}$ suitable notion of „surface area" of a graph $\mathcal{G}$?

Also: Asymptotic result has a pendant for discrete graphs $G = (V, E)$:

Theorem (*Trace formula*)

$G = (V, E)$ *discrete graph,* $\sigma := \mathrm{diag}(\sigma_1, \ldots, \sigma_{|V|}) \in \mathbb{R}^{|V|\times|V|}$.
$\Delta_{G;\sigma} := I - D^{-\frac{1}{2}}(A - \sigma)D^{-\frac{1}{2}}$ *normalized Schrödinger op. Then:*

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References

[1] B. and J. Kerner, *Comparing the spectrum of Schrödinger operators on quantum graphs*, arXiv:2212.13954 [math-ph] (2022).

[2] B. and J. Kerner, *On notions of a surface area of a graph and related bounds on Schrödinger operators*, work in progress.

[3] R. Band, H. Schanz and G. Sofer, *Differences between Robin and Neumann eigenvalues on metric graphs*, arXiv:2212.12531 [math-ph] (2022).

[4] G. Riviére and J. Royer, *Spectrum of a non-selfadjoint quantum star graph*, J. Phys. A: Mathematical and Theoretical **53** (2020), 495202.

[5] Z. Rudnick, I. Wigman, and N. Yesha *Differences between Robin and Neumann eigenvalues*, Commun. Math. Phys. **388** (2021), 1603-1635.

Thank you for your attention!