UNIVERSITÄT PASSAU

Fakultät für Informatik und Mathematik

Topology

Jochen Glück

Lecture notes

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Contents

Introduction

Organisation of the manuscript

The manuscript is organised as follows:

- The manuscript consists of chapters that are enumerated as 1, 2, etc., and each chapter consists of sections that are enumerated as 1.1, 1.2., etc.

- Definitions, theorems, remarks, etc., are enumerated consecutively within each section.

- Each chapter starts with list of *Opening Questions* that are intended as motivations for the concepts and results presented in this chapter.

  Whenever you start to read a new chapter, please take some time to read these questions and to think about them for a few minutes. This will probably help you to understand the subsequent notions and results and to put them into context.

- Throughout the manuscript, you will find several brief exercises. These exercises are not part of your weekly problem sheets. Most of them are relatively simple, and they are supposed to assist you with your learning process. Please take the time to solve these exercise while you are working with the manuscript.

- Each chapter contains a final section called *Addenda*. In these sections you can read about further results that are related to the lecture but that are not contents of the lecture on their own.

  These addenda might help you to better understand the notions and results discussed in the lecture, or they might arouse your curiosity to learn more about related topics. Please decide on your own which of these addenda you would like to read.

These are lecture notes

Within the notion *lecture notes*, there is particular emphasis on the word *lecture*. This implies the following:

- It is recommended (although not obligatory) that you watch the video lectures, and that you participate in the live sessions.

- No less important: Do the problems on the weekly problem sheets.
- In case that you have any questions: Please well free to contact me at your convenience!

Beware of errors!

Without doubt this manuscript contains errors. In fact, it is very likely that there are many of them.

- If you believe you have found an error, please let me know (really!). The easiest way to do so is via email.
- Email address: jochen.glueck@uni-passau.de

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1 Topologies and topological spaces: basic notions

Opening Questions.

(a) When is a subset $S \subseteq \mathbb{R}^d$ called *open*? Is the union of open sets again open? Is the intersection of open sets again open?

(b) What does it mean to say that a point x is located *on the boundary* of a set $S \subseteq \mathbb{R}^d$?

(c) Can we also speak about open and closed sets in metric spaces?

What structure do we need on a set X in order to speak about open and closed subsets of X?

1.1 Topologies and topological spaces

Two of the most fundamental notions in analysis are *convergence of sequences* and *continuity of functions*. Let us briefly recall how those notions can be characterised on the space $\mathbb{R}^d$:

- A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ *converges* to a vector $x \in \mathbb{R}^d$ iff[1] for every neighbourhood $U \subseteq \mathbb{R}^d$ of x there exists an index $n_0 \in \mathbb{N}_0$ such that $x_n \in U$ for all $n \geq n_0$.
- A function $f : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2}$ is *continuous* iff the pre-image $f^{-1}(U)$ of every open subset $U \subseteq \mathbb{R}^{d_2}$ is open in $\mathbb{R}^{d_1}$.

Depending on the lecturer (or book author), you might have seen convergence and continuity defined in a slightly different way, but the two descriptions above are equivalent to the many other possible definitions of convergence and continuity.

You also know that these notions can be generalized to other spaces than $\mathbb{R}^d$, for instance to so-called *metric spaces*. The main idea of mathematical *topology* is to take this even further: we want to define concepts such as convergence and continuity in very general situations where no metric might be available.[2] The brief reminder above suggests what we actually need in

[1] We use the rather common abbreviation "iff" for "if and only if".

[2] This is not done out of mere mathematical curiosity; in fact, one encounters many situations in mathematics where convergence and continuity cannot be described by means of a metric.

order to talk about convergence and continuity: we need to know what an *open set* is and what a *neighbourhood* of a given point is.

This latter observation is the starting point of topology: We begin with a set X and with a collection τ of subsets of X which we simply call the *open sets* in X.[3] Of course, we will need the collection τ to satisfy certain axioms in order to built a rich and useful theory. The axioms that we shall impose on τ are again suggested by the spaces $\mathbb{R}^d$. Recall the following properties of open subsets on $\mathbb{R}^d$:

- The empty set is open and the entire space $\mathbb{R}^d$ is open.
- The intersection of finitely many open sets in $\mathbb{R}^d$ is again open.
- The union of arbitrarily many open sets in $\mathbb{R}^d$ is again open.

Let us take those three properties as a blueprint of what we require from the collection τ of open sets on our set general X; this guides us to the following definition:

Definition 1.1.1 (Topologies, topological spaces and open sets).

(a) Let X be a set. A *topology* on X is a subset τ of the power set 2^X of X with the following properties:[4]

  (I) We have $\emptyset \in \tau$ and $X \in \tau$.

  (II) The set τ is stable with respect to finite intersections, i.e.: for each $n \in \mathbb{N}$ and all $U_1, \dots, U_n \in \tau$ we have $U_1 \cap \dots \cap U_n \in \tau$.[5]

  (III) The set τ is stable with respect to arbitrary unions, i.e.: for each index set I and each family $(U_i)_{i\in I}$ such that $U_i \in \tau$ for each i, we have have $\bigcup_{i\in I} U_i \in \tau$.

(b) A *topological space* is a pair (X, τ) such that X is a set and τ is a topology on X.

(c) Let (X, τ) be a topological space. A subset $U \subseteq X$ is called *open* iff $U \in \tau$.

On each set X there are two very simply topologies:

Examples 1.1.2 (Discrete and indiscrete topology). Let X be a set.

[3] The second notion that we have just mentioned – *neighbourhoods* – will be of relevance later, namely from Section 1.3 on.

[4] Throughout the course, we use the notation 2^X for the *power set* of a set X, i.e., for the set of all subsets of X.

[5] Note that, by a simple induction argument, this property is equivalent to requiring that $U_1 \cap U_2 \in \tau$ for all $U_1, U_2 \in \tau$.

(a) The power set 2^X itself is a topology on X. It is called the *discrete topology* on X.

(b) The set $\{\emptyset, X\}$ is a topology on X. It is called the *indiscrete topology* on X.

Both those examples are, in a sense, trivial; but we shall always keep those two examples in mind as extreme cases that can occur. Here is a slightly more interesting example of a topology:

Example 1.1.3 (The topology of co-finite sets). Let X be a set. We call a subset $U \subseteq X$ *co-finite* iff its complement $U^c := X \setminus U$ is finite. The set

$$\tau := \{U \subseteq X : U \text{ is co-finite}\} \cup \{\emptyset\}$$

is a topology on X.

Exercise 1.1.4 (The topology of co-finite sets). Let X be a set. Prove that the set τ of co-finite subsets of X is indeed a topology on X, as claimed in Example 1.1.3.

Finally, we come back to the example $\mathbb{R}^d$ which motivated our definition of a topology:

Example 1.1.5 (Euclidean topology on $\mathbb{R}^d$). In the course Analysis 2, a subset U of $\mathbb{R}^d$ is called *open* if it has the following property: for each $x \in U$ there exists a number $\varepsilon > 0$ such that all vectors $y \in \mathbb{R}^d$ with $\left\|y - x\right\|_2 < \varepsilon$ are also in U.[6]

Let τ denote the set of all subsets of $\mathbb{R}^d$ that are open in the sense of Analysis 2. Then it is not difficult to show that τ is a topology[7] – the so-called *Euclidean topology* on $\mathbb{R}^d$.

According to Definition 1.1.1(c) the sets in τ are called *open sets*. Since we defined the Euclidean topology τ to consist of all sets that are open in the sense of Analysis 2, this means that for the Euclidean topology our definition of the notion *open* is consistent with the notion *open* from your Analysis 2 course.

Next, we introduce the notion of a *closed set*:

Definition 1.1.6 (Closed sets). Let (X, τ) be a topological space. A subset $C \subseteq X$ is called *closed* if and only if its complement $C^c = X \setminus C$ is open.

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[6] Here, $\|\cdot\|_2$ denotes the *Euclidean norm*, i.e., $\|x\|_2 = \left(x_1^2 + \dots + x_d^2\right)^{1/2}$ for each $x \in \mathbb{R}^d$.

[7] We shall not show this here, because we are going to prove this in more generality in Section 1.4

It follows readily from the above definition that a subset S in a topological space is open if and only if its complement is closed.

The set of closed subsets of a topological space has stability properties which are, in a sense, complementary to those of the open sets:

Proposition 1.1.7 (Stability properties of closed sets). *Let (X, τ) be a topological space.*

(a) *The sets $\emptyset$ and X are closed.*

(b) *Let $n \in \mathbb{N}$ and let $C_1, \dots, C_n \subseteq X$ be closed. Then the union $C_1 \cup \dots \cup C_n$ is closed, too.*

(c) *Let I be an arbitrary index set, and for each $i \in I$ let $C_i \subseteq X$ be closed. Then the intersection $\bigcap_{i \in I} C_i$ is closed, too.*

Proof. (a) The set X is closed since its complement $\emptyset$ is open; and the set $\emptyset$ is closed because its complement X is open.

(b) If $C_1, \dots, C_n \subseteq X$ are closed, then their complements ín X are open. Hence, the set

$$\left(C_1 \cup \dots \cup C_n\right)^{\mathrm{c}} = (C_1)^{\mathrm{c}} \cap \dots \cap (C_n)^{\mathrm{c}}$$

is open, so $C_1 \cup \dots \cup C_n$ is closed.

(c) Since each set C_i is closed, its complement is open. Thus, the set

$$\left(\bigcap_{i \in I} C_i\right)^{\mathrm{c}} = \bigcup_{i \in I} (C_i)^{\mathrm{c}}$$

is open, so we conclude that $\bigcap_{i \in I} C_i$ is closed. □

Let us list a few examples of closed sets in certain topological spaces:

Examples 1.1.8 (Closed sets in various topological spaces).

(a) If a set X is endowed with the discrete topology, every subset of X is closed.

(b) If a set X is endowed with the indiscrete topology, only the sets $\emptyset$ and X are closed.

(c) Let X be a set and let τ be the topology on X that consists of co-finite sets (see Example 1.1.3). Then the closed sets in the topological space (X, τ) are precisely the finite subsets of X and the set X itself.

(d) Let $\mathbb{R}^2$ be endowed with the Euclidean topology. For instance, the following subsets of $\mathbb{R}^2$ are closed: every finite set; the unit circle; every coordinate axis.[8]

The fact that every finite set is closed can, for instance, be seen as follows: one first shows that the complement of every one-point set is open, and thus concludes that every one-point set is closed. Hence, it follows from Proposition 1.1.7(b) that every finite set is closed.

An explanation why the coordinate axis and the unit circle are open will be given later, in Example 2.3.6.

1.2 Interior and closure

In order to further develop the theory of topological spaces, we need a few more concepts that can be defined in every topological space; we begin with the interior and the closure of a set.

Definition 1.2.1 (Interior and closure). Let (X,τ) be a topological space and let $S \subseteq X$.

(a) The *interior* of S is defined to be the union of all open sets in X that are contained in S; we denote it by S^{o}.

(b) The *closure* of S is defined to be the intersection of all closed sets in X that contain S; we denote it by $\overline{S}$.

Let (X,τ) be a topological space and let $S \subseteq X$. By definition, we have $S^{\mathrm{o}} \subseteq S \subseteq \overline{S}$. Moreover, the interior S^{o} is an open set (since the union of arbitrarily many sets is open according to the definition of a topology), and the closure $\overline{S}$ is closed (since the intersection of arbitrarily many closed sets is closed according to Proposition 1.1.7(c)). The following exercise and the subsequent proposition give a bit more precise information:

Exercise 1.2.2 (Open and closed sets via interior and closure). Let (X,τ) be a topological space and let $S \subseteq X$.

(a) Prove that S is open if and only if $S = S^{\mathrm{o}}$.

(b) Prove that S is closed if and only if $S = \overline{S}$.

Proposition 1.2.3 (Characterization of interior and closure). *Let (X,τ) be a topological space and let $S \subseteq X$.*

[8] Of course there are many more closed subsets of $\mathbb{R}^2$; later on in this course, in particular in Proposition 2.3.4, you will learn about efficient ways to identify many closed subsets of a topological space.

(a) *The interior S^{o} is the largest open subset of S. More precisely: S^{o} is an open subset of S, and if U is another open subset of S then $U \subseteq S^{\mathrm{o}}$.*

(b) *The closure $\overline{S}$ is the smallest closed superset of S. More precisely: $\overline{S}$ is a closed superset of S, and if C is another closed superset of S, then $\overline{S} \subseteq C$.*

Proof. In the discussion preceding the proposition we have already noted that S^{o} is an open subset of S and that $\overline{S}$ is a closed superset of S.

Now, let U be an open subset of S. Per definition, S^{o} is the union of all open subsets of S; in particular, U is one of the sets that are unified to obtain S^{o}. Hence, $U \subseteq S^{\mathrm{o}}$.

Similarly, let C be a closed superset of S. Per definition, S^{o} is the intersection of all closed supersets of S. In particular, C is one of the sets that are intersected to obtain $\overline{S}$. Hence, $\overline{S} \subseteq C$. □

As a consequence of the preceding proposition we obtain the following "monotonicity" properties of the interior and the closure:[9]

Corollary 1.2.4 (Interior and closure are monotone set operations). *Let (X, τ) be a topological space and let $S_1 \subseteq S_2 \subseteq X$. Then*

$$(S_1)^{\mathrm{o}} \subseteq (S_2)^{\mathrm{o}} \qquad \textit{and} \qquad \overline{S_1} \subseteq \overline{S_2}.$$

Proof. As $(S_1)^{\mathrm{o}}$ is an open subset of S_2 and $(S_2)^{\mathrm{o}}$ is the largest open subset of S_2, we have $(S_1)^{\mathrm{o}} \subseteq (S_2)^{\mathrm{o}}$.

Similarly, as $\overline{S_2}$ is a closed superset of S_1 and $\overline{S_1}$ is the smallest closed superset of S_1, we have $\overline{S_1} \subseteq \overline{S_2}$. □

The following proposition describes how interiors and closures are related via complements:

Proposition 1.2.5 (Complements of interiors and closures). *Let (X, τ) be a topological space and let $S \subseteq X$.*

(a) *We have*

$$(S^{\mathrm{o}})^{\mathrm{c}} = \overline{(S^{\mathrm{c}})} \qquad \textit{and} \qquad \left(\overline{S}\right)^{\mathrm{c}} = (S^{\mathrm{c}})^{\mathrm{o}}.$$

(b) *Consequently, we have the representations*

$$S^{\mathrm{o}} = \left(\overline{(S^{\mathrm{c}})}\right)^{\mathrm{c}} \qquad \textit{and} \qquad \overline{S} = ((S^{\mathrm{c}})^{\mathrm{o}})^{\mathrm{c}}$$

for the interior and the closure of S.

[9] Alternatively, one could infer those monotonicity properties directly from the definition of the interior and the closure.

Proof. (a) We proceed in four steps:

- *Step 1: Let us first show that* $(S^{\mathrm{o}})^{\mathrm{c}} \supseteq \overline{(S^{\mathrm{c}})}$*:* The set $(S^{\mathrm{o}})^{\mathrm{c}}$ is closed and a superset of S^{c} (since $S^{\mathrm{o}} \subseteq S$). Hence, $(S^{\mathrm{o}})^{\mathrm{c}}$ is also a superset of the closure $\overline{(S^{\mathrm{c}})}$ (see Proposition 1.2.3(b)).

- *Step 2: Next we show* $\left(\overline{S}\right)^{\mathrm{c}} \supseteq (S^{\mathrm{c}})^{\mathrm{o}}$*:* By substituting S with S^{c} in the formula $(S^{\mathrm{o}})^{\mathrm{c}} \supseteq \overline{(S^{\mathrm{c}})}$ that we have shown in Step 1, we obtain

$$((S^{\mathrm{c}})^{\mathrm{o}})^{\mathrm{c}} \supseteq \overline{S}$$

  and thus, by taking complements, $\left(\overline{S}\right)^{\mathrm{c}} \supseteq (S^{\mathrm{c}})^{\mathrm{o}}$.

- *Step 3: Now we prove that* $\left(\overline{S}\right)^{\mathrm{c}} \subseteq (S^{\mathrm{c}})^{\mathrm{o}}$*:* The set $\left(\overline{S}\right)^{\mathrm{c}}$ is open and a subset of S^{c} (since $\overline{S} \supseteq S$). Hence, $\left(\overline{S}\right)^{\mathrm{c}}$ is also a subset of the interior of S^{c} (see Proposition 1.2.3(a)).

- *Step 4: Finally, we show that* $(S^{\mathrm{o}})^{\mathrm{c}} \subseteq \overline{(S^{\mathrm{c}})}$*:* By substituting S with S^{c} in the formula $\left(\overline{S}\right)^{\mathrm{c}} \subseteq (S^{\mathrm{c}})^{\mathrm{o}}$ that we have shown in Step 3, we obtain

$$\left(\overline{(S^{\mathrm{c}})}\right)^{\mathrm{c}} \subseteq S^{\mathrm{o}}$$

  and thus, by taking complements, $(S^{\mathrm{o}})^{\mathrm{c}} \subseteq \overline{(S^{\mathrm{c}})}$

(b) This follows immediately from the formulae in (a) by taking complements. □

The stability properties of the open and closed sets with respect to intersections and unions lead to certain rules for the manipulation of interiors and closures of sets. Let us sum up those rules in the following two propositions:

Proposition 1.2.6 (Rules for manipulating interiors). *Let (X,τ) be a topological space.*

(a) *Let I be an arbitrary index set and let $S_i \subseteq X$ for each $i \in I$. Then*

$$\left(\bigcap_{i\in I} S_i\right)^{\mathrm{o}} \subseteq \bigcap_{i\in I} (S_i)^{\mathrm{o}} \qquad \textit{and} \qquad \left(\bigcup_{i\in I} S_i\right)^{\mathrm{o}} \supseteq \bigcup_{i\in I} (S_i)^{\mathrm{o}}$$

(b) *Let $n \in \mathbb{N}$ and let $S_1,\dots,S_n \subseteq X$. Then*

$$(S_1 \cap \cdots \cap S_n)^{\mathrm{o}} = (S_1)^{\mathrm{o}} \cap \cdots \cap (S_n)^{\mathrm{o}}.$$

Proof. (a) First, we prove the inclusion on the left. Let us abbreviate $U := (\bigcap_{i\in I} S_i)^\circ$. Then U is an open set, and for each $i \in I$ we have $U \subseteq S_i$ and thus also $U \subseteq (S_i)^\circ$ according to Proposition 1.2.3(a).

Now we prove the inclusion on the right. The set $\bigcup_{i\in I}(S_i)^\circ$ is open and a subset of $\bigcup_{i\in I} S_i$; hence, $\bigcup_{i\in I}(S_i)^\circ$ is contained in the interior of $\bigcup_{i\in I} S_i$ according to Proposition 1.2.3(a).

(b) Above, we have already shown the inclusion "$\subseteq$" for arbitrarily many sets. To show the converse inclusion "$\supseteq$", note that the set

$$U := (S_1)^\circ \cap \dots \cap (S_n)^\circ$$

is open as an intersection of finitely many open sets; moreover, U is clearly as subset of $S_1 \cap \dots \cap S_n$ and thus also of its interior (due to Proposition 1.2.3(a)). □

Proposition 1.2.7 (Rules for manipulating closures). *Let (X,τ) be a topological space.*

(a) *Let I be an arbitrary index set and let $S_i \subseteq X$ for each $i \in I$. Then*

$$\overline{\bigcap_{i\in I} S_i} \subseteq \bigcap_{i\in I} \overline{S_i} \qquad \text{and} \qquad \overline{\bigcup_{i\in I} S_i} \supseteq \bigcup_{i\in I} \overline{S_i}$$

(b) *Let $n \in \mathbb{N}$ and let $S_1,\dots,S_n \subseteq X$. Then*

$$\overline{S_1 \cup \dots \cup S_n} = \overline{S_1} \cup \dots \cup \overline{S_n}.$$

Proof. There are two ways to prove this: We can either argue similarly as in the proof of Proposition 1.2.6. Alternatively, we can employ Proposition 1.2.5 – which tells us how to switch between interiors and closures by taking complements – in order to derive our claims from the assertions of Proposition 1.2.6.

We pose the details as an exercise. □

Let us now discuss a few examples to illustrate the concepts of the interior and the closure of a set.

Example 1.2.8 (Topological properties of intervals). We endow the set $\mathbb{R}$ with the Euclidean topology.[10] Let $a,b \in \mathbb{R}$ and $a < b$. Let us consider the four intervals (a,b), $(a,b]$, $[a,b)$ and $[a,b]$. These sets have the following properties:

(a) The set (a,b) is open but not closed; the set $[a,b]$ is closed but not open; the sets $(a,b]$ and $[a,b)$ are neither open nor closed.

[10] See Example 1.1.5, where the Euclidean topology was introduced (on $\mathbb{R}^d$ for a general dimension d).

(b) The interior of each of the four sets is (a, b). The closure of each of the four sets is $[a, b]$.

Proof. (a) We first show that (a, b) is open. Let $x \in (a, b)$. If we choose ε to be the minimum of $x - a$ and $b - x$, then $\varepsilon > 0$, and every point $y \in \mathbb{R}$ that is strictly closer than ε to x is still in (a, b). Hence, (a, b) is open.

Next we show that $[a, b]$ is closed. To this end, we must prove that $[a, b]^c = (-\infty, a) \cup (b, \infty)$ is open. So let $x \in [a, b]^c$. If we choose ε to be the minimum of $|a - x|$ and $|b - x|$, then $\varepsilon > 0$, and every point $y \in \mathbb{R}$ that is strictly closer than ε to x is also in $[a, b]^c$. Hence, $[a, b]^c$ is open.

Next we show that $(a, b]$ is not open. To this end, consider the point $b \in (a, b]$. If $\varepsilon > 0$ is an arbitrary number, there is always a point $y \in \mathbb{R}$ which is strictly closer then ε to b, but which is not contained in $(a, b]$ (for instance the point $y := b + \varepsilon/2$). Hence, $(a, b]$ is not open.

For a similar reason, the complement $(a, b]^c = (-\infty, a] \cup (b, \infty)$ is not open, so we conclude that $(a, b]$ is not closed.

By analogous arguments one can also show that $[a, b)$ is neither open nor closed.

(b) Let S be one of our four intervals. We first show that $(a, b) = S^\circ$. To this end, note that (a, b) is an open set that is contained in S; hence, $(a, b) \subseteq S^\circ \subseteq S$. On the other hand, if this inclusion was strict – i.e., if $(a, b) \subsetneq S^\circ$ –, then S° would have to be one of the sets $(a, b]$, $[b, a)$ and $[a, b]$; but this cannot be true since none of these sets is open.

Now we show that $\overline{S} = [a, b]$. To this end, we first observe that $[a, b]$ is a closed set that contains S, so $[a, b] \supseteq \overline{S} \supseteq S$. If this inclusion was strict, $\overline{S}$ would have to be one of the sets (a, b), $(a, b]$ and $[a, b)$; but this cannot be true since none of these sets is closed. □

Example 1.2.9 (Interior and closure in the co-finite topology). Let X be a set and let τ be the topology of co-finite sets that we introduced in Example 1.1.3. Thus, a subset of X is open if and only if it is co-finite or empty; and a subset of X is closed if and only if it is finite or equal to X.

Let $S \subseteq X$. Then:

(a) If S is co-finite, then $S^\circ = S$; otherwise, $S^\circ = \emptyset$.

(b) If S is finite, then $\overline{S} = S$; otherwise, $\overline{S} = X$.

Proof. (a) If S is co-finite, then S is open, so $S^\circ = S$ (Exercise 1.2.2(a)). If S is not co-finite, then no subset of S is co-finite, either. Hence, the only open subset of S is $\emptyset$, which shows that $S^\circ = \emptyset$.

(b) If S is finite, then S is closed, so $\overline{S} = S$ (Exercise 1.2.2(b)). If S is not finite, then no superset of S is finite, either. Hence, the only closed superset of S is X, so $\overline{S} = X$. □

Examples 1.2.8 and 1.2.9 can be used to show that the inclusions in Propositions 1.2.6 and 1.2.7 are not equalities, in general. We discuss this in an exercise.

1.3 Neighbourhoods and boundary

A neighbourhood of a point x in a topological space is simply a set that contains x in its interior. Let us make this precise in the following definition:

Definition 1.3.1 (Neighbourhoods). Let (X,τ) be a topological space and let $x \in X$. A set $N \subseteq X$ is called a *neighbourhood* of x if $x \in N^{\circ}$.

Exercise 1.3.2 (Neighbourhoods via open subsets). Let (X,τ) be a topological space, let $x \in X$ and $N \subseteq X$. Prove that N is a neighbourhood of x if and only if there exists an open set $U \subseteq X$ such that

$$x \in U \subseteq N.$$

Example 1.3.3 (Some neighbourhoods in $\mathbb{R}$). Endow $\mathbb{R}$ with the Euclidean topology. From Example 1.2.8 and Exercise 1.3.2 one can readily infer the following properties:

(a) Each of the sets $(0,2)$, $(0,2]$, $[0,2)$ and $(0,2)$ is a neighbourhood of the point 1.

(b) The set $(0,2] \cup [4,5]$ is a neighbourhood of the point 1.

(c) The set $(0,2]$ is not a neighbourhood of the point 2.

The next result demonstrates how the closure of a set can be described by means of neighbourhoods:

Proposition 1.3.4 (The closure via neighbourhoods). *Let (X,τ) be a topological space, let $x \in X$ and let $S \subseteq X$. We have $x \in \overline{S}$ if and only if every neighbourhood of x has non-empty intersection with S.*

Proof. "$\Rightarrow$" Let $x \in \overline{S}$ and let $N \subseteq X$ be a neighbourhood of x. Assume towards a contradiction that N does not intersect S. Then, in particular, N° does not intersect S, so S is contained in the set $(N^{\circ})^{c}$. Since the latter set is closed, we even have

$$\overline{S} \subseteq (N^{\circ})^{c}$$

Hence, $\overline{S}$ does not intersect N° either; but this is a contradiction since x is contained in both $\overline{S}$ and N°.

"$\Leftarrow$" Let us again argue via contraposition: we assume that $x \notin \overline{S}$. Then the open set $N := \left(\overline{S}\right)^{c}$ contains x and is thus a neighbourhood of x that does not intersect S (since it does not intersect $\overline{S}$). $\square$

Now we introduce one further notion which is important from a geometric point of view:

Definition 1.3.5 (Boundary of a set). Let (X, τ) be a topological space and let $S \subseteq X$. The set

$$\partial S := \overline{S} \setminus S^{\mathrm{o}}$$

is called the *boundary* of S.

Let us prove a few properties of the boundary of a set:

Proposition 1.3.6 (Properties of the boundary). *Let (X, τ) be a topological space and let $S \subseteq X$. Then:*

(a) *We have $\partial S = \overline{S} \cap \overline{S^{c}}$.*

(b) *The boundary ∂S is closed.*

(c) *We have $\partial S = \partial(S^{c})$.*

(d) *We have $\overline{S} = (S^{\mathrm{o}}) \cup (\partial S) = S \cup \partial S$.*

Proof. (a) We have

$$\partial S = \overline{S} \setminus S^{\mathrm{o}} = \overline{S} \cap (S^{\mathrm{o}})^{c} = \overline{S} \cap \overline{S^{c}},$$

where the last equality follows from Proposition 1.2.5(a).

(b) According to assertion (a), the boundary of S is the intersection of the closed sets $\overline{S}$ and $\overline{S^{c}}$; thus, ∂S is itself closed.

(c) This follows since the formula in (a) yields the same result if we replace S with S^{c}.

(d) Since $S^{\mathrm{o}} \subseteq \overline{S}$, we have

$$\overline{S} = (S^{\mathrm{o}}) \cup \left(\overline{S} \setminus S^{\mathrm{o}}\right) = (S^{\mathrm{o}}) \cup (\partial S);$$

this proves the first equality. Moreover,

$$(S^{\mathrm{o}}) \cup (\partial S) \subseteq S \cup \partial S \subseteq \overline{S},$$

so the identical sets $\overline{S}$ and $(S^{\mathrm{o}}) \cup (\partial S)$ are also equal to $S \cup \partial S$. $\square$

Next, we get back to our interval examples and discuss their boundary:

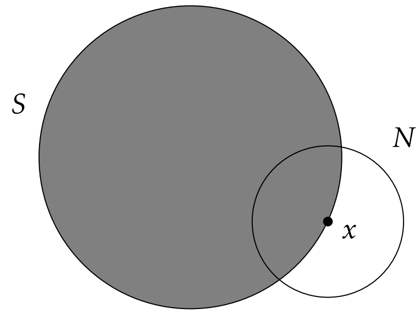

Figure 1.3.1: Illustration of Proposition 1.3.8: The point x is located on the boundary of S. Every neighbourhood N of x intersects both S and S^c.

Example 1.3.7 (The boundary of an interval). Endow $\mathbb{R}$ with the Euclidean topology and let $a, b \in \mathbb{R}$ such that $a < b$. If S is one of the intervals (a,b), $(a,b]$, $[a,b)$ and $[a,b]$, then it follows from Example 1.2.8(b) that

$$\partial S = \overline{S} \setminus S^{\circ} = [a,b] \setminus (a,b) = \{a,b\};$$

so the boundary of S consists precisely of the points a and b.

Recall that, in Proposition 1.3.4, we described the elements x of the closure of a set S by means of the neighbourhoods of x. A similar description is possible for the points in the boundary of S:

Proposition 1.3.8 (Boundary via neighbourhoods). *Let (X,τ) be a topological space, let $x \in X$ and let $S \subseteq X$. We have $x \in \partial S$ if and only if every neighbourhood of x has non-empty intersection with S and non-empty intersection with S^c.*

Proof. This is an immediate consequence of the description of the points in the closure that we gave in Proposition 1.3.4 and of the formula for the boundary that we gave in Proposition 1.3.6(a). □

The situation in Proposition 1.3.8 is illustrated in Figure 1.3.1.

1.4 Metric spaces as topological spaces

In this section we give a reminder of the notion *metric space* and we explain that there is a natural topology associated to every metric space.

Definition 1.4.1 (Metrics and metric spaces). (a) Let M be a set. A *metric* on M is a mapping $\mathrm{d} : M \times M \to [0,\infty)$ that satisfies the following properties:

(I) *Positive definiteness:* For all $x, y \in M$ we have the following equivalence: $\mathrm{d}(x,y) = 0$ if and only $x = y$.

(II) *Symmetry:* For all $x, y \in M$ we have $\mathrm{d}(x,y) = \mathrm{d}(y,x)$.

(III) *Triangle inequality:* For all $x, y, z \in M$ we have

$$\mathrm{d}(x,z) \le \mathrm{d}(x,y) + \mathrm{d}(y,z).$$

(b) A *metric space* is a pair (M, d) where M is a set and d is a metric on M.

Here is a list of various examples of metric spaces:

Example 1.4.2 (Discrete metric). Let X be a set. We define a mapping $\mathrm{d} : X \times X \to [0,\infty)$ by

$$\mathrm{d}(x,y) := \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y \end{cases}$$

for all $x, y \in X$. Then one can easily check that d is a metric on X – the so-called *discrete metric*.

Example 1.4.3 (Euclidean metric on $\mathbb{R}^d$). We define a mapping $\mathrm{d} : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ by

$$\mathrm{d}(x,y) = \left(\sum_{k=1}^{d} (x_k - y_k)^2 \right)^{1/2}$$

for all $x, y \in \mathbb{R}^d$. As you know from your Analysis 2 course, d is a metric[11] on $\mathbb{R}^d$ – the so-called *Euclidean metric* on $\mathbb{R}^d$.

Example 1.4.4 (SNCF metric). Let $n \in \mathbb{N}$ and let us consider a set F with precisely $n+1$ elements which we denote by p and $s_1, \dots, s_n$. Moreover, fix a finite sequence of numbers $d_1, \dots, d_n \in (0,\infty)$. We define a metric $\mathrm{d} : F \times F \to [0,\infty)$ by the formulas

$$\begin{aligned}
&\mathrm{d}(p,p) = 0, \\
&\mathrm{d}(s_j, s_j) = 0 \quad \text{for all } j \in \{1,\dots,n\}, \\
&\mathrm{d}(s_j, s_k) = d_j + d_k \quad \text{for all distinct } j,k \in \{1,\dots,n\}, \\
&\mathrm{d}(s_j, p) = \mathrm{d}(p, s_j) = d_j \quad \text{for all } j \in \{1,\dots,n\}.
\end{aligned}$$

It is not difficult to check that d is indeed a metric on F. Cynics used to called it the *French railway metric*, or for short the *SNCF metric*. A visualisation of this metric is shown in Figure 1.4.1.

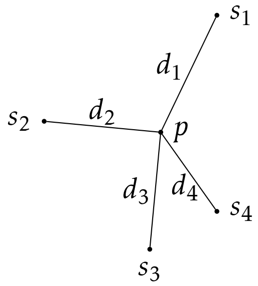

Figure 1.4.1: Visualisation of the SNCF metric from Example 1.4.4 for the case $n = 4$.

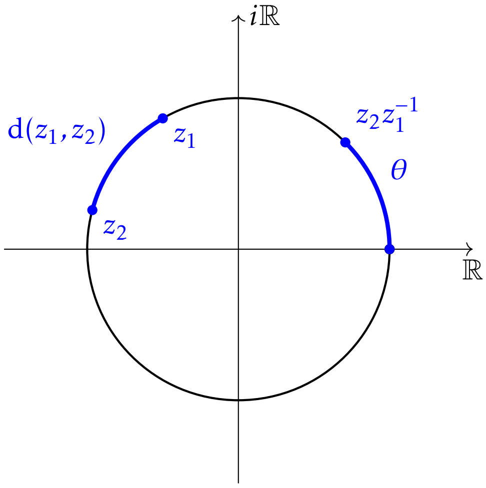

Figure 1.4.2: Illustration of Example 1.4.5: Distance $\mathrm{d}(z_1, z_2)$ of the points z_1 and z_2 on the complex unit circle.

Example 1.4.5 (Metric on the unit circle). Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ denote the complex unit circle. For all $z_1, z_2 \in \mathbb{T}$ we define $\mathrm{d}(z_1, z_2) \in [0, \infty)$ as the length of the shortest path on the unit circle that it takes to get from z_1 to z_2 (or vice versa).

More formally, this number can be computed as follows: let θ denote that unique number in the interval $[-\pi, \pi)$ such that $e^{i\theta} = z_2 z_1^{-1}$; then $\mathrm{d}(z_1, z_2) = |\theta|$.

The mapping $\mathrm{d} : \mathbb{T} \times \mathbb{T} \to [0, \infty)$ is a metric.

An illustration of this example is given in Figure 1.4.2.

[11] Recall that for this metric, the triangle inequality is a consequence of the Cauchy–Schwarz inequality.

Example 1.4.6 (Supremum metric). Let $\mathrm{C}([0,1];\mathbb{C})$ denote the set of all continuous functions from $[0,1]$ to $\mathbb{C}$. If we define

$$\mathrm{d}(f,g) := \|f-g\|_\infty := \sup_{x\in[0,1]} |f(x)-g(x)|$$

for all $f,g \in \mathrm{C}([0,1];\mathbb{C})$, then d is a metric on $\mathrm{C}([0,1];\mathbb{C})$, which we call the *supremum metric*.

As you can see in the above examples, we often use the same symbol d to denote a metric in various different examples. So whenever we use a specific metric in a specific example it is important to clarify which metric precisely we are using.

The Euclidean metric in Example 1.4.3 and the supremum metric in Example 1.4.6 are both metrics which are induced by a norm on a vector space.

On metric spaces, the notion of a *ball* is particularly important; thus, we introduce the following notation for balls:

Definition 1.4.7 (Balls on metric spaces). Let (M,d) be a metric space. Let $x \in M$ and $r \in [0,\infty)$.

(a) The *open ball* in M with center x and radius r is the set

$$\mathrm{B}_{<r}(x) := \{y \in M : \mathrm{d}(y,x) < r\}$$

(b) The *closed ball* in M with center x and radius r is the set

$$\mathrm{B}_{\le r}(x) := \{y \in M : \mathrm{d}(y,x) \le r\}$$

In the following proposition, we list a few properties of closed and open balls in metric spaces.

Proposition 1.4.8 (Properties of balls). *Let (M,d) be a metric space and let $x \in M$.*

(a) *We have $\mathrm{B}_{<0}(x) = \emptyset$ and $\mathrm{B}_{\le 0}(x) = \{x\}$.*

(b) *For each $r \in [0,\infty)$ we have $\mathrm{B}_{<r}(x) \subseteq \mathrm{B}_{\le r}(x)$.*

(c) *For each $r \in [0,\infty)$ we have $x \in \mathrm{B}_{\le r}(x)$ and for each $r \in (0,\infty)$ we have $x \in \mathrm{B}_{<r}(x)$.*

(d) *For all numbers $0 \le s < r < \infty$ we have $\mathrm{B}_{\le s}(x) \subseteq \mathrm{B}_{<r}(x)$. In fact, we even have*

$$\mathrm{B}_{<r}(x) = \bigcup_{s\in[0,r)} \mathrm{B}_{\le s}(x)$$

for each $r \in (0,\infty)$.

The proof of Proposition 1.4.8 is not difficult; it is left as an exercise for you in order to check your understanding of metric spaces and open and closed balls:

Exercise 1.4.9 (Properties of balls). Prove Proposition 1.4.8.

On every metric space, the metric induces a topology in a canonical way. Thus, metric spaces constitute an important class of examples for topological spaces. Let us discuss this in more detail in the following proposition and the subsequent definition.

Proposition 1.4.10 (Induced topology on a metric space). *Let* (M, d) *be a metric space. Let* $\tau \subseteq 2^M$ *denote the set of all subsets* $U \subseteq M$ *that have the following property:*

$$\forall x \in U \; \exists \varepsilon > 0 \text{ such that } \mathrm{B}_{<\varepsilon}(x) \subseteq U. \tag{1.4.1}$$

Then τ *is a topology on* M.

Proof. The empty set and the entire space: Obviously, both the empty set and the entire space M satisfy condition (1.4.1), so $\emptyset, M \in \tau$.

Stability with respect to finite intersection: Let $U_1, U_2 \in \tau$; in order to prove that the set $U_1 \cap U_2$ satisfies the condition (1.4.1), let $x \in U_1 \cap U_2$. Since U_1 and U_2 both satisfy (1.4.1), there exist numbers $\varepsilon_1, \varepsilon_2 > 0$ such that $\mathrm{B}_{<\varepsilon_1}(x) \subseteq U_1$ and $\mathrm{B}_{<\varepsilon_2}(x) \subseteq U_2$. So if we set $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$, we conclude that

$$\mathrm{B}_{<x}(\varepsilon) \subseteq \mathrm{B}_{<x}(\varepsilon_1) \cap \mathrm{B}_{<x}(\varepsilon_2) \subseteq U_1 \cap U_2.$$

Hence, $U_1 \cap U_2$ satisfies (1.4.1) and is thus in τ.

Stability with respect to arbitrary unions: Let I be a set, and for each $i \in I$ let $U_i \in \tau$. We have to show that the union $\bigcup_{i \in I} U_i$ satisfies (1.4.1). So let $x \in \bigcup_{i \in I} U_i$. Then there exists an index $i_0 \in I$ such that $x \in U_{i_0}$. As U_{i_0} satisfies (1.4.1), there exists a number $\varepsilon > 0$ such that $\mathrm{B}_{<x}(\varepsilon) \subseteq U_{i_0}$, and consequently,

$$\mathrm{B}_{<x}(\varepsilon) \subseteq \bigcup_{i \in I} U_i.$$

Thus, the union $\bigcup_{i \in I} U_i$ satisfies condition (1.4.1), too, and is therefore in τ. □

Definition 1.4.11 (Induced topology on a metric space). Let (M, d) be a metric space. The topology τ from Proposition 1.4.10 is called the *topology induced by* d.

Note that if (M,d) is a metric space, then by Proposition 1.4.10 and Definition 1.4.11, a set $U \subseteq M$ is open (with respect to the topology induced by d) if and only if for each $x \in U$ there exists an open ball with center x and non-zero radius that is contained in U. It is a good exercise to show that the same assertion remains true if we replace open balls with closed balls:

Exercise 1.4.12 (Open sets via closed balls). Let (M,d) be a metric space. Show that a set $U \subseteq M$ is open if and only if it satisfies the following property:

$$\forall x \in U \; \exists \varepsilon > 0 \text{ such that } \mathrm{B}_{\le \varepsilon}(x) \subseteq U.$$

Next we discuss a few nice topological properties of balls:

Proposition 1.4.13 (Topological properties of balls in metric spaces). *Let (M,d) be a metric space, let $x \in M$ and $r \in [0,\infty)$.*

(a) *The open ball $\mathrm{B}_{<r}(x)$ is open.*

(b) *The closed ball $\mathrm{B}_{\le r}(x)$ is closed.*

(c) *We always have $\overline{\mathrm{B}_{<r}(x)} \subseteq \mathrm{B}_{\le r}(x)$.*

Proof. (a) Let $y \in \mathrm{B}_{<r}(x)$ and set $\delta := \mathrm{d}(x,y)$; then $\delta < r$. It follows from the triangle inequality that

$$\mathrm{B}_{<r-\delta}(y) \subseteq \mathrm{B}_{<r}(x);$$

hence, $\mathrm{B}_{<r}(x)$ is indeed open.

(b) We show that the complement $U := (\mathrm{B}_{\le r}(x))^{\mathrm{c}}$ is open. So let $y \in U$. Then the number $s := \mathrm{d}(x,y)$ satisfies $s > r$. By the triangle inequality, the open ball $\mathrm{B}_{<s-r}(y)$ cannot intersect $\mathrm{B}_{\le r}(x)$; hence, $\mathrm{B}_{<s-r}(y) \subseteq U$.

(c) Since $\mathrm{B}_{\le r}(x)$ is closed and contains $\mathrm{B}_{<r}(x)$, the first named set also contains the closure of $\mathrm{B}_{<r}(x)$ (Proposition 1.2.3(b)). □

Note that we do not have equality in Proposition 1.4.13(c), in general. An easy counterexample can be found by choosing $r = 0$: Then we have $\overline{\mathrm{B}_{<0}(x)} = \overline{\emptyset} = \emptyset$, but $\mathrm{B}_{\le 0}(x) = \{x\}$. For non-zero radius, the same problem will be discussed in an exercise.

Throughout the course, we will often come back to metric spaces as important examples for topological spaces. We shall see that, as a rule, the situation is often a bit simpler on metric spaces than on general topological spaces – i.e., metric spaces do often have “better” or “simpler” properties.

1.5 Separation axioms

When we introduced the notion of a topological space, you might have noticed that we did not make any assumptions on the behaviour of one-point sets. In particular, in a general topological space (X,τ), the set $\{x\}$ for a point $x \in X$ need not be closed, in general.

For instance, if X is a set with two or more elements and τ denotes the indiscrete topology on X (Example 1.1.2(b)), then no one-point set is closed.

In the following definition we introduce so-called *separations axioms* for topological spaces; such axioms are satisfied by many – though not all – spaces, and they are related to the question whether one-point sets are closed and whether one can separate any two distinct points by open sets.

Definition 1.5.1 (Separation axioms $\mathbf{T_0}$ – $\mathbf{T_2}$). Let (X,τ) be a topological space.

(a) We say that (X,τ) satisfies the *separation axiom $\boldsymbol{T}_2$* if and only if for any two distinct points $x_1, x_2 \in X$ there exist disjoint open sets $U_1, U_2 \subseteq X$ such that $x_1 \in U_1$ and $x_2 \in U_2$.

A topological space is called a *Hausdorff topological space* if and only if it satisfies the separation axiom $\mathbf{T}_2$.

(b) We say that (X,τ) satisfies the *separation axiom $\boldsymbol{T}_1$* if and only if for any two distinct points $x_1, x_2 \in X$ both of the following two assertions hold:[12]

- There exists an open set $U \subseteq X$ such that $x_1 \in U$ but $x_2 \notin U$.
- There exists an open set $U \subseteq X$ such that $x_2 \in U$ but $x_1 \notin U$.

(c) We say that (X,τ) satisfies the *separation axiom $\boldsymbol{T}_0$* if and only if for any two distinct points $x_1, x_2 \in X$ at least one of the following two assertions holds:

- There exists an open set $U \subseteq X$ such that $x_1 \in U$ but $x_2 \notin U$.
- There exists an open set $U \subseteq X$ such that $x_2 \in U$ but $x_1 \notin U$.

We shall often use the following kind of abbreviations: instead of, for instance, saying that a topological space (X,τ) satisfies the separation axiom $\mathbf{T}_0$, we simply say that (X,τ) is a $\mathbf{T}_0$-space (and similarly for the axioms $\mathbf{T}_1$ and $\mathbf{T}_2$).

[12] Due to the symmetry of these two assertions, it is actually equivalent to assume merely that the first assertion does always hold.

It follows readily from Definition 1.5.1 that we have the implications

$$\mathbf{T}_2 \quad \Rightarrow \quad \mathbf{T}_1 \quad \Rightarrow \quad \mathbf{T}_0.$$

We shall soon see by a couple of concrete examples that none of the converse implications is true. Before we discuss such examples, we prove two simple but useful propositions.

Proposition 1.5.2 (Metric spaces are Hausdorff). *Let (M, d) be a metric space and let τ denote the topology on M induced by* d. *Then (M, τ) is Hausdorff, i.e., it satisfies the separation axiom $\boldsymbol{T}_2$.*

Proof. Let $x_1, x_2 \in M$ be two distinct points. Then it follows from the positive definiteness of the metric that the number $\delta := \mathrm{d}(x_1, x_2)$ is non-zero.[13] The two balls $U_1 := \mathrm{B}_{<\delta/2}(x_1)$ and $U_2 := \mathrm{B}_{<\delta/2}(x_2)$ are open and contain x_1 and x_2, respectively. Moreover, it follows from the triangle inequality that they do not intersect. □

Later on in the course we will see that there exist Hausdorff topological spaces whose topology is not induced by a metric.

Proposition 1.5.3 (Characterisation of $\mathbf{T}_1$-spaces by closed one-point sets). *For each topological space (X, τ) the following assertions are equivalent:*

(i) *The space (X, τ) satisfies the separation axiom $\boldsymbol{T}_1$.*

(ii) *For each $x \in X$ the set $\{x\}$ is closed.*

Proof. "(i) ⇒ (ii)" Let $x \in X$. In order to show that $\{x\}$ is closed, we show that the complement $V := \{x\}^{\mathrm{c}}$ is open. And to this end, it suffices to prove that each element of V is contained in the interior V^{o} (Exercise 1.2.2(a))

So let $y \in V$. Then $y \neq x$, so the separation axiom $\mathbf{T}_1$ implies that the existence of an open set $U \subseteq X$ that contains y, but not x. In particular, the open set U is a subset of V, and thus also of the interior V^{o} of V. Hence, $y \in U \subseteq V^{\mathrm{o}}$. We have proved that $V = V^{\mathrm{o}}$, i.e., V is open.

"(ii) ⇒ (i)" In order to prove that X satisfies the separation axiom $\mathbf{T}_1$, let $x_1, x_2 \in X$ be two distinct points. Assertions (ii) implies that the sets $U_1 := \{x_2\}^{\mathrm{c}}$ and $U_2 := \{x_1\}^{\mathrm{c}}$ are open. Moreover, U_1 contains x_1 but not x_2, and similarly, U_2 contains x_2 but not x_1. □

Note that Proposition 1.5.3 implies that in a $\mathbf{T}_1$-space every finite set is closed. Now we discuss a few examples, as promised:

[13] This is actually the first time that we us the positive definiteness of the metric; everything also that we have discussed from the beginning of Section 1.5 until now also works without the assumption of positive definiteness, i.e., for so-called *pseudo-metrics*.

Example 1.5.4 (A space that is not $\mathbf{T_0}$). If X is an arbitrary set with at least two elements and τ denotes the indiscrete topology on X (Example 1.1.2(b)), then (X,τ) does not satisfy the separation axiom $\mathbf{T}_0$.

Example 1.5.5 ($\mathbf{T_0}$ does not imply $\mathbf{T_1}$). Consider the set $X = \{1,2\}$ and the topology $\tau = \{\emptyset, \{1\}, X\}$ on X.[14] Then one can readily check the space (X,τ) satisfies the separation axiom $\mathbf{T}_0$, but not the separation axiom $\mathbf{T}_1$.

Example 1.5.6 ($\mathbf{T_1}$ does not imply $\mathbf{T_2}$). Let X be an infinite set and let τ denote the topology of co-finite sets on X (Example 1.1.3). Then the topological space (X,τ) satisfies the separation axiom $\mathbf{T}_1$, but not the separation axiom $\mathbf{T}_2$.

We conclude this section with a little exercise in which you should revisit the above examples once again.

Exercise 1.5.7 (Claims in preceding examples). In Examples 1.5.4, 1.5.5 and 1.5.6 several claims have been made about separation properties of various topological spaces. Check whether you can justify each of these claims in detail.

1.6 Addenda: Alternative axiomatizations of topological spaces

The axioms of topologies in Definition 1.1.1 are rather concise and very efficient to work with.[15] On the other hand, the motivation provided in Section 1.1 for this choice of axioms is a bit shallow. The fact that the open sets in $\mathbb{R}^d$ or in metric spaces satisfy a couple of properties is hardly a good reason for choosing precisely the properties that we mentioned as axioms for general topologies.

One good reason why the axioms are actually well-chosen is that they allow the development of a rich and non-trivial theory which is applicable in many situations throughout mathematics.

On the other hand, there are various alternative sets of axioms that can be used to define topological spaces (and which afterwards turn out to be equivalent to the definition we gave). Here are a few examples; let X be a fixed set.

(a) Instead of defining the axioms of the family of *open* sets on X, when can define the axioms that the family of *closed* sets should satisfy (namely, the properties listed in Proposition 1.1.7).

[14] It is easy to see that τ is indeed a topology.

[15] Thus, hardly surprising, it is also the standard definition of a topology that is used in many books on the topic.

(b) Instead of starting with a family of sets, one can fix several axioms that a *closure operator* $2^X \ni S \mapsto \overline{S} \in 2^X$ should satisfy (from this, on can than derive a family of closed sets be calling a set $S \subseteq X$ *closed* iff $S = \overline{S}$).

Alternatively, one can also begin by defining a set-valued mapping whose properties imitate that of the interior $2^X \ni S \mapsto S^\circ \in 2^X$.

(c) One can start off "locally" and first define neighbourhoods of points.

(d) One can begin by defining a notion of *convergence* for nets, and impose several axioms on this concept of convergence.

(e) And many more...

A very interesting discussion of various more or less intuitive axiomatisations of topology can be found in the answers to this MathOverflow post.

Convergence and continuity

2

Opening Questions.

(a) For a function $f : \mathbb{R} \to \mathbb{R}$, what precisely does it mean to say that $\lim_{x\to\infty} f(x) = 0$?

(b) For all positive integers $m, n \in \mathbb{N}$, consider a real number $x_{m,n}$. How can we make sense of the notation $\lim_{m,n\to\infty} x_{m,n}$?

(c) What is your intuitive interpretation of the notion *continuity* for a function $f : \mathbb{R}^d \to \mathbb{R}^d$ (for a fixed number $d \in \mathbb{N}$)?

2.1 Sequences and nets

Let us recall a concept that you are already very familiar with: if X is a set and we write down an infinite "list" of elements of X, say

$$(x_1, x_2, x_3, \dots),$$

then we call this "list" a *sequence* in X. We often denote it by $(x_k)_{k\in\mathbb{N}}$. Clearly, it does not really matter whether we use the index set $\mathbb{N}$ to enumerate the elements of our sequence or whether we use, for instance, the index set $\mathbb{N}_0$.

From a more technical point of view, a sequence – say, indexed over $\mathbb{N}$ – is simply a mapping from $\mathbb{N}$ to X; thus, we could also write $x(k)$ instead of x_k.

You should keep those basics about sequences in mind in this section, since we are going to vastly generalise the concept of a sequence. First however, let us clarify in the following definition that, in the setting of a topological space, one can speak of *convergence* of a sequence:

Definition 2.1.1 (Convergence of sequences). Let (X, τ) be a topological space and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in X.

(a) Let $x \in X$. The sequence $(x_k)_{k\in\mathbb{N}}$ is said to *converge to x* (or to *be convergent to x*) if and only if the following condition is satisfied:

For each neighbourhood[1] N of x there exists an index $k_0 \in \mathbb{N}$ such that $x_k \in N$ for all integers $k \geq k_0$.

In this case we call x a *limit*[2] of the sequence $(x_n)_{n\in\mathbb{N}}$, and we denote this by $x_k \overset{k\to\infty}{\to} x$.[3]

[1] Recall that we defined the notion *neighbourhood* in Definition 1.3.1

[2] It is important here to speak of *a* limit rather than *the* limit, since limits need not be unique in topological spaces; see Theorem 2.1.18 for more information about the question when limits are unique.

[3] Or simply by $x_k \overset{k}{\to} x$.

(b) The sequence $(x_k)_{k\in\mathbb{N}}$ is said to *converge* (or to be *convergent*) if there exists $x \in X$ such that the sequence converges to x.

In your Analysis 2 course you have probably defined convergence of sequences in metric spaces by means of an "ε-k_0-criterion". The following exercise is simple but important as it shows that this definition is equivalent to the one given above.

Exercise 2.1.2 (ε-k_0-criterion for convergence of sequences). Let (M, d) be a metric space, let $x \in X$ and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in M. Show that the following assertions are equivalent:

(i) The sequence $(x_k)_{k\in\mathbb{N}}$ converges to x (in the sense of Definition 2.1.1)

(ii) For each $\varepsilon > 0$ there exists an index $k_0 \in \mathbb{N}$ such that $\mathrm{d}(x_k, x) < \varepsilon$ for each integer $k \geq k_0$.

(iii) For each $\varepsilon > 0$ there exists an index $k_0 \in \mathbb{N}$ such that $\mathrm{d}(x_k, x) \leq \varepsilon$ for each integer $k \geq k_0$.

As you probably know from your Analysis 2 course, in metric spaces sequences are very useful in order to describe, for instance, closedness of sets and continuity of functions.

Later on in this course you will learn about examples which show that this does no longer work in general topological spaces. The reason of the problem is that sequences are, in a way, too "small". A common way to solve this is the introduction of a concept which is a much more general than a sequence: the notion of a *net*. Loosely speaking, a net is similar to a sequence, but one allows for more general index sets than just $\mathbb{N}$ or $\mathbb{N}_0$.

Remember that, when dealing with sequences, we often use the order on the index set $\mathbb{N}$ or $\mathbb{N}_0$ (for instance, this just occurred in Definition 2.1.1, where we use the assertion $k \geq k_0$ in the definition of convergence). So if we want to use more general index sets than $\mathbb{N}$ or $\mathbb{N}_0$ but still want to do similar things as for sequences, then we will need some kind of order on our more general index sets, too.

The precise properties of this order, and the precise definition of the notion *net* are the contents of the following two definitions.

Definition 2.1.3 (Directed sets). A *directed set* is a pair $(I, \preceq)$ where I is a set and $\preceq$ is a relation on I that satisfies the following properties:

(I) *Reflexivity:* For each $i \in I$ we have $i \preceq i$.

(II) *Transitivity:* For all $i, j, k \in I$ the conditions $i \preceq j$ and $j \preceq k$ imply $j \preceq k$.

(III) *Directedness:* For all $i, j \in I$ there exists an element $k \in I$ such that $i \preceq k$ and $j \preceq k$.

If $(I, \preceq)$ is a directed set, then we call $\preceq$ the *direction*[4] on this directed set.

Note that we do not require the direction to be anti-symmetric, i.e., it might happen that $i \preceq j$ and $j \preceq i$, but $i \neq j$.

Remarks 2.1.4. (a) If $(I, \preceq)$ is a directed set and $i, j \in I$, then we often use the notation $j \succeq i$ synonymously with $i \preceq j$.

(b) We often suppress the direction when we write down a directed set, i.e., we write for instance "let I be a directed set" instead of "let $(I, \preceq)$ be a directed set".

Definition 2.1.5 (Nets). Let X be a set. A *net in X* is a family $(x_i)_{i \in I}$ of elements x_i of X, where the index set I is a non-empty directed set.

Equivalently, we can simply say that a net in X is a mapping from a non-empty directed set I to X.

The seemingly harmless notion of a net is in fact an extremely useful concept which gives us much more freedom than the rather restricted setting of sequences. As a first illustration of this, let us list a few examples of how a net can look like.

Example 2.1.6 (Every sequence is a net). Let X be a set. If we endow $\mathbb{N}$ with its usual order $\leq$, then $(\mathbb{N}, \leq)$ is a directed set. Hence, each sequence $(x_n)_{n \in \mathbb{N}}$ in X is also a net.

From now on we will, unless otherwise stated, always tacitly assume that $\mathbb{N}$ is endowed with its usual order; therefore, we consider every sequence as a net in a canonical way.

The next example is of a somewhat different nature:

Example 2.1.7 (A net indexed by the finite subsets of a given set). Let S be an arbitrary set, and let $\mathcal{F}$ denote the set of all finite subsets of S. The inclusion $\subseteq$ is a relation on $\mathcal{F}$ which is reflexive, transitive and directed; hence, $(\mathcal{F}, \subseteq)$ is a directed set.

A simple but illuminating example of a net in the set $\mathbb{N}_0$ with index set $\mathcal{F}$ is given by $(\#F)_{F \in \mathcal{F}}$, where we use the symbol $\#F$ to denote the cardinality of of the set F.

The index set $\mathcal{F}$ from Example 2.1.7, which is directed by set inclusion, will occur again later in the course.

[4] Or the *order*, but this terminology might not be optimal since we do not require the relation $\preceq$ to by anti-symmetric.

Example 2.1.8 (Functions on $\mathbb{R}$ as nets). Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and endow $\mathbb{R}$ with its usual order $\leq$. Then $(\mathbb{R}, \leq)$ is a directed set and $(f(x))_{x\in\mathbb{R}}$ is a net.

The next example will turn out to be very important throughout the entire course.

Example 2.1.9 (The set of neighbourhoods is directed). Let (X, τ) be a topological space, let $x \in X$ and let $\mathcal{N}(x) \subseteq 2^X$ denote the set of all neighbourhoods of x. We endow $\mathcal{N}(x)$ with the order $\preceq$ that we define to be the converse set inclusion, i.e., we define $N_1 \preceq N_2$ for two neighbourhoods N_1 and N_2 of x iff $N_1 \supseteq N_2$.

Then $(\mathcal{N}(x), \preceq)$ is a directed set. Indeed, the relation $\preceq$ is obviously reflexive and transitive. To show that it is also directed, let $N_1, N_2 \in \mathcal{N}(x)$. Then x is an element of the interior of both N_1 and N_2, so there exist open sets U_1 and U_2 such that

$$x \in U_1 \subseteq N_1 \qquad \text{and} \qquad x \in U_2 \subseteq N_2.$$

Since the intersection of two open sets is again open, it follows that $U_1 \cap U_2$ is a neighbourhood of x. Moreover, $U_1 \cap U_1$ is a subset of both N_1 and N_2, so $U_1 \cap U_2$ is larger than both N_1 and N_2 with respect to $\preceq$.

Hence, if we choose a point $x_N \in X$ for each $N \in \mathcal{N}(x)$, then $(x_N)_{N\in\mathcal{N}(x)}$ is a net in X.

Example 2.1.10 (A net indexed over $\mathbb{N}^2$). Endow the set $\mathbb{N}^2$ with the order $\preceq$ given by

$$(i_1, i_2) \preceq (j_1, j_2) \qquad \text{if and only if} \qquad i_1 \leq j_1 \quad \text{and} \quad i_2 \leq j_2.$$

Then $(\mathbb{N}^2, \preceq)$ is a directed set. Thus, we can consider nets that are indexed over $\mathbb{N}^2$, for instance the net $\left(\frac{1}{i} - \frac{1}{j}\right)_{(i,j)\in\mathbb{N}^2}$ in $\mathbb{R}$.

The previous example can be extended to are more general construction:

Proposition 2.1.11 (Product of two directed sets). *Let $(I_1, \preceq_1)$ and $(I_2, \preceq_2)$ be directed sets. We define a relation $\preceq$ on the product set $I_1 \times I_2$ by*

$$(i_1, i_2) \preceq (j_1, j_2) \qquad \textit{if and only if} \qquad i_1 \preceq_1 j_1 \quad \textit{and} \quad i_2 \preceq_2 j_2.$$

Then $(I_1 \times I_2, \preceq)$ is a directed set.

The proof is left to you in order to check your understanding of the concept *directed set*:

Exercise 2.1.12 (Product of two directed sets). Prove Proposition 2.1.11.

You probably have noticed that in all examples above, as well as in Proposition 2.1.11, we have not really said much about nets; in fact, the more interesting part up to now was the discussion of several directed sets which serve as index sets for nets.

The next definition is the reason why nets are actually interesting in topology: similarly as we did for sequences in Definition 2.1.1, we are now going to define the notion of *convergence* for nets.

Definition 2.1.13 (Convergence of nets). Let (X,τ) be a topological space and let $(x_i)_{i\in I}$ be a net in X.

(a) Let $x \in X$. The net $(x_i)_{i\in I}$ is said to *converge to* x (or to *be convergent to* x) if and only if the following condition is satisfied:

For each neighbourhood[5] N of x there exists an index $i_0 \in I$ such that $x_i \in N$ for all $i \in I$ that satisfy $i \geq i_0$.

In this case we call x a *limit* of the net $(x_i)_{i\in I}$, and we denote this by $x_i \overset{i}{\to} x$.[6]

(b) The net $(x_i)_{i\in I}$ is said to *converge* (or to be *convergent*) if there exists $x \in X$ such that the net converges to x.

Since every sequence is a net, we now have two definitions of convergence for sequences: Definition 2.1.1 and Definition 2.1.13.

It is easy but important to observe that those two definitions are consistent; this is a consequence of the fact that we always endow $\mathbb{N}$ with its usual order when we consider a sequence as a net.

If the topology on our space is induced by a metric, one can also describe convergence by means of an *ε-i_0-criterion*, just as you did for sequences in Exercise 2.1.2:

Exercise 2.1.14 (ε-i_0-criterion for convergence of nets). Consider a metric space (M,d). Formulate and prove an analogue of the equivalence from Exercise 2.1.2 for nets instead of sequences.

Several of the examples of nets that we discussed above can be used as a first illustration of the convergence of nets.

Examples 2.1.15 (Some examples of convergent nets). (a) Let (X,τ) be a topological space, let $x \in X$ and let $(I,\preceq)$ be a non-empty directed set. Then the constant net $(x)_{i\in I}$ converges to x.

[5] Recall that we defined the notion *neighbourhood* in Definition 1.3.1

[6] We could also write $x_i \overset{i\to\infty}{\to} x$, but this notation is somewhat uncommon when talking about nets.

(b) Endow $\mathbb{R}$ with the Euclidean metric and the topology induced by this metric. In the situation of Example 2.1.8, the net $(f(x))_{x\in\mathbb{R}}$ converges to a number $y \in \mathbb{R}$ if and only if the following condition is satisfied:

For each $\varepsilon > 0$ there exists a real number x_0 such that $\left|f(x) - y\right| < \varepsilon$ for all real numbers $x \geq x_0$.[7]

This characterisation follows easily from the result of Exercise 2.1.14.

(c) Again, endow $\mathbb{R}$ with the Euclidean metric. The net $\left(\frac{1}{i} - \frac{1}{j}\right)_{(i,j)\in\mathbb{N}^2}$ from Example 2.1.10 converges to 0. This can, for instance, be shown by using the result of Exercise 2.1.14

(d) In the situation of Example 2.1.9, assume that the points x_N are chosen such that $x_N \in N$ for each $N \in \mathcal{N}(x)$. Then the net $(x_N)_{N\in\mathcal{N}(x)}$ converges to x.

Indeed, let N be a neighbourhood of x. For all $\tilde{N} \in \mathcal{N}(x)$ with $\tilde{N} \geq N$ we then have $x_{\tilde{N}} \in \tilde{N} \subseteq N$; this proves the claim by the very definition of the convergence of nets.[8]

One reason why nets are useful is that they can be used to characterise the closure of a set. This is the content of the following theorem:

Theorem 2.1.16 (Closures via nets). *Let (X,τ) be a topological space and let $S \subseteq X$. For each point $x \in X$ the following assertions are equivalent:*

(i) *The point x is an element of the closure $\overline{S}$.*

(ii) *There exists a net $(x_i)_{i\in I}$ in S that converges to x.*

The situation in Theorem 2.1.16 is illustrated by a simple example in $\mathbb{R}$ in Figure 2.1.1.

Proof of Theorem 2.1.16. "(i) $\Rightarrow$ (ii)" As we have $x \in \overline{S}$, every neighbourhood of x intersects S (Proposition 1.3.4). Now, let $I := \mathcal{N}(x)$ denote the set of all neighbourhoods of x; we order $\mathcal{N}(x)$ by converse set inclusion, which renders it a directed set (Example 2.1.9). For each $N \in \mathcal{N}(x)$, there exists a point $x_N \in N \cap S$. Then $(x_N)_{N\in\mathcal{N}(x)}$ is a net which, as shown in Example 2.1.15(d), converges to x. Moreover, each element of the net is in S.

"(ii) $\Rightarrow$ (i)" We assume towards a contradiction that $x \notin \overline{S}$. Since the complementary set $U := \left(\overline{S}\right)^{\mathrm{c}}$ is open, it is a neighbourhood of x. By the convergence of $(x_i)_{i\in I}$ to x, this implies that there exists an index $i_0 \in I$ such

[7] We point out that this means precisely that $\lim_{x\to\infty} f(x) = y$ in the sense as it is usually introduced in Analysis 1.

[8] More graphically speaking, we choose the index i_0 from Definition 2.1.13(a) as $i_0 := \tilde{N}$.

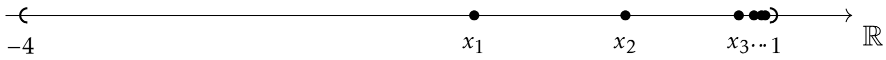

Figure 2.1.1: Illustration of Theorem 2.1.16 on the space $\mathbb{R}$ with the Euclidean topology: A sequence $(x_n)_{n\in\mathbb{N}}$ in the set $(-4,1)$ that converges to the point 1 in the closure of $(-4,1)$.

that $x_i \in U$ for all indices $i \in I$ that satisfy $i \geq i_0$. In particular, we have $x_{i_0} \in U$ and thus, $x_{i_0} \notin S$. This is a contradiction. $\square$

As a simple consequence of the previous theorem, we obtain a characterisation of closed sets in terms of the limits of nets:

Corollary 2.1.17 (Closed sets via nets). *Let (X,τ) be a topological space and let $C \subseteq X$. The following assertions are equivalent:*

(i) *The set C is closed.*

(ii) *For every net $(x_i)_{i\in I}$ in C that converges to a point $x \in X$, we also have $x \in C$.*

Proof. "(i) $\Rightarrow$ (ii)" Let $(x_i)_{i\in I}$ be a convergent net in X whose elements are all contained in C; we denote the limit of this net by x. According to Theorem 2.1.16 the point x is an element of the closure $\overline{C}$. But the latter set coincides with C as C is closed (Exercise 1.2.2(b)).

"(ii) $\Rightarrow$ (i)" It suffices to prove that $\overline{C} = C$ (Exercise 1.2.2(b)). So let $x \in \overline{C}$. According to Theorem 2.1.16 there exists a net $(x_i)_{i\in I}$ that converges to x and whose elements are all contained in C. By condition (ii) this implies that $x \in S$. $\square$

We will see later on that the implication "(ii) $\Rightarrow$ (i)" in the corollary does not remain true in general topological spaces if we only consider sequences instead of nets.

It is an interesting observation that sequences and nets in general topological spaces can have more than one limit. For instance, endow a space X with the indiscrete topology (Example 1.1.2(b)). Then it is easy to see that every net in X converges to each point in X. The question whether limits are unique is closely related to separation axioms. We have:

Theorem 2.1.18 (Limits are unique in $\mathbf{T}_2$-spaces). *For each topological space (X,τ) the following assertions are equivalent:*

(i) *The space (X,τ) is Hausdorff, i.e., it satisfies the separation axiom $\boldsymbol{T}_2$.*

(ii) *Each net in X has at most one limit.*

Proof. "(i) $\Rightarrow$ (ii)" Let $(x_i)_{i\in I}$ be a net in X and assume that this net converges to two points $x, y \in X$. We assume that $x \neq y$ and show that this leads to a contradiction.

Due to the Hausdorff property, we can find open sets U, V that contain x and y, respectively, but that do not intersect. As our net converges to both x and y, we can find indices $i_x, i_y \in I$ such that $x_i \in U$ for all $i \geq i_x$ and such that $x_i \in V$ for all $i \geq i_y$.

But since I is directed, there exists an index $i_1 \in I$ which is larger than both i_x and i_y. Hence, $x_{i_1} \in U \cap V$, which contradicts the fact that U and V do not intersect.

"(ii) $\Rightarrow$ (i)" Assume towards a contradiction that our space is not Hausdorff. Then there exist two distinct points $x_1, x_2 \in X$ such that each open set that contains x_1 intersects each open set that contains x_2. Consequently, each neighbourhood of x_1 intersects each neighbourhood of x_2. Now, let $\mathcal{N}(x_1)$ and $\mathcal{N}(x_2)$ denote the sets of all neighbourhoods of x_1 and x_2, respectively.

We direct both sets $\mathcal{N}(x_1)$ and $\mathcal{N}(x_2)$ by converse set inclusion as in Example 2.1.9, and then we endow the product $\mathcal{N}(x_1) \times \mathcal{N}(x_2)$ with the product direction $\preceq$ described in Proposition 2.1.11. In other words, this means that

$$(N_1, N_2) \preceq (\tilde{N}_1, \tilde{N}_2) \qquad \text{if and only if} \qquad N_1 \supseteq \tilde{N}_1 \quad \text{and} \quad N_2 \supseteq \tilde{N}_2.$$

for all neighbourhoods $N_1, \tilde{N}_1$ of x_1 and $N_2, \tilde{N}_2$ of x_2.

For each pair $(N_1, N_2) \in \mathcal{N}(x_1) \times \mathcal{N}(x_2)$ we can find a point $x_{(N_1,N_2)}$ in $N_1 \cap N_2$ as we know that the latter intersection is non-empty. Then

$$\big(x_{(N_1,N_2)}\big)_{(N_1,N_2)\in\mathcal{N}(x_1)\times\mathcal{N}(x_2)}$$

is a net in X which can easily be checked to converge both to x_1 and x_2. But this contradicts assertion (ii) since $x_1 \neq x_2$. $\square$

Exercise 2.1.19 (A detail in the proof of Theorem 2.1.18). In the proof of the implication "(ii) $\Rightarrow$ (i)" of Theorem 2.1.18 it was claimed that the net

$$\big(x_{(N_1,N_2)}\big)_{(N_1,N_2)\in\mathcal{N}(x_1)\times\mathcal{N}(x_2)}$$

converges to x_1 and to x_2. Give all the missing details of this argument.

We close the chapter with the definition of the following notation: if a (X, τ) is a topological Hausdorff space, then – as shown in Theorem 2.1.18 – each net in X has at most one limit. This makes it reasonable to use the notation lim, to which you are already used from Analysis.

Definition 2.1.20 (Limit notation). Let (X, τ) be a topological Hausdorff space. If $(x_i)_{i\in I}$ is a convergent net in X, then we denote its limit by $\lim_i x_i$.

2.2 Subnets

When one deals with sequences, the notion of a *subsequence* is extremely important. There is a similar notion for nets, namely that of a *subnet*, which we explore in this section.

In order to define this concept, we first need to introduce the notion of a *co-final* mapping between two directed sets:

Definition 2.2.1 (Co-final mappings). Let $(I, \preceq_I)$ and $(J, \preceq_J)$ be directed sets.[9] A mapping $\varphi : I \to J$ is called *co-final* iff for each $j_0 \in J$ there exists $i_0 \in I$ such that $\varphi(i) \succeq_J j_0$ for all $i \succeq_I i_0$.

Examples 2.2.2 (Some examples of co-final mappings). (a) Endow $\mathbb{N}$ and $\mathbb{R}$ with their usual orders and consider the following mappings from $\mathbb{N}$ to $\mathbb{R}$:

$$\begin{aligned}
\varphi_1 &: n \mapsto n, \\
\varphi_2 &: n \mapsto n + 2 \cdot (-1)^n, \\
\varphi_3 &: n \mapsto (-1)^n \cdot n, \\
\varphi_4 &: n \mapsto \arctan(n).
\end{aligned}$$

The mappings φ_1 and φ_2 are co-final, while the mappings φ_3 and φ_4 are not.[10]

(b) Let $(I_1, \preceq_1)$ and $(I_2, \preceq_2)$ be directed set and endow the product $I_1 \times I_2$ with the product direction $\preceq$ that we discussed in Proposition 2.1.11. Then both mappings

$$\begin{aligned} \varphi_1 : I_1 \times I_2 &\to I_1 \\ (i_1, i_2) &\mapsto i_1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \varphi_2 : I_1 \times I_2 &\to I_2 \\ (i_1, i_2) &\mapsto i_2 \end{aligned}$$

are co-final.

Exercise 2.2.3 (Composition of co-final mappings). Prove that the composition of two co-final mappings is co-final.

As promised, we can now define what we mean by a subnet of a net.

[9] For the sake of clarity, we use the different symbols $\preceq_I$ and $\preceq_J$ here to denote the directions on I and J, respectively. However, as the course proceeds, we will more and more adopt to the common convention to denote – if there is no risk of confusion – the direction on two directed sets both by the same symbol.

[10] It is instructive to sketch the graph of each of these mappings in order to better understand their behaviour.

Definition 2.2.4 (Subnets). Let X be a set and let $(x_i)_{i\in I}$ and $(y_j)_{j\in J}$ be nets in X. We say that $(x_i)_{i\in I}$ is a *subnet* of $(y_j)_{j\in J}$ iff there exists a co-final mapping $\varphi : I \to J$ such that $y_{\varphi(i)} = x_i$ for each $i \in I$.

Let us note that a subnet of a subnet is always a subnet of the original net; more precisely:

Remark 2.2.5 (Subnets of subnets). Let X be a set and let $(x_i)_{i\in I}$, $(y_j)_{j\in J}$ and $(z_k)_{k\in K}$ by nets in X. If $(x_i)_{i\in I}$ is a subnet of $(y_j)_{j\in J}$ and $(y_j)_{j\in J}$ is a subnet of $(z_k)_{k\in K}$, then $(x_i)_{i\in I}$ is a subnet of $(z_k)_{k\in K}$. This follows from Exercise 2.2.3.

Definition 2.2.4 has a rather axiomatic taste: we are given two nets, and the question whether the first one is a subnet of the second one depends on the existence of a certain co-final map. In concrete situations, we will often apply this definition in a more "constructive" way: suppose we are given a net $(y_j)_{j\in J}$ in a set X; depending on what precisely we intend to do, we will then choose an appropriate directed set I and a co-final mapping $\varphi : I \to J$ and thus construct a subnet $(x_i)_{i\in I}$ of $(y_j)_{j\in J}$ by setting $x_i := y_{\varphi(i)}$ for each $i \in I$. Of course, it is not really necessary to introduce an extra symbol for x_i. Instead, we may simply say that $(y_{\varphi(i)})_{i\in I}$ is a subnet of $(y_j)_{j\in J}$.

Note that this is not much different – just more general – than the procedure that we perform when we choose a subsequence of a given sequence. Say, we are given a sequence

$$(y_j)_{j\in\mathbb{N}} = (y_1, y_2, y_3, y_4, \dots)$$

in a set X. If, for some reason, we would like to consider the subsequence that consists of each second element of the original sequence, this means formally that we consider the sequence $(x_i)_{i\in\mathbb{N}}$, where x_i is given by $x_i = y_{2i}$ for each $i \in \mathbb{N}$. This is a special case of the situation in Definition 2.2.4: we now have $I = J = \mathbb{N}$, and $\varphi(i) = 2i$ for each $i \in I$.

More generally, we observe:

Remark 2.2.6 (Subsequences as subnets). Let X be a set and let $(x_i)_{i\in\mathbb{N}}$ and $(y_j)_{j\in\mathbb{N}}$ be sequences in X. If $(x_i)_{i\in\mathbb{N}}$ is a subsequence of $(y_j)_{j\in\mathbb{N}}$, then $(x_i)_{i\in\mathbb{N}}$ is also a subnet of $(y_j)_{j\in\mathbb{N}}$.

It is worthwhile pointing out that the remark above has no converse: it may happen that a sequence $(x_i)_{i\in\mathbb{N}}$ is a subnet, but not a subsequence, of another sequence $(y_j)_{j\in\mathbb{N}}$. We discuss an example of such a situation on Problem Sheet 3.

Throughout the course we will encounter many constructions of subnets. As a start, let us now discuss a particularly simple class of examples of subnets:

Example 2.2.7 (Tails of a net). Let X be a set and let $(x_i)_{i\in I}$ be a net in X. Fix an index $i_0 \in I$. Then the subset

$$I_{\geq i_0} := \{i \in I : i \geq i_0\}$$

of I is also a directed set, and the inclusion mapping $I_{\geq i_0} \hookrightarrow I$ is co-final. Hence, the so-called *tail* $(x_i)_{i\in I_{\geq i_0}}$ of $(x_i)_{i\in I}$ is a subnet of $(x_i)_{i\in I}$.

To keep the notation simple, we will often denote this tail simply by $(x_i)_{i\geq i_0}$.

The next proposition says that convergence of a net to a point is inherited by subnets.

Proposition 2.2.8 (Convergence is inherited by subnets). *Let (X,τ) be a topological space and let $(x_i)_{i\in I}$ be a net in X that converges to a point $x \in X$. Then every subnet of $(x_i)_{i\in I}$ converges to x, too.*

Proof. Let $(w_h)_{h\in H}$ by a subnet of $(x_i)_{i\in I}$. Then there exists a co-final mapping $\varphi : H \to I$ such that $w_h = x_{\varphi(h)}$ for each $h \in H$.

Now, let N be a neighbourhood of x. As $(x_i)_{i\in I}$ converges to x, there exists an index $i_0 \in I$ such that $x_i \in N$ for all $i \geq i_0$. Since φ is co-final, we can moreover find an index $h_0 \in H$ such that $\varphi(h) \geq i_0$ for all $h \geq h_0$.

For each $h \geq h_0$ this implies that $w_h = x_{\varphi(h)} \in N$; this proves that $(w_h)_{h\in H}$ does indeed converge to x. □

We will employ subnets in several situations throughout the course. In this section, we present two applications: the *hair-splitting lemma* that characterises convergence of a net, and the definition of *accumulation points* of a net.

Lemma 2.2.9 (Hair-splitting lemma). *Let (X,τ) be a topological space, let $x \in X$ and let $(x_i)_{i\in I}$ be a net in X. The following assertions are equivalent:*

(i) *The net $(x_i)_{i\in I}$ converges to x.*

(ii) *Every subnet of $(x_i)_{i\in I}$ has a subnet that converges to x.*

Proof. "(i) ⇒ (ii)" Every subnet of $(x_i)_{i\in I}$ is a subnet of itself. Thus, the implication follows readily from the fact that convergence to x is inherited by subnets (Proposition 2.2.8).

"(ii) ⇒ (i)" Assume that $(x_i)_{i\in I}$ does not converge to x. It is now our task to find a subnet of $(x_i)_{i\in I}$ which has no subnet that converges to x.

Since $(x_i)_{i\in I}$ does not converge to x, there exists a neighbourhood N of x with the following property: for each index $i \in I$ there exists another index in I – which we denote by $\varphi(i)$ – such that $\varphi(i) \geq i$ and $(x_{\varphi(i)}) \notin N$. Then $(x_{\varphi(i)})_{i\in I}$ is a subnet of $(x_i)_{i\in I}$ since $\varphi : I \to I$ is co-final.

But none of the elements $x_{\varphi(i)}$ is in N, so no subnet of $(x_{\varphi(i)})_{i\in I}$ converges to x. □

The hair splitting lemma above does, of course, also apply if the net $(x_i)_{i\in I}$ is a sequence. On the other hand, one can also give other versions of the hair splitting lemma which are specifically adapted to sequences; see Lemma 2.7.1 in the Addenda at the end of Chapter 2 for details.

Now we come to the second application of subnets that we promised, namely the definition of accumulation points:

Definition 2.2.10 (Accumulation points of nets). Let (X, τ) be a topological space and let $(x_i)_{i\in I}$ be a net in X. A point $x \in X$ is a called an *accumulation point* of the net $(x_i)_{i\in I}$ iff there exists a subnet of $(x_i)_{i\in I}$ that converges to x.

The set of all accumulation points of a given net can be described by intersecting the closures of all "tail sets" of the net; more precisely speaking, the following holds:

Theorem 2.2.11 (Structure of the set of accumulation points). *Let (X, τ) be a topological space and let $(x_i)_{i\in I}$ be a net in X. Let $A \subseteq X$ denote the set of all accumulation points of this net. Then*

$$A = \bigcap_{i_0 \in I} \overline{\{x_i : i \geq i_0\}}.$$

In particular, A is closed.

The proof of the theorem is not particularly difficult. However, due to time constraints, we do not include it in the lecture; instead, we defer the proof to the Addenda presented in Section 2.7 at the end of this chapter. If you are interested in the proof, you can find it there.

2.3 Continuous functions

One of the most important concepts in topology is that of a continuous function. Let us start this section right away with the definition of this notion:

Definition 2.3.1 (Continuous functions and homeomorphisms). Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let $f : X_1 \to X_2$ be a function.

(a) Let $x \in X_1$. The function f is said to be *continuous at x* iff the following condition is satisfied:

For each neighbourhood $N_2 \subseteq X_2$ of $f(x)$ there exists a neighbourhood $N_1 \subseteq X_1$ of x such that $f(N_1) \subseteq N_2$.

(b) The function f is said to be *continuous*[11] iff it is continuous at every point $x \in X_1$.

[11] We could also say – a bit more precisely – in this case that f is *globally continuous*.

(c) The function f is called a *homeomorphism* if f is bijective and both f and its inverse function f^{-1} are continuous.

For metric spaces, the definition of continuity given above is equivalent to the ε-δ-condition that you probably know from your Analysis 2 course. The details are discussed in the following exercise:

Exercise 2.3.2 (Continuity in metric spaces). Let (M_1, d_1) and (M_2, d_2) be metric spaces and let $f : M_1 \to M_2$ be a function. Let $x \in M_1$. Show that the following assertions are equivalent:

1. The function f is continuous at x (in the sense specified in Definition 2.3.1(a)).

2. For each number $\varepsilon > 0$ there exists a number $\delta > 0$ such that we have $\mathrm{d}_2\big(f(\tilde{x}), f(x)\big) < \varepsilon$ for all $\tilde{x} \in M_1$ that satisfy $\mathrm{d}_1(\tilde{x}, x) < \delta$.

Continuity of complicated functions is typically not easy to show by a direct application of the definition of continuity. Instead one uses certain "stability properties" of the class of continuous functions. For instance you know from course Analysis 1 and 2 courses that the sum of two continuous $\mathbb{R}^d$-valued functions (which are both defined on, say, a metric space) is again continuous. We will have more to say about this "continuity of sums" as soon as we have discussed product topologies in Chapter 3.

For the moment, we only prove another type of stability property of continuity, namely stability with respect to the composition of functions:

Proposition 2.3.3 (Continuity of compositions). *Let* (X_1, τ_1), (X_2, τ_2) *and* (X_3, τ_3) *be topological spaces and consider two functions* $X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3$.

(a) *Let* $x \in X_1$. *If* f *is continuous at* x *and* g *is continuous at* $f(x)$, *then the composition* $g \circ f$ *is continuous at* x, *too.*

(b) *If* f *and* g *are continuous, then so is the composition* $g \circ f$.

Proof. (a) Let $N_3 \subseteq X_3$ by a neighbourhood of $(g \circ f)(x) = g(f(x))$. As g is continuous at $f(x)$, there exists a neighbourhood $N_2 \subseteq X_2$ of $f(x)$ such that $g(N_2) \subseteq N_3$. And as f is continuous at x, there exists a neighbourhood $N_1 \subseteq X_1$ of x such that $f(N_1) \subseteq N_2$. In total, we thus have

$$(g \circ f)(N_1) = g(f(N_1)) \subseteq g(N_2) \subseteq N_3.$$

This proves that $g \circ f$ is indeed continuous at x.

(b) This is an immediate consequence of (a). $\square$

Global continuity of a function can be characterised by the property that pre-images of open sets be open, or alternatively by the property that pre-images of closed sets be closed:

Proposition 2.3.4 (Global continuity via pre-images). *Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let $f : X_1 \to X_2$ be a function. The following assertions are equivalent:*

(i) *The function f is continuous.*

(ii) *For each open set U in X_2 the pre-image $f^{-1}(U)$ is open in X_1.*

(iii) *For each closed set C in X_2 the pre-image $f^{-1}(C)$ is closed in X_1.*

Proof. "(i) $\Rightarrow$ (ii)" Let $U \subseteq X_2$ be open. It suffices to show that each point x in $f^{-1}(U)$ is located in the interior of the latter set; so fix $x \in f^{-1}(U)$.

Since U is open, it is a neighbourhood of $f(x)$; as f is continuous at x, we can thus find a neighbourhood $N \subseteq X_1$ of x such that $f(N) \subseteq U$. Hence, $N \subseteq f^{-1}(U)$. This shows that

$$x \in N^{\circ} \subseteq (f^{-1}(U))^{\circ},$$

where the inclusion follows from the fact that taking the interior is a monotone set operation (Corollary 1.2.4).

"(ii) $\Rightarrow$ (i)" Let $x \in X_1$. We need to show that f is continuous at x, so let $N_2 \subseteq X_2$ be a neighbourhood of $f(x)$.

Then $f(x) \in (N_2)^{\circ}$ and hence, x is an element of the pre-image $N_1 := f^{-1}((N_2)^{\circ})$. The set N_1 is open according to assertion (ii), hence it is a neighbourhood of x; and obviously, $f(N_1) \subseteq (N_2)^{\circ} \subseteq N_2$.

"(ii) $\Leftrightarrow$ (iii)" This equivalence follows from the facts that a subset of of a topological space is open iff its complement is closed (and vice versa) and that $\left(f^{-1}(S)\right)^{\mathrm{c}} = f^{-1}(S^{\mathrm{c}})$ for each subset S of X_2. $\square$

It is instructive to apply Proposition 2.3.4 to show that the identity map on a topological space is continuous and that constant mappings are always continuous; this is the content of the following exercise:

Exercise 2.3.5. (a) Let (X, τ) be a topological space. Show that the identity map $\mathrm{id} : X \to X$ is continuous.

(b) Let (X, τ_X) and (X, τ_Y) be topological spaces. Fix $y_0 \in Y$ and consider the constant function

$$\begin{aligned} f : X &\to Y, \\ x &\mapsto y_0. \end{aligned}$$

Prove that f is continuous.

Proposition 2.3.4 can be quite useful to show that a given set is open or closed. Let us demonstrate this by means of a simple example:

Example 2.3.6 (Closed sets in $\mathbb{R}^2$). Let $\mathbb{R}^2$ be endowed with the Euclidean metric and the induced topology. In Example 1.1.8 it was claimed that the unit circle and each coordinate axis in $\mathbb{R}^d$ is closed.

Proposition 2.3.4 provides us with an easy way to check this:

- The unit circle $\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$ is the pre-image of the closed set[12] $\{1\} \subseteq \mathbb{R}$ under the continuous mapping

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R}, \\ x &\mapsto x_1^2 + x_2^2. \end{aligned}$$

- The x_1-axis in $\mathbb{R}^2$ is the pre-image of the closed set $\{0\} \subseteq \mathbb{R}$ under the continuous mapping

$$\begin{aligned} g : \mathbb{R}^2 &\to \mathbb{R}, \\ x &\mapsto x_2. \end{aligned}$$

Note however, that the above arguments used that fact that f and g are continuous, which we have not proved in this course (but which you know from your Analysis courses). The most important observation which is needed to prove that the function f is continuous is that sums and products of continuous functions are again continuous functions. In fact, it would not be too difficult to show this right here; however we prefer to postpone this to the point in the course where we discuss *product topologies* since this makes the continuity of sums and products much clearer.

You most likely know from your Analysis courses that continuity of a function $f : \mathbb{R} \to \mathbb{R}$ – or more generally, of a function between two metric spaces – at a point x can be checked by considering the action of f on sequences that converge to x.

In the more general setting of topological spaces, a similar result holds – however, we have to replace sequences with nets:[13]

Theorem 2.3.7 (Continuity via nets). *Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let $f : X_1 \to X_2$ be a function. For each point $x \in X_1$ the following assertions are equivalent:*

(i) *The function f is continuous at x.*

[12] Where we endow $\mathbb{R}$ with the Euclidean metric, too.

[13] This is very similar to the characterisation of closures in Theorem 2.1.16 where we also needed nets rather than sequences.

(ii) *For each net $(x_i)_{i\in I}$ in X_1 that converges to x, the net $\big(f(x_i)\big)_{i\in I}$ in X_2 converges to $f(x)$.*

Note that this theorem implies that, if both (X_1,τ_1) and (X_2,τ_2) are Hausdorff spaces, if f is a continuous function between them and $(x_i)_{i\in I}$ is a convergent net in X, then $\lim_i f(x_i) = f(\lim_i x_i)$.

Proof of Theorem 2.3.7. "(i) $\Rightarrow$ (ii)" Let $(x_i)_{i\in I}$ be a net in X_1 that converges to x. In order to prove that $\big(f(x_i)\big)_{i\in I}$ converges to $f(x)$, let $N_2 \subseteq X_2$ be a neighbourhood of $f(x)$.

Due to the continuity of f at x, there exists a neighbourhood $N_1 \subseteq X_1$ of x such that $f(N_1) \subseteq N_2$. As $(x_i)_{i\in I}$ converges to x, there exists an index $i_0 \in I$ such that $x_i \in N_1$ for all $i \geq i_0$. Consequently, $f(x_i) \in f(N_1) \subseteq N_2$ for all $i \geq i_0$. This proves that $f(x_i) \to f(x)$.

"(ii) $\Rightarrow$ (i)" Assume towards a contradiction that f is not continuous at x. Then there exists a neighbourhood $N_2 \subseteq X_2$ of $f(x)$ such that $f(N_1) \not\subseteq N_2$ for each neighbourhood $N_1 \subseteq X_2$ of x.

So, if $\mathcal{N}(x)$ denotes the set of all neighbourhoods of x, we can find for each $N_1 \in \mathcal{N}(x)$ a point $x_{N_1} \in N_1$ such that $f(x_{N_1}) \notin N_2$. We know that $\mathcal{N}(x)$ is a directed set with respect to converse set inclusion (Example 2.1.9), so $(x_{N_1})_{N_1\in\mathcal{N}(x)}$ is a net. Moreover, this net converges to x since each point x_{N_1} is in N_1 (Example 2.1.15(d)).

However, the net $\big(f(x_{N_1})\big)_{N_1\in\mathcal{N}(x)}$ does not converge to $f(x)$ as none of its elements is located in the neighbourhood N_2 of $f(x)$. $\square$

2.4 Filters

In this Section we briefly introduce and discuss concept which is closely related to nets – although it looks rather different at first glance: the notion of a *filter*.[14]

Definition 2.4.1 (Filters). Let X be a set. A non-empty set $\mathcal{F} \subseteq 2^X$ is called a *filter* on X if it satisfies the following conditions:

(I) The empty set $\emptyset$ is not an element of $\mathcal{F}$.

(II) For all $F_1, F_2 \in \mathcal{F}$ the intersection $F_1 \cap F_2$ is also an element of $\mathcal{F}$.

[14] In fact, it turns out that many things that can be done with filters can also be done with nets, and vice versa (although there are some situations where one of the two concepts is slightly better suited). But as it would probably lead too far to discuss both nets and filters in detail in this course, we mostly focus on one of the concepts – which are in our case nets (a choice which is mainly a matter of personal taste). Still, we explain the basic properties of filters in this section (and in Section 2.6) so that you get at least a brief impression of them. Additional information on filters can be found in the Addenda at the end of this Chapter.

(III) For all $F \in \mathcal{F}$ and each set $G \subseteq X$ with $G \supseteq F$ we also have $G \in \mathcal{F}$.

We have already used an important example of a filter several times in this course (although we have not yet used the word *filter* to describe it):

Example 2.4.2 (The neighbourhoods of a point constitute a filter). Let (X, τ) be a topological space and let $x \in X$. Let $\mathcal{N}(x)$ denote the set of all neighbourhoods of x. Then $\mathcal{N}(x)$ is a filter.

Proof. We first note that $\mathcal{N}(x)$ is non-empty since $X \in \mathcal{N}(x)$. Now we show that $\mathcal{N}(x)$ satisfies the axioms in Definition 2.4.1:

Axiom (I)*:* The empty set is not an element of $\mathcal{N}(x)$ since every neighbourhood of x contains x.

Axiom (II)*:* For $N_1, N_2 \in \mathcal{N}(x)$ we have $x \in (N_1)^{\mathrm{o}} \cap (N_2)^{\mathrm{o}} = (N_1 \cap N_2)^{\mathrm{o}}$, where the equality follows from Proposition 1.2.6(b). Hence, $N_1 \cap N_1 \in \mathcal{N}(x)$

Axiom (III)*:* If $N \in \mathcal{N}(x)$ and if $\tilde{N} \subseteq X$ is a superset of N, then

$$x \in N^{\mathrm{o}} \subseteq \tilde{N}^{\mathrm{o}},$$

so $\tilde{N}$ is also a neighbourhood of x. □

The preceding example motivates us to give the following name to the set of all neighbourhoods of a point:

Definition 2.4.3 (Neighbourhood filter). Let (X, τ) be a topological space and let $x \in X$. The set of all neighbourhoods of x is called the *neighbourhood filter* of x.

Note that filters are directed with respect to converse set inclusion, which often makes them good choices for index sets of nets. In fact, in several proofs we have already used the neighbourhood filter of a point as index set of a net (although we did not use the word *filter* then).[15]

Another important example of a filter is the so-called *Fréchet filter*:

Example 2.4.4 (Fréchet filter). Let X be an infinite set and let $\mathcal{F}$ denote the set of all co-finite subsets of X.[16] Then it is easy to check that $\mathcal{F}$ is a filter. We call it the *Fréchet filter* on X.

Exercise 2.4.5 (Fréchet filter on finite sets?). Why do we need the assumption that X be infinite in Exercise 2.4.4?

[15] Explicit examples of this are the proof of the implication "(ii) ⇒ (i)" in Theorem 2.3.7 and the proof of the implication "(i) ⇒ (ii)" in Theorem 2.1.16.

[16] I.e., a set $S \subseteq X$ is in $\mathcal{F}$ if an only if S^{c} is finite.

Much more can be said about filters – for instance, we can define convergence of filters in topology spaces and prove similar results as we did about nets. We do not treat those results within the course; however, if you are interested in these concepts, you can find more information about it in the Addenda at the end of this chapter.

The reasons why we gave a very brief introduction to filters above are (i) in order to give you some background about the notion *neighbourhood filter* which is very common terminology in topology and (ii) in order to set the stage for the usage of so-called *ultrafilters* which we will introduce in Section 2.6 in order to prove the existence of so-called *universal nets*.

2.5 Interlude: Zorn's lemma

In many situations in mathematics, one faces the task to show the existence of an object that is *maximal* with respect to a certain kind of order. For instance, if one wants to prove that every vector space V has a basis, one needs to show that there exists a *maximal linearly independent set* in V. Another example stems from basic algebra: in order to show that every field has an algebraic closure, one needs to prove that every commutative unital ring contains a maximal ideal.[17] As another example we mention the Hahn–Banach extension theorem for linear functionals in functional analysis, whose proof is based on the existence of a maximal vector subspace with certain properties.

Existence proofs of maximal elements often work in a very non-constructive way: they employ *Zorn's lemma* (which is equivalent to the axiom of choice); we discuss the lemma in this section. We will use Zorn's lemma in the subsequent Section 2.6 in order to prove the existence of so-called *ultrafilters*.

Let us first recall the notion *partial order* which you most likely know from your Analysis 1 course.

Definition 2.5.1 (Partial order). Let X be a set. A relation $\leq$ on X is called a *partial order* iff it satisfies the following axioms:[18]

(I) *Reflexivity:* For each $x \in X$ we have $x \leq x$.

(II) *Anti-symmetry:* If $x, y \in X$ satisfy both $x \leq y$ and $y \leq x$, then $x = y$.

(III) *Transitivity:* If $x, y, z \in X$ such that $x \leq y$ and $y \leq z$, then $x \leq z$.

[17] A similar argument also occurs in the representation theory of commutative Banach algebras: in order to show that the character space of a unital commutative Banach algebra is non-empty, one needs to show the existence of a maximal proper ideal in such an algebra.

[18] Note that the axioms *reflexivity* and *transitivity* also occurred in the definition of directed sets (Definition 2.1.3)

A *partially ordered set* is a pair $(X, \leq)$, where X is a set and $\leq$ is a partial order on X.[19]

If $(X, \leq)$ is a partially ordered set and $x, y \in X$, then we use the notation $y \geq x$ synonymously with $x \leq y$.

We are interested in *maximal elements* and in *upper bounds* within a partially ordered set. Let us recall these two notations, along with the concept of a *chain*, in the following definition:

Definition 2.5.2 (Maximal elements, upper bounds, and chains). Let $(X, \leq)$ be a partially ordered set and let $S \subseteq X$.

(a) An element $s \in S$ is called a *maximal element of S* iff the only element $t \in S$ that satisfies $s \leq t$ is $t = s$.

(b) An element $x \in X$ is called an *upper bound of S* iff $s \leq x$ for each $s \in S$.

(c) The set S is called a *chain* iff all elements in S are comparable, by which we mean that for all $s, t \in S$ we have $s \leq t$ or $t \leq s$.

It is instructive to briefly compare the notions *maximal element* and *upper bound*: Let $(X, \leq)$ be a partially ordered set and let $S \subseteq X$.

- On the one hand, the concept *maximal element* is a bit more restrictive than the notion *upper bound* since each maximal element of S must, by definition, be an element of S – where an upper bounded of S need not be an element of S.

- With respect to another aspect, the concept *upper bound* is more restrictive than the notion *maximal element*, though: if $x \in X$ is an upper bound of S, then x dominates all elements of S; however, if $x \in S$ is a maximal element of S, it does not necessarily dominate all elements of S – we can only be sure that there are no other elements in S which are larger than x.

This might be a bit confusing at first glance, so it is a good idea to do the following exercise in order to get more familiar with the concepts *maximal element* and *upper bound*:

Exercise 2.5.3. (a) Give an example of a partially ordered set $(X, \leq)$ and a subset $S \subseteq X$ such that S has at least two distinct upper bounds in X.

(b) Give an example of a partially ordered set $(X, \leq)$ and a subset $S \subseteq X$ such that S has at least two distinct maximal elements.

[19] We will sometimes by slightly imprecise in our notation and say things like "let X be a partially ordered set", thereby suppressing the symbol $\leq$ in the notation.

(c) Show that if $(X, \leq)$ is a partially ordered set and $S \subseteq X$, then S has at most one upper bound that is contained in S.

For many partially ordered sets $(X, \leq)$ the following situation occurs:[20]

- One would like to prove the existence of a maximal element of X itself.
- It is, due to the structure of $(X, \leq)$, not difficult to show that every chain $S \subseteq X$ has an upper bound in X.

This is where Zorn's lemma enters the game: it says that the second bullet point gives us a solution to the first bullet point:

Lemma 2.5.4 (Zorn). *Let $(X, \leq)$ be a non-empty partially ordered set and assume that each non-empty chain $C \subseteq X$ has an upper bound in X. Then X has a maximal element.*

Note that, in Zorn's lemma, we are only talking about maximal elements of the set X itself (although the notion *maximal element* is defined for subsets of X, too).

We will not prove the lemma here; instead, we now proceed to the next section where we use Zorn's lemma in order to show the existence of so-called *ultrafilters*, which we then in turn use to prove the existence of so-called *universal nets*.

2.6 Universal nets and ultrafilters

We introduced *filters* in Definition 2.4.1. There is a particularly curious – and highly useful – class of filters which we discuss now: the so-called *ultrafilters*. Let X be a set, and let Γ denote the set of all filters on X (note that Γ is a subset of 2^{2^X}).[21] Hence, the set Γ is partially ordered by set inclusion.

Definition 2.6.1 (Ultrafilters). Let X be a set and let Γ denote the set of all filters on X.

A filter $\mathcal{U}$ on X is called an *ultrafilter* if it is a maximal element of Γ (with respect to set inclusion).

Ultrafilters can be characterised by the following curious property:

Proposition 2.6.2 (Characterisation of ultrafilters). *Let X be a set and let $\mathcal{U}$ be a filter on X. The following assertions are equivalent:*

[20] We will see an example of this situation in the subsequent Section 2.6

[21] If you like power sets from the very bottom of your heart, you could alternatively say that Γ is an element of $2^{2^{2^X}}$.

(i) *The filter $\mathcal{U}$ is an ultrafilter.*

(ii) *For each subset $S \subseteq X$ we have either $S \in \mathcal{U}$ or $S^c \in \mathcal{U}$.*

Proof. "(i) $\Rightarrow$ (ii)" Let $\mathcal{U}$ be an ultrafilter and let $S \subseteq X$. Assume that $S^c \notin \mathcal{U}$; we have to show that $S \in \mathcal{U}$.

To this end, observe that each set $U \in \mathcal{U}$ intersects S; indeed, if a set $U \in \mathcal{U}$ did not intersect S, then U would be a subset of S^c, which would imply $S^c \in \mathcal{U}$. Now we can define the family of sets

$$\tilde{\mathcal{U}} := \{V \subseteq X : \exists U \in \mathcal{U} \text{ such that } U \cap S \subseteq V\}.$$

Then $\tilde{\mathcal{U}}$ is easily checked to be a filter, where Axiom (I) in Definition 2.4.1 follows from the fact that S intersects each $U \in \mathcal{U}$. Moreover, we clearly have $\tilde{\mathcal{U}} \supseteq \mathcal{U}$, so it follows from the maximality of the ultrafilter $\mathcal{U}$ that actually $\mathcal{U} = \tilde{\mathcal{U}}$.

But obviously $S \in \tilde{\mathcal{U}}$; so we also have $S \in \mathcal{U}$.

"(ii) $\Rightarrow$ (i)" Assume that assertion (ii) holds and let $\tilde{\mathcal{U}}$ be a filter on X with the property $\tilde{\mathcal{U}} \supseteq \mathcal{U}$. We have to show that each $\tilde{U} \in \tilde{\mathcal{U}}$ is also an element of $\mathcal{U}$.

So fix $\tilde{U} \in \tilde{\mathcal{U}}$. We know that either $\tilde{U} \in \mathcal{U}$ or $\tilde{U}^c \in \mathcal{U}$; but the latter case would imply $\tilde{U}^c \in \tilde{U}$, which is a contradiction since the intersection $\tilde{U}^c \cap \tilde{U} = \emptyset$ is not in $\tilde{\mathcal{U}}$. Hence, we indeed have $\tilde{U} \in \mathcal{U}$. $\square$

Simple examples of ultrafilters are the so-called *fixed ultrafilters*:

Example 2.6.3 (Fixed ultrafilters). Let X be a set and let $x \in X$. Then

$$\mathcal{U} := \{U \subseteq X : x \in U\}$$

is an ultrafilter on X.

An ultrafilter $\mathcal{U}$ on X is called *fixed* iff there exists a point $x \in X$ such that $\mathcal{U}$ is given by the formula above.

Ultrafilters that are not fixed are much more interesting; they thus have a name on their own:

Definition 2.6.4 (Free ultrafilter). Let X be a set. An ultrafilter $\mathcal{U}$ on X is called *free* if it is not fixed.

The question whether an ultrafilter is free or fixed is closely related to the Fréchet filter that was introduced in Example 2.4.4:

Proposition 2.6.5 (Characterisation of free ultrafilters). *Let X be a set and let $\mathcal{U}$ be an ultrafilter on X. If X is infinite, then the following assertions are equivalent:*

(i) *The ultrafilter $\mathcal{U}$ is free.*

(ii) *We have* $\bigcap_{U \in \mathcal{U}} U = \emptyset$.

(iii) *The ultrafilter* $\mathcal{U}$ *contains the Fréchet filter on* X.

Proof. We pose this as an exercise on a problem sheet. □

Given that all ultrafilters satisfy the rather strange property (ii) in Proposition 2.6.2 one might be tempted to expect that no free ultrafilters can exist at all.

However, things are actually quite different: by using Zorn's lemma, we now show that each filter is contained in an ultrafilter:

Proposition 2.6.6 (Existence of ultrafilters). *Let* X *be a set and let* $\mathcal{F}$ *be a filter on* X*. Then there exists an ultrafilter* $\mathcal{U}$ *on* X *such that* $\mathcal{F} \subseteq \mathcal{U}$.

Proof. Let $\Gamma_{\mathcal{F}} \subset 2^{2^X}$ denote the set of all filters on X that contain $\mathcal{F}$. Then $\Gamma_{\mathcal{F}}$ is partially ordered with respect to set inclusion. Let us show that $\Gamma_{\mathcal{F}}$ has a maximal element by showing that we can apply Zorn's lemma:

If Ψ is a non-empty chain in $\Gamma_{\mathcal{F}}$, then one can readily check that the union $\mathcal{H} := \bigcup_{\mathcal{G} \in \Psi} \mathcal{G}$ is a filter on X that contains $\mathcal{F}$. Hence, $\mathcal{H} \in \Gamma_{\mathcal{F}}$ and clearly, $\mathcal{H}$ contains each filter in Ψ. Therefore, $\mathcal{H}$ is an upper bound of Ψ in $\Gamma_{\mathcal{F}}$.

Thus, it follows from Zorn's lemma that $\Gamma_{\mathcal{F}}$ has a maximal element $\mathcal{U}$. By the very definition of $\Gamma_{\mathcal{F}}$, $\mathcal{U}$ is a filter on X that contains $\mathcal{F}$. Moreover, $\mathcal{U}$ is even maximal among all filters on X (i.e., not only among all filters on X that contain $\mathcal{F}$), for if $\tilde{\mathcal{U}} \supseteq \mathcal{U}$ is another filter, then $\tilde{\mathcal{U}}$ also contains $\mathcal{F}$, so $\tilde{\mathcal{U}} \in \Gamma_{\mathcal{F}}$, and hence $\tilde{\mathcal{U}} = \mathcal{U}$ by the maximality of $\mathcal{U}$ in F. Thus, $\mathcal{U}$ is an ultrafilter on X. □

It is important to note that it follows from the previous proposition that there exists a free ultrafilter on each infinite set X: indeed, it follows from Proposition 2.6.6 that there exists an ultrafilter $\mathcal{U}$ on X that contains the Fréchet filter on X. Then by Proposition 2.6.5, the ultrafilter $\mathcal{U}$ is free.

Ultrafilters are an extremely powerful tool in set theory and in topology. However, we use them in this course mainly in order to prove the existence of another – similarly powerful – tool: *universal nets*. These nets are defined by a property that is very similar to property (ii) in Proposition 2.6.2:

Definition 2.6.7 (Universal nets). Let X be a set.

(a) A net $(x_i)_{i \in I}$ is called *universal* iff for each subset $S \subseteq X$ the net $(x_i)_{i \in I}$ is eventually in S or eventually in S^c.

(b) Let $(x_i)_{i \in I}$ and $(y_j)_{j \in J}$ be nets in X. We call $(x_i)_{i \in I}$ a *universal subnet of* $(y_j)_{j \in J}$ iff $(x_i)_{i \in I}$ is a universal net and a subnet of $(y_j)_{j \in J}$.

By the usage of ultrafilters we can now prove the existence of universal nets; more generally, we will even prove that every net has a universal subnet:

Proposition 2.6.8 (Existence of universal subnets). *Let X be a set. Then every net in X has a universal subnet.*

Proof. Let $(y_j)_{j\in J}$ be a net. Consider the filter

$$\mathcal{F} := \{F \subseteq J : \exists j_0 \in J \text{ such that all } j \geq j_0 \text{ are in } F\}$$

on J.[22] As shown in Proposition 2.6.6, there exists an ultrafilter $\mathcal{U}$ on J that contains $\mathcal{F}$. Each filter is directed with respect to converse set inclusion, and we use the directed set $(\mathcal{U}, \supseteq)$ as index of the universal subset that we seek.

For each $U \in \mathcal{U}$, choose an element in U – which exists since U is non-empty, and which we denote by $\varphi(U)$ to stress its dependence on U.

Then the mapping $\varphi : \mathcal{U} \to J$ is co-final. Indeed, for each $j_0 \in J$, consider the subset

$$T := \{j \in J : j \geq j_0\}$$

of J; then T is clearly in $\mathcal{F}$ and hence in $\mathcal{U}$. But for every $U \in \mathcal{U}$ which satisfies $U \subseteq T$ we have $\varphi(U) \in U$ and thus $\varphi(U) \in T$, so $\varphi(U) \geq j_0$.

Hence, $(y_{\varphi(U)})_{U\in\mathcal{U}}$ is a subnet of $(y_j)_{j\in J}$. Finally, let us show that the net $(y_{\varphi(U)})_{U\in\mathcal{U}}$ is universal: Let $S \subseteq X$. We consider the set

$$R := \{j \in J : y_j \in S\} \subseteq J.$$

Since $\mathcal{U}$ is an ultrafilter on J, we have either $R \in \mathcal{U}$ or $R^c \in \mathcal{U}$ (Proposition 2.6.2).

If $R \in \mathcal{U}$ then, for each $U \in \mathcal{U}$ that is a subset of R, we have $\varphi(U) \in U \subseteq R$, so $y_{\varphi(U)} \in S$; thus, our subnet $(y_{\varphi(U)})_{U\in\mathcal{U}}$ is eventually in S. If, on the other hand, $R^c \in \mathcal{U}$, then, for each $U \in \mathcal{U}$ that is a subset of R^c, we have $\varphi(U) \in U \subseteq R^c$, so $y_{\varphi(U)} \in S^c$; thus, our subnet $(y_{\varphi(U)})_{U\in\mathcal{U}}$ is eventually in S^c. ☐

It is easy to show, but quite important to observe, that each subnet of a universal net is itself universal:

Proposition 2.6.9 (Subnets of universal nets are universal). *Let X be a set and let $(x_i)_{i\in I}$ be a universal net. Then every subnet of $(x_i)_{i\in I}$ is universal, too.*

Proof. Let $(w_h)_{h\in H}$ be a subnet of $(x_i)_{i\in I}$, and let $S \subseteq X$. If $(x_i)_{i\in I}$ is eventually in S, then so is $(w_h)_{h\in H}$; and if $(x_i)_{i\in I}$ is eventually in S^c, then so is $(w_h)_{h\in H}$, too. Thus, $(w_h)_{h\in H}$ is indeed universal. ☐

[22] Note that Axion (II) in the definition of filters (Definition 2.4.1) is satisfied since J is directed.

Here is another nice consistency property of universal nets – the image of a universal net under a function is again a universal net:

Proposition 2.6.10 (Images of universal nets are universal). *Let X_1, X_2 be sets, let $f : X_1 \to X_2$ be a function and let $(x_i)_{i \in I}$ be a universal net in X_1. Then $\big(f(x_i)\big)_{i \in I}$ is a universal net in X_2.*

Proof. Let $S_2 \subseteq X_2$ and define $S_1 := f^{-1}(S_2) \subseteq X_1$. If $(x_i)_{i \in I}$ is eventually in S_1, then $\big(f(x_i)\big)_{i \in I}$ is eventually in S_2. If, on the other hand, $(x_i)_{i \in I}$ is eventually in $(S_1)^c$, then $\big(f(x_i)\big)_{i \in I}$ is eventually in $(S_2)^c$. $\square$

Universal nets are an extremely efficient tool when studying *compactness* of topological spaces – which we will do in Chapter 4.

For the moment, we start with a rather simple application of universal nets: in the next result, we show how they can be used to simplify the hair-splitting lemma (Lemma 2.2.9):

Lemma 2.6.11 (Universal hair-splitting lemma). *Let (X, τ) be a topological space, let $x \in X$ and let $(x_i)_{i \in I}$ be a net in X. The following assertions are equivalent:*

(i) *The net $(x_i)_{i \in I}$ converges to x.*

(ii) *Every universal subnet of $(x_i)_{i \in I}$ converges to x.*

Proof. "(i) $\Rightarrow$ (ii)" This implication is obvious.

"(ii) $\Rightarrow$ (i)" According to the hair-splitting lemma (Lemma 2.2.9) it suffices to show that every subnet of $(x_i)_{i \in I}$ has a subnet that converges to x.

So let $(w_h)_{h \in H}$ be a subnet of $(x_i)_{i \in I}$. As shown in Proposition 2.6.8, $(w_h)_{h \in H}$ has a universal subnet $(v_g)_{g \in G}$. Clearly, $(v_g)_{g \in G}$ is a universal subnet of $(x_i)_{i \in I}$, too, so it converges to x according to assertion (ii). $\square$

A nice feature of the universal hair splitting Lemma 2.6.11 compared to the hair splitting Lemma 2.2.9 is that we only have to deal with universal subnets of $(x_i)_{i \in I}$ instead of subnets of subnets. Apart from aesthetic considerations, this has the major advantage that we do not have to decide for any particular subnet of a given subnet. When we study compactness in Chapter 4, it will become apparent that the option to avoid any decision for an arbitrary subnet of a given subnet sometimes greatly facilitates arguments.

Next, we note that one can also formulate the characterisation of continuity in Theorem 2.3.7 in terms of universal nets:

Theorem 2.6.12 (Continuity via universal nets). *Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let $f : X_1 \to X_2$ be a function. For each point $x \in X_1$ the following assertions are equivalent:*

(i) *The function f is continuous at x.*

(ii) *For each universal net $(x_i)_{i\in I}$ in X_1 that converges to x, the net $\big(f(x_i)\big)_{i\in I}$ in X_2 converges to $f(x)$.*

Proof. "(i) $\Rightarrow$ (ii)" This follows from Theorem 2.3.7.

"(ii) $\Rightarrow$ (i)" Assume that f is not continuous at x and let $\mathcal{N}(x)$ denote the neighbourhood filter of x in X_1. Then $\mathcal{N}(x)$ is a directed set with respect to converse set inclusion.

Since f is not continuous at x, there exists a neighbourhood N_2 of $f(x)$ such that $f(N_1) \nsubseteq N_2$ for each $N_1 \in \mathcal{N}(x)$. Hence, for each $N_1 \in \mathcal{N}(x)$ there exists a point $z_{N_1} \in N_1$ such that $f(z_{N_1}) \notin N_2$. The net $(z_{N_1})_{N_1\in\mathcal{N}(x)}$ converges to x (Example 2.1.15(d)),

Now we choose a universal subnet $(x_i)_{i\in I}$ of $(z_{N_1})_{N_1\in\mathcal{N}(x)}$ (which exists according to Proposition 2.6.8). Then $(x_i)_{i\in I}$ converges to x, too, but we have $f(x_i) \notin N_2$ for each $i \in I$. Thus, the net $\big(f(x_i)\big)_{i\in I}$ does not converge to $f(x)$, which shows that assertion (ii) fails. □

Similarly, there are also versions of the description of closed sets from Theorem 2.1.16 and Corollary 2.1.17 which employ only universal nets, and accumulation points of nets can also be described in terms of universal subnets. The proofs of these fact are not difficult, but in order to proceed to the next topic, we defer these facts to the Addenda in Section 2.7.

2.7 Addenda: More on nets

Hair-splitting for sequences

Since every sequence is a net, the hair splitting Lemma 2.2.9 is in particular true if $(x_i)_{i\in I}$ is a sequence. However, part (ii) then still talks about subnets of subnets of this sequence.

Now, we have already pointed out on several occasions that sequences alone are often not sufficient when we deal with general topological spaces, and that it may even happen that a sequence is a subnet but not a subsequence of another sequence.

Thus, the following version of the hair splitting lemma might come as a surprise: it says that, if we want to characterise convergence of a sequence to a given point, then it actually suffices to do the "hair splitting" with subsequences of subsequences or with subnets of subsequences.

Lemma 2.7.1 (Hair-splitting lemma for sequences). *Let (X,τ) be a topological space, let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. The following assertions are equivalent:*

(i) *The sequence* $(x_n)_{n\in\mathbb{N}}$ *converges to* x.

(ii) *Every subsequence of* $(x_n)_{n\in\mathbb{N}}$ *has a subsequence that converges to* x.

(iii) *Every subsequence of* $(x_n)_{n\in\mathbb{N}}$ *has a subnet that converges to* x.

Please note that – although this might seem a bit counter-intuitive at first glance – assertion (iii) in the lemma is formally *weaker* than assertion (ii) (i.e., assertion (iii) is, at least formally, easier to check).[23]

Proof. "(i) $\Rightarrow$ (ii)" Each subsequence of $(x_n)_{n\in\mathbb{N}}$ is a subsequence of itself and converges to x according to Proposition 2.2.8.

"(ii) $\Rightarrow$ (iii)" This implication is obvious.

"(iii) $\Rightarrow$ (i)" Assume that $(x_n)_{n\in\mathbb{N}}$ does not converge to x. We have to find a subsequence of $(x_n)_{n\in\mathbb{N}}$ that has no subnet that converges to x.

Since $(x_n)_{n\in\mathbb{N}}$ does not converge to x, there exists a neighbourhood N of x with the following property: for each index $n \in \mathbb{N}$ there exists another index $\tilde{n} \geq n$ in $\mathbb{N}$ such that $(x_{\tilde{n}}) \notin N$.

Thus, we can recursively construct indices $n_1 < n_2 < n_3 \dots$ such that $x_{n_k} \notin N$ for each $k \in \mathbb{N}$. Hence, $(x_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ that has no subnet that converges to x. □

Sets of accumulation points: a proof of Theorem 2.2.11

In the following we provide a proof of Theorem 2.2.11 which we did not include in the lecture due to time constraints.

Proof of Theorem 2.2.11. Once the claimed equality is shown, it follows that A is closed as the intersection of closed sets (Proposition 1.1.7(c)). So we now show the claimed equality of sets by proving both inclusions.

"$\subseteq$" Let $x \in A$ and let $i_0 \in I$.

We have to show that $x \in \overline{\{x_i : i \geq i_0\}}$. As x is an accumulation point of $(x_i)_{i\in I}$, the latter net has a subnet $(w_h)_{h\in H}$ that converges to x. Hence, there exists a co-final mapping $\varphi : H \to I$ such that $x_{\varphi(h)} = w_h$ for all $h \in H$.

Due to the co-finality of φ, we can find an index $h_0 \in H$ such that $\varphi(h) \geq i_0$ for all $h \geq h_0$. Thus, all elements of the tail $(w_h)_{h\geq h_0}$ are located in the set $\{x_i : i \geq i_0\}$. Moreover, the tail $(w_h)_{h\geq h_0}$ is a subnet of $(w_h)_{h\in H}$ (Example 2.2.7) and thus converges to x, too (Proposition 2.2.8). As shown in Theorem 2.1.16, this implies that x is in the closure of $\{x_i : i \geq i_0\}$.

"$\supseteq$" Let $x \in \overline{\{x_i : i \geq i_0\}}$. We have to find a subnet of $(x_i)_{i\in I}$ that converges to x.

We define the index set H of this desired subnet as follows: let $\mathcal{N}(x)$ denote the set of all neighbourhoods of x, directed by converse set inclusion

[23] But of course, the lemma says that both assertions are actually equivalent.

(as discusses in Example 2.1.9). We then set $H := I \times \mathcal{N}(x)$ and endow this set with the product direction that we discussed in Proposition 2.1.11.

Now we construct a subnet of $(x_i)_{i\in I}$ with index set H as follows: let $j \in I$ and $N \in \mathcal{N}(x)$. We know from the characterisation of closures in Proposition 1.3.4 that the set $\{x_i : i \geq j\}$ intersects N (as x is contained in the closure of the former set). Thus, there exists an index in I – that depends on j and N and that we thus denote by $\varphi(j,N)$ – which is larger than j and which satisfies $x_{\varphi(j,N)} \in N$. We simply define

$$w_{(j,N)} := x_{\varphi(j,N)}.$$

Since the mapping $\varphi : H \to I$ is co-final, $(w_{(j,N)})_{(j,N)}$ is indeed a subnet of $(x_i)_{i\in I}$. Moreover, since $w_{(j,N)} \in N$ for each $N \in \mathcal{N}(x)$, it follows that the net $(w_{(j,N)})_{(j,N)\in H}$ converges to x. Thus, x is indeed an accumulation point of $(x_i)_{i\in I}$. □

The construction of the subnet $(w_{(j,N)})_{(j,N)\in H}$ in the proof of the inclusion "$\supseteq$" above is quite instructive since it shows how flexible the concept of directed sets is when we try to construct nets with certain properties: the index set of our subnet is $H = I \times \mathcal{N}(x)$; the first component of this product is responsible for making that mapping φ co-final, and the second component of the product is responsible for ensuring convergence of $(w_{(j,N)})_{(j,N)\in H}$ to x. This is a very typical situation when we construct nets that we wish to be adapted to specific purposes: one can use various directed sets to ensure different properties and then take the product of these sets as the desired index set.

More on universal nets

Just as we rephrased the characterisation of continuity via nets from Theorem 2.3.7 by means of universal nets in Theorem 2.6.12, we can also rephrase the description of closures and closed sets via nets (Theorem 2.1.16 and Corollary 2.1.17) by means of universal nets:

Theorem 2.7.2 (Closures via universal nets). *Let (X,τ) be a topological space and let $S \subseteq X$. For each point $x \in X$ the following assertions are equivalent:*

(i) *The point x is an element of the closure $\overline{S}$.*

(ii) *There exists a universal net $(x_i)_{i\in I}$ in S that converges to x.*

Proof. This is an immediate consequence of the description of closures by means of nets (Theorem 2.1.16) and of the fact that each net has a universal subnet (Proposition 2.6.8). □

Similarly, we have:

Corollary 2.7.3 (Closed sets via universal nets). *Let (X, τ) be a topological space and let $C \subseteq X$. The following assertions are equivalent:*

(i) *The set C is closed.*

(ii) *For every universal net $(x_i)_{i \in I}$ in C that converges to a point $x \in X$, we also have $x \in C$.*

For a proper understanding of Theorem 2.7.2(ii) and Corollary 2.7.3(ii) it is important to note that the question whether a net in a set $S \subseteq X$ is universal does not depend on whether we consider it as a net in S or as a net in X. This is the content of the following exercise:

Exercise 2.7.4 (Universal nets in subsets). Let X be a set and let $S \subseteq X$. Let $(x_i)_{i \in I}$ be a net in S. Show that $(x_i)_{i \in I}$ is a universal net in S if and only if it is a universal net in X.

Finally, let us note that accumulation points of a net can also be characterised by means of universal subnets:

Proposition 2.7.5 (Accumulation points via universal nets). *Let $(x_i)_{i \in I}$ be a net in a topological space (X, τ). A point $x \in X$ is an accumulation point of $(x_i)_{i \in I}$ if and only if there exists a universal subnet of $(x_i)_{i \in I}$ that converges to x.*

Proof. This is a consequence of the definition of the notion *accumulation point* (Definition 2.2.10) and of the fact that each net has a universal subnet (Proposition 2.6.8). □

2.8 Addenda: More on filters

Much more can be said about filters then we have done in this chapter. In particular, one can define a notion of *convergence* of filters which is closely related to convergence of nets. For a detailed treatment of these topics, we refer for instance to [Sch97, Chapters 5 and 7].

3 Constructing new topologies from given ones

Opening Questions.

(a) If (X,τ) is a topological space and $S \subseteq X$, is there a natural way to define a topology on S, too?

(b) Let (X,τ) be a topological space and let $(x_j)_{j\in J}$ and $(y_j)_{j\in J}$ be nets in X (with the same index set J) and assume that $x_j \to x$ and $y_j \to y$ for two points $x,y \in X$. Does it make sense to say that $(x_j,y_j) \to (x,y)$ in $X \times X$?

(c) Consider functions $f, f_n : [0,1] \to \mathbb{R}$ for $n \in \mathbb{N}$. What does it mean to say that "the sequence (f_n) converges to f"?

(d) Can we "factor out" an equivalence relation from a topological space?

3.1 Smallest topologies and bases

In this section we discuss how we can, given an arbitrary family $\mathcal{R}$ of subsets of a given set X, construct a topology τ with respect to which all the sets in $\mathcal{R}$ open (i.e., $\mathcal{R} \subseteq \tau$). Of course, we could always choose $\tau = 2^X$, but this choice of τ is rather uninteresting and does not have much to do with $\mathcal{R}$.

Thus, we actually intend to discuss the *smallest* topology τ that contains $\mathcal{R}$. This is done in the following proposition and the subsequent definition.

Proposition 3.1.1 (Existence of smallest topologies). *Let X be a set.*

(a) *Let I be a set and for each $i \in I$, let $\tau_i \subseteq 2^X$ be a topology on X. Then $\bigcap_{i\in I} \tau_i \subseteq 2^X$ is also a topology on X.*

(b) *Let $\mathcal{R} \subseteq 2^X$. Then there exists a (unique) smallest topology τ on X that contains $\mathcal{R}$, and this smallest topology is given by*

$$\tau := \bigcap_{\substack{\rho \supseteq \mathcal{R}, \\ \rho \text{ is a topology on } X}} \rho.$$

Proof. You have proved assertion (a) in Problem 3(c) on Problem Sheet 1, and assertion (b) is an immediate consequence of (a). $\square$

The previous proposition allows us to introduce the following terminology:

Definition 3.1.2 (Generated topology, bases and subbases). Let X be a set.

(a) Let $\mathcal{R} \subseteq 2^X$. We call the smallest topology τ on X that contains $\mathcal{R}$ the *topology generated by* $\mathcal{R}$.

(b) Let $\mathcal{S} \subseteq 2^X$. We call $\mathcal{S}$ a *subbasis* iff the union of all sets in $\mathcal{S}$ equals X.

Now assume in addition that τ is a topology on X. We call $\mathcal{S}$ a *subbasis of the topology* τ iff it is a subbasis and the topology generated by $\mathcal{S}$ equals τ.

(c) Let $\mathcal{B} \subseteq 2^X$. We call $\mathcal{B}$ a *basis* iff it is a subbasis and for all $B_1, B_2 \in \mathcal{B}$ and each $x \in B_1 \cap B_2$ there exists a set $B_0 \in \mathcal{B}$ such that $x \in B_0 \subseteq B_1 \cap B_2$.

Now assume in addition that τ is a topology on X. We call $\mathcal{B}$ a *basis of the topology* τ iff it is a basis and the topology generated by $\mathcal{B}$ equals τ.

The question whether a given set $\mathcal{B} \subseteq 2^X$ is a basis is closely related to the particular form of the topology generated by $\mathcal{B}$:

Proposition 3.1.3 (Characterisation of bases). *Let X be a set and let $\mathcal{B} \subseteq 2^X$. Then the subsets*

$$\begin{aligned} \tau_1 &:= \Big\{ U \subseteq X : \forall x \in U \; \exists B \in \mathcal{B} \text{ such that } x \in B \subseteq U \Big\}, \\ \tau_2 &:= \Big\{ \cup \mathcal{A} : \mathcal{A} \subseteq \mathcal{B} \Big\} \end{aligned}$$

of 2^X coincide. Moreover, the following assertions are equivalent:

(i) *The set $\mathcal{B}$ is a basis.*

(ii) *The set τ_1 (equivalently: τ_2) is a topology on X.*

If the equivalent conditions above are satisfied, then the topology $\tau_1 = \tau_2$ coincides with the topology generated by $\mathcal{B}$.

Proof. We first prove the equality $\tau_1 = \tau_2$:

"$\subseteq$" Let $U \in \tau_1$. We set $\mathcal{A} := \{B \in \mathcal{B} : B \subseteq U\}$. Then $\mathcal{A} \subseteq \mathcal{B}$, and it follows from $U \in \tau_1$ that the set $\cup\mathcal{A} = \bigcup_{B \in \mathcal{A}} B$ equals U. Hence, $U \in \tau_2$.

"$\supseteq$" Let $U \in \tau_2$, i.e., there exists a set $\mathcal{A} \subseteq \mathcal{B}$ such that $U = \cup\mathcal{A}$. For each $x \in U$ we can thus find a set $B \in \mathcal{A}$ such that $x \in B$, and hence we have $x \in B \subseteq U$; this shows that $U \in \tau_1$.

Now we prove the claimed equivalence:

"(i) $\Rightarrow$ (ii)" Obviously, the empty set is in τ_1, and since τ_1 is a subbasis, X is also in τ_1.

It is also clear that τ_1 is stable with respect to arbitrary unions. Finally, if $U_1, U_2 \in \tau_1$, then it follows from the definition of a basis that also $U_1 \cap U_2 \in \tau_1$.

"(ii) $\Rightarrow$ (i)" It follows from $X \in \tau_1$ that the set X is the union all sets in $\mathcal{B}$, so $\mathcal{B}$ is a subbasis. Now, let $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$. We have $B_1, B_2 \in \tau_1$ and thus, as τ_1 is a topology, also $B_1 \cap B_2 \in \tau_1$. Hence, by the very definition of τ_1 there exists a set $B_0 \in \mathcal{B}$ such that $x \in B_0 \subseteq B_1 \cap B_2$. This proves that $\mathcal{B}$ is a basis.

Finally, assume that the equivalent conditions (i) and (ii) are satisfied; let $\tau \subseteq 2^X$ denote the topology on X generated by $\mathcal{B}$. Since τ_2 is a topology on 2^X that contains $\mathcal{B}$, we have $\tau \subseteq \tau_2$. But on the other hand, since $\mathcal{B} \subseteq \tau$ and since τ is stable with respect to arbitrary unions, we also have $\tau_2 \subseteq \tau$. □

Let us now consider the preceding proposition from a slightly different viewpoint: assume that we are already given a topology τ on X and a set $\mathcal{B} \subseteq 2^X$. The following corollary reformulates Proposition 3.1.3 by characterising whether $\mathcal{B}$ is a basis *of the given topology* τ:

Corollary 3.1.4 (Characterisations of bases of a given topology). *Let* (X, τ) *be a topological space and let* $\mathcal{B} \subseteq 2^X$. *The following assertions are equivalent:*

(i) *The set* $\mathcal{B}$ *is a basis of the topology* τ.

(ii) *A set* $U \subseteq X$ *is open if and only if for each* $x \in U$ *there exists* $B \in \mathcal{B}$ *such that* $x \in B \subseteq U$.

(iii) *Each set* $B \in \mathcal{B}$ *is open, and each open set* $U \subseteq X$ *is the union of all* $B \in \mathcal{B}$ *that are subsets of* U.

Proof. Let τ_1 and τ_2 denote the same sets as in Proposition 3.1.3. Then assertion (ii) of the corollary says precisely that $\tau = \tau_1$, and assertion (iii) of the corollary says that $\tau = \tau_2$. The claimed equivalence thus follows from Proposition 3.1.3. □

Here are a few examples of bases of topologies:

Examples 3.1.5 (Bases in various topological spaces). (a) Let X be a set and let $\tau = 2^X$ denote the discrete topology on X. Then the set

$$\mathcal{B} := \Big\{ \{x\} : x \in X \Big\}$$

of all singletons in X is a basis of τ.

(b) Let (M, d) denote a metric space. Then the set

$$\mathcal{B} := \Big\{ \mathrm{B}_{<r}(x) : x \in M, \ r \in (0, \infty) \Big\}$$

of all open balls in M with non-zero radius is a basis of the topology induced by the metric d.

(c) On the real line, the set

$$\mathcal{B} := \Big\{[a,b) \subseteq \mathbb{R} : a,b \in \mathbb{R}\Big\}$$

is a basis of the Sorgenfrey topology.[1]

Exercise 3.1.6 (Closed balls in metric spaces). Let (M,d) be a metric space. The set of all closed balls in M is not a basis of the induced topology, in general (since closed balls need not be open). Why doesn't this contradict Exercise 1.4.12?

The following two results show that, in order to study convergence and continuity, one can often restrict to the basis of a topology:

Proposition 3.1.7 (Convergence via bases). *Let (X,τ) be a topological space, and let $\mathcal{B} \subseteq 2^X$ be a basis of τ. A net $(x_i)_{i\in I}$ converges to a point $x \in X$ if and only if for each $B \in \mathcal{B}$ that contains x, the net $(x_i)_{i\in I}$ is eventually in B.*

Proof. This is a consequence of Corollary 3.1.4 (ii). □

Proposition 3.1.8 (Continuity via bases). *Let (X_1,τ_1) and (X_2,τ_2) be topological spaces, and let $\mathcal{B}_1 \subseteq 2^{X_1}$ and $\mathcal{B}_2 \subseteq 2^{X_2}$ be bases of τ_1 and τ_2, respectively. Consider a function $f : X_1 \to X_2$.*

(a) *Let $x \in X_1$. The following assertions are equivalent:*

   (i) *The function f is continuous at x.*

   (ii) *For each $B_2 \in \mathcal{B}_2$ that contains $f(x)$ there exists a neighbourhood $N_1 \subseteq X_1$ of x such that $f(N_1) \subseteq B_2$.*

   (iii) *For each $B_2 \in \mathcal{B}_2$ that contains $f(x)$ there exists a set $B_1 \in \mathcal{B}_1$ that contains x such that $f(B_1) \subseteq B_2$.*

(b) *The function f is continuous if and only if $f^{-1}(B_2)$ is open in X_1 for each $B_2 \in \mathcal{B}_2$.*

Proof. Again, this is a simple consequence of Corollary 3.1.4(ii). □

We close this section with a few remarks on the comparison between topologies on a given set:

Definition 3.1.9 (Finer and coarser topologies). Let X be a set and let τ_1 and τ_2 be topologies on X. We say that τ_1 is *finer* than τ_2, or that τ_2 is *coarser* then τ_1 iff $\tau_2 \subseteq \tau_1$.

[1] Recall that the Sorgenfrey topology was introduced in Problem 6 on Problem Sheet 2.

If we use the terminology from Definition 3.1.9, then the topology on a set X generated by a set $\mathcal{R} \subseteq 2^X$ is the coarsest topology that contains $\mathcal{R}$.

If X is a set and $\mathcal{R}_2 \subseteq \mathcal{R}_1 \subseteq 2^X$, then the topology generated by $\mathcal{R}_1$ is clearly finer than the topology generated by $\mathcal{R}_2$.

Here is are a few criteria that characterise whether a given topology τ_1 is finer than another given topology τ_2:

Proposition 3.1.10 (Comparison of topologies). *Let τ_1 and τ_2 be topologies on a set X. The following assertions are equivalent:*

(i) *The topology τ_1 is finer than τ_2, i.e., every set in X that is open with respect to τ_2 is also open with respect to τ_1.*

(ii) *Every set in X that is closed with respect to τ_2 is also closed with respect to τ_1.*

(iii) *The identity mapping* $\mathrm{id} : (X, \tau_1) \to (X, \tau_2)$ *is continuous.*

(iv) *If a net $(x_i)_{i \in I}$ converges to $x \in X$ with respect to τ_1, then it also converges to x with respect to τ_2.*

Proof. We pose the proof as an exercise on Problem Sheet 6. $\square$

Proposition 3.1.3 in this section describes how one can construct a topology on a set X from a basis on X. We use this, for instance, on Problem Sheet 5 in order to define the so-called *order topology* on a linearly ordered set.

In the subsequent two sections we present two more ways to define new topologies. In both section, we assume that we are already given one or several topological spaces, and we use functions to "transfer" these topologies to other sets.

3.2 Initial topologies

The subject of this section are topologies that are transferred by a family of functions from their co-domains to their common domain; these are called *initial topologies*. We divide the section in four non-enumerated subsections, starting with the definition and the basic properties of initial topologies.

Basics on initial topologies

The details are explained in the following proposition and in the subsequent definition.

Proposition 3.2.1 (Initial topology). *Let X be a set and let A be a non-empty (index) set. For each $\alpha \in A$, let (X_α, τ_α) be a topological space and let $f_\alpha : X \to X_\alpha$ be a function.*

Then there exists a coarsest topology τ on X with respect to which all the functions f_α are continuous; in fact, τ is the topology generated by the subbasis

$$\mathcal{S} := \bigcup_{\alpha \in A} \left\{ f_\alpha^{-1}(U) : U \in \tau_\alpha \right\}$$

Moreover, the set

$$\mathcal{B} := \left\{ \bigcap_{\alpha \in F} f_\alpha^{-1}(U_\alpha) : F \subseteq A \text{ is finite, and } U_\alpha \in \tau_\alpha \text{ for each } \alpha \in F \right\}$$

is a basis of τ.

Proof. We first note that $\mathcal{S}$ is a indeed a subbasis on X since, for any $\alpha_0 \in A$, we have $X = f_{\alpha_0}^{-1}(X_{\alpha_0}) \in \mathcal{S}$ – so the union of all sets in $\mathcal{S}$ is obviously equal to X.

Next we show that $\mathcal{B}$ is indeed a basis on X. Clearly, it is a subbasis since it contains the subbasis $\mathcal{S}$. Now, let $B_1, B_2 \in \mathcal{B}$. It suffices to prove that $B_1 \cap B_2 \in \mathcal{B}$. By the definition of $\mathcal{B}$, there exist finite sets $F_1, F_2 \subseteq A$ and sets U_α for $\alpha \in F_1$ and $\tilde{U}_\alpha$ for $\alpha \in F_2$ such that

$$B_1 = \bigcap_{\alpha \in F_1} f_\alpha^{-1}(U_\alpha) \qquad \text{and} \qquad B_2 = \bigcap_{\alpha \in F_2} f_\alpha^{-1}(\tilde{U}_\alpha).$$

We now write the set $F := F_1 \cup F_2$ as the disjoint union $F = G_0 \cup G_1 \cup G_2$, where

$$G_0 := F_1 \cap F_2, \qquad G_1 := F_1 \setminus F_2, \qquad G_2 := F_2 \setminus F_1.$$

Then we obtain

$$\begin{aligned} B_1 \cap B_2 &= \left(\bigcap_{\alpha \in G_0} f_\alpha^{-1}(U_\alpha \cap \tilde{U}_\alpha) \right) \cap \left(\bigcap_{\alpha \in G_1} f_\alpha^{-1}(U_\alpha) \right) \cap \left(\bigcap_{\alpha \in G_2} f_\alpha^{-1}(\tilde{U}_\alpha) \right) \\ &= \bigcap_{\alpha \in F} f^{-1}(V_\alpha), \end{aligned}$$

where we define $V_\alpha = U_\alpha \cap \tilde{U}_\alpha$ if $\alpha \in G_0$, $V_\alpha = U_\alpha$ if $\alpha \in G_1$ and $V_\alpha = \tilde{U}_\alpha$ if $\alpha \in G_2$. Hence, $B_1 \cap B_2 \in \mathcal{B}$.

Next we note that $\mathcal{S}$ and $\mathcal{B}$ generated the same topology; this is true since $\mathcal{S}$ is a subset of $\mathcal{B}$ and since each set in $\mathcal{B}$ is an intersection of finitely many sets from $\mathcal{S}$.

Finally, let τ denote the topology generated by $\mathcal{S}$ (respectively, by $\mathcal{B}$). Obviously, every function f_α is continuous with respect to this topology on

X. If, on the other hand, $\tilde{\tau}$ is another topology on X such that each function f_α is continuous with respect to $\tilde{\tau}$, then $\tilde{\tau}$ obviously contains $\mathcal{S}$. Hence, $\tilde{\tau} \supseteq \tau$, which shows that τ is the coarsest topology on X with respect to which all functions f_α are continuous. $\square$

Definition 3.2.2 (Initial topology). In the situation of Proposition 3.2.1, the topology τ is called the *initial topology* of the family $(f_\alpha)_{\alpha \in A}$.

In order to efficiently work with initial topologies, the following theorem is extremely useful.

Theorem 3.2.3 (Properties of the initial topology). *Let X be a set and let A be a non-empty (index) set. For each $\alpha \in A$, let (X_α, τ_α) be a topological space and let $f_\alpha : X \to X_\alpha$ be a function. We endow X with the initial topology τ of the family $(f_\alpha)_{\alpha \in A}$.*

(a) *A net $(x_i)_{i \in I}$ in X converges to a point $x \in X$ if and only if, for each $\alpha \in A$, the net $\big(f_\alpha(x_i)\big)_{i \in I}$ converges to $f_\alpha(x)$.*

(b) *Let (W, τ_W) be a topological space and let $w \in W$. A mapping $e : W \to X$ is continuous at w if and only if the composition $f_\alpha \circ e : W \to X_\alpha$ is continuous at w for each $\alpha \in A$.*

The situation in Theorem 3.2.3(b) is illustrated by the following commutative diagram:

$$\begin{tikzcd} W \arrow[rr, "f_\alpha \circ e"] \arrow[dr, "e"'] & & X_\alpha \\ & X \arrow[ur, "f_\alpha"'] & \end{tikzcd}$$

Proof of Theorem 3.2.3. (a) Let $(x_i)_{i \in I}$ be a net in X and let $x \in X$.

"$\Rightarrow$" This implication is clear since each function f_α is continuous.

"$\Leftarrow$" This implication follows from the fact that convergence can be characterised by considering only sets from a basis (Proposition 3.1.7), and by the description of the basis $\mathcal{B}$ of τ in Proposition 3.2.1.

(b) Let $e : W \to X$ be a function.

"$\Rightarrow$" This is clear since the composition of continuous functions is continuous.

"$\Leftarrow$" This is a consequence of (a) together with the characterisation of continuity by means of net convergence (Theorem 2.3.7). $\square$

In the rest of this section we discuss to important initial topologies: the *product topology* and the *subspace topology*, as well as a few further examples of initial topologies at the very and of the section.

Product topologies

Before we define general product topologies, let us start with the following instructive special case:

Example 3.2.4 (Product of two topological spaces). Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and set $X := X_1 \times X_2$. Consider the projection mappings

$$\begin{aligned} & p_1 : X \to X_1, \quad (x_1, x_2) \mapsto x_1, \\ \text{and} \quad & p_2 : X \to X_2, \quad (x_1, x_2) \mapsto x_2. \end{aligned}$$

The initial topology τ on X of the family (p_1, p_2) is called the *product topology* of τ_1 and τ_2.

In this situation, assertions (a) and (b) of Theorem 3.2.3 mean the following:

(a) A net $\big((y_i, z_i)\big)_{i\in I}$ in $X = X_1 \times X_2$ converges to a point $(y, z) \in X$ if and only if the net $(y_i)_{i\in I}$ in X_1 converges to y and the net $(z_i)_{i\in I}$ in X_2 converges to z.

(b) For a topological space (W, τ_W), a function $e : W \to X$ – which is always given by $e(w) = \big(e_1(w), e_2(w)\big)$ for two component functions $e_1 : W \to X_1$ and $e_2 : W \to X_2$ and for all $w \in W$ – is continuous if and only if both e_1 and e_2 are continuous.

Exercise 3.2.5. Show that property (a) in Example 3.2.4 characterises the product topology on $X_1 \times X_2$, i.e., show that if $\tilde{\tau}$ is a topology on $X_1 \times X_2$ such that convergence of nets with respect to $\tilde{\tau}$ is equivalent to componentwise convergence, then $\tilde{\tau}$ coincides with the product topology.

Here are a few examples to highlight the role of the product of two topological spaces in several situations.

Example 3.2.6 (The metric is continuous on a metric space). Let (M, d) be a metric space, which we endow – as usual – with the induced topology. In addition, let the product $M^2 = M \times M$ be endowed with the product topology, and let $\mathbb{R}$ be endowed with the Euclidean topology.

Then the function $\mathrm{d} : M \times M \to \mathbb{R}$ is continuous.

Proof. For the proof we need the inequality

$$\big|\mathrm{d}(x_1, y_1) - \mathrm{d}(x_2, y_2)\big| \le \mathrm{d}(x_1, x_2) + \mathrm{d}(y_1, y_2), \tag{3.2.1}$$

which holds for all $x_1, x_2, y_1, y_2 \in M$ and which follows easily from the triangle inequality that is satisfied by the metric d.

Now, let $\big((x_i, y_i)\big)_{i \in I}$ be a net in $M \times M$ that converges to a point $(x, y) \in M \times M$. We have to prove that the net $\big(\mathrm{d}(x_i, y_i)\big)_{i \in I}$ in $\mathbb{R}$ converges to the number $d := \mathrm{d}(x, y)$. According to Exercise 2.1.14 it suffices to prove that, for each $\varepsilon > 0$, there exists $i_0 \in I$ such that $\mathrm{d}(x_i, y_i) \in (d - \varepsilon, d + \varepsilon)$. So fix $\varepsilon > 0$.

We know from point (a) in Example 3.2.4 that the net $(x_i)_{i \in I}$ converges to x and that the net $(y_i)_{i \in I}$ converges to y. Hence, there exists an index $i_0 \in I$ such that

$$\mathrm{d}(x_i, x) < \varepsilon \qquad \text{and} \qquad \mathrm{d}(y_i, y) < \varepsilon$$

for all $i \geq i_0$. Together with the inequality (3.2.1) this yields, again for $i \geq i_0$,

$$\Big|\mathrm{d}(x_i, y_i) - \mathrm{d}(x, y)\Big| \leq \mathrm{d}(x_i, x) + \mathrm{d}(y_i, y) \leq \varepsilon,$$

so indeed $\mathrm{d}(x_i, y_i) \in (d - \varepsilon, d + \varepsilon)$. □

Exercise 3.2.7 (The product of Euclidean spaces). Let $d_1, d_2 \in \mathbb{N}$ and endow both $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ with the Euclidean topology, repectively. Show that the product topology on

$$\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^{d_1 + d_2}$$

coincides with the Euclidean topology on this space.

Example 3.2.8 (Sums and products of continuous functions). Let $\mathbb{R}^2$ be endowed with the Euclidean topology, which is – according to Exercise 3.2.7 – the product topology on $\mathbb{R} \times \mathbb{R}$ (where $\mathbb{R}$ is also endowed with the Euclidean topology). You know from your Analysis courses that the functions

$$s : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto x_1 + x_2$$

and

$$m : \mathbb{R}^2 \to \mathbb{R}, \quad (x_1, x_2) \mapsto x_1 \cdot x_2$$

are continuous.

Let (W, τ_W) be a topological space and let $f, g : W \to \mathbb{R}$ be continuous. Then it follows from Example 3.2.4(b) that the mapping

$$h : W \to \mathbb{R}^2, \quad w \mapsto \big(f(w), g(w)\big)$$

is continuous. Thus, the compositions $s \circ h$ and $m \circ h$, which map from W to $\mathbb{R}$, are also continuous. If we note that these compositions are actually given by

$$(s \circ h)(w) = f(w) + g(w) \qquad \text{and} \qquad (m \circ h)(w) = f(w) \cdot g(w)$$

for all $w \in W$, we can thus see that sums and products of continuous real-valued functions are continuous.

We had already anticipated (a special case of) the above example when we discussed in Example 2.3.6 that the unit circle in $\mathbb{R}^2$ is closed.

Example 3.2.9 (Topological vector spaces). Let $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$ be endowed with the Euclidean topology, let V be a vector space over $\mathbb{F}$, and let τ be a topology on V. The pair (V, τ) is called a *topological vector space* iff the mappings

$$\begin{aligned} & V \times V \ni (v_1, v_2) \mapsto v_1 + v_2 \in V \\ \text{and} \quad & \mathbb{F} \times V \ni (\lambda, v) \mapsto \lambda v \in V \end{aligned}$$

are continuous. Here, the product spaces $V \times V$ and $\mathbb{F} \times V$ are endowed with the product topology, respectively.

After this quite extensive discussion of the product of two topological spaces, we now come to the general case: products of arbitrary families of topological spaces.

Example 3.2.10 (Product topology of arbitrarily many spaces). Let A be a non-empty (index) set. For each $\alpha \in A$, let (X_α, τ_α) be a topological space. We consider the Cartesian product $X := \prod_{\alpha \in A} X_\alpha$; for each index $\beta \in A$, let $p_\beta : X \to X_\beta$ denote the projection onto the β-th component, i.e.,

$$p_\beta : (x_\alpha)_{\alpha \in A} \mapsto x_\beta.$$

The initial topology of the family $(p_\beta)_{\beta \in A}$ is called the *product topology* on X.

In this situation, assertions (a) and (b) of Theorem 3.2.3 mean the following:

(a) A net $\big((x_{\alpha,i})_{\alpha \in A}\big)_{i \in I}$ in $X = \prod_{\alpha \in A} X_\alpha$ converges to a point $(x_\alpha)_{\alpha \in A}$ if and only if it converges componentwise, i.e., if and only if, for each $\beta \in A$, the net $(x_{\beta,i})_{i \in I}$ in X_β converges to x_β.

(b) For a topological space (W, τ_W), a function $e : W \to X$ – which is always given by $e(w) = (e_\alpha(w))_{\alpha \in A}$, with functions $e_\alpha : W \to X_\alpha$ for each $\alpha \in A$ – is continuous if and only if each function e_α is continuous.

An important special case of the above situation is when all the spaces X_α coincide:

Example 3.2.11 (Functions and sequences: The product topology on power spaces). Let (X, τ) be a topological space.

(a) The space $X^{\mathbb{N}}$ of all sequences in X is simply the product $\prod_{n \in \mathbb{N}} X$, where all factors are equal to X. Hence, we can endow $X^{\mathbb{N}}$ with the product topology.

(b) The space $X^{\mathbb{R}}$ of all functions from $\mathbb{R}$ to X is the product $\prod_{r\in\mathbb{R}} X$, where all factors are equal to X. Thus, we can endow $X^{\mathbb{R}}$ with the product topology. Convergence of a net $(f_i)_{i\in I}$ in $X^{\mathbb{R}}$ to a function $f \in X^{\mathbb{R}}$ means precisely that, for each $r \in \mathbb{R}$, the net $(f_i(r))_{i\in I}$ converges to $f(r)$; in other words, converges with respect to the product topology is simply pointwise convergence in this situation.

Subspaces topologies

Now we come to our second important example class of initial topologies – namely subspace topologies:

Example 3.2.12 (Subspace topology). Let (X, τ) be a topological space and let $S \subseteq X$. Let $j : S \to X$ denote the embedding of S into X.[2] Then the initial topology of j is called the *subspace topology on S*;[3] we denote it by $\tau|_S$.

In this situation, assertions (a) and (b) of Theorem 3.2.3 say the following:

(a) A net $(s_i)_{i\in I}$ in S converges to a point $s \in S$ with respect to $\tau|_S$ if and only if the same net converges to s in X with respect to τ.

(b) For a topological space (W, τ_W), a function $e : W \to S$ is continuous (with respect to $\tau|_S$) if and only if e is continuous (with respect to τ) when considered as a function from W to X.

The subspace topology can be described in quite an explicit way. In order to do so, we need the following simple observation about initial topologies of a single mapping, which you have already proved in one of the live lectures:

Proposition 3.2.13 (Initial topology of a single mapping). *Let X be a set, let (Y, τ_Y) be a topological space and let $f : X \to Y$ be a mapping. Then the initial topology τ_X of f is given by*

$$\tau_X = \left\{ f^{-1}(U) : U \in \tau_Y \right\}.$$

Similarly, the closed sets in X are precisely the sets of the form $f^{-1}(C)$, where C is closed in Y.

Proof. The set $\left\{ f^{-1}(U) : U \in \tau_Y \right\}$ is a subbasis of τ_X according to the definition of the initial topology (Proposition 3.2.1 and Definition 3.2.2). Moreover, it is easy to check that this set is itself a topology, which implies that it actually equal to τ_X. This proves the first claim.

Consequently, all sets in X that are closed with respect to τ_X are of the form $(f^{-1}(U))^{\mathrm{c}} = f^{-1}(U^{\mathrm{c}})$ for $U \in \tau_Y$, which proves the second claim. □

[2] By which we simply mean that $j(s) = s$ for each $s \in S$.

[3] Sometimes, the subspace topology is also called *trace topology* or *relative topology*.

Proposition 3.2.14 (Explicit description of the subspace topology). *Let (X,τ) be a topological space and endow $S \subseteq X$ with the subspace topology $\tau|_S$.*

A set $R \subseteq S$ is...

(a) *...open if and only if there exists an open set $U \subseteq X$ such that $R = U \cap S$.*

(b) *...closed if and only if there exists an closed set $C \subseteq X$ such that $R = C \cap S$.*

Proof. Let $j : S \to X$ denote the canonical embedding.

(a) By Proposition 3.2.13, the set R is open if and only if there exists an open set $U \subseteq X$ such that $R = j^{-1}(U)$. But the latter set is precisely the set $U \cap S$.

(b) This follows from an analogous argument as in (a). □

It is important to note that, for a subset S of a metric space, it does not matter whether we endow S with the subspace topology or whether we endow S with the topology induced by the restriction of the metric to S. This is the content of the next exercise:

Exercise 3.2.15 (The subspace topology on metric spaces). Let (M,d) be a metric space and let $S \subseteq M$. Endow M with the topology τ induced by the metric d.

Show that the subspace topology $\tau|_S$ coincides with the topology induced by the metric $\mathrm{d}|_{S\times S}$ on S.

For instance, the preceding exercise implies the following: if S is a subset of $\mathbb{R}^d$, it does not matter whether we consider the Euclidean metric on S and endow S with the induced topology or whether we endow the entire space $\mathbb{R}^d$ with the Euclidean topology and S with the subspace topology.

Further examples of initial topologies

In the following, we briefly discuss two more example of initial topologies:

Example 3.2.16 (Pointwise convergence of continuous functions). Consider the vector space $C([0,1])$ of all real-valued continuous functions on the interval $[0,1]$. We would like to define a topology τ on $C([0,1])$ such that convergence of nets in $C([0,1])$ is equivalent to pointwise convergence. There are two ways to achieve this:

- *Option 1:* We can consider $C([0,1])$ as a subset of the space $\mathbb{R}^{[0,1]}$ of all real-valued functions on $[0,1]$. The latter space is equal to the product $\prod_{\alpha\in[0,1]}\mathbb{R}$ and may thus be endowed with the product topology of the Euclidean topology on $\mathbb{R}$. Finally, we then endow $C([0,1])$ with the subspace topology of this product topology.

- *Option 2:* We can consider, for each $\alpha \in [0,1]$ the mapping

$$\begin{aligned} f_\alpha : C([0,1]) &\to \mathbb{R}, \\ f &\mapsto f(\alpha), \end{aligned}$$

  and then consider the initial topology of $(f_\alpha)_{\alpha\in[0,1]}$ on $C([0,1])$.

Both options yields a topology on $C([0,1])$ such that a net $(f_i)_{i\in I}$ in $C([0,1])$ converges to a function $f \in C([0,1])$ if and only if, for each $\alpha \in [0,1]$, the net $(f_i(\alpha))_{i\in I}$ converges to $f(\alpha)$. In particular, the topologies that we obtain by these two options coincide (Proposition 3.1.10(i) and (iv)).

Example 3.2.17 (Uniform convergence on compact sets). On the space $C(\mathbb{R}^d)$ of all real-valued continuous functions on $\mathbb{R}^d$ we would like to find a topology τ such that a net $(f_i)_{i\in I}$ converges to f if and only if, for each compact subset $K \subseteq \mathbb{R}^d$, the net $(f_i|_K)_{i\in I}$ converges uniformly to $f|_K$.

To this end, let $\mathcal{K}$ denote the set of all non-empty compact subsets of $\mathbb{R}^d$. For each $K \in \mathcal{K}$, let $C(K)$ denote the space of all real-valued continuous functions on K. This is a metric space with respect to the metric d that is given by

$$\mathrm{d}(f,g) := \|f-g\|_\infty = \sup_{\omega\in K} |f(\omega) - g(\omega)|.$$

Convergence with respect to the topology induced by this metric means precisely uniform convergence.

Now we consider, for each $K \in \mathcal{K}$, the restriction mapping

$$\begin{aligned} \varphi_K : C(\mathbb{R}^d) &\to C(K), \\ f &\mapsto f|_K. \end{aligned}$$

Let τ denote the initial topology on $C(\mathbb{R}^d)$ of the family $(f|_K)_{K\in\mathcal{K}}$. Then convergence with respect to τ means precisely uniform convergence on each compact subset of $\mathbb{R}^d$.

3.3 Final topologies

Just as we transferred the topologies of a family of given spaces (X_α, τ_α) back to a set X via mappings $f_\alpha : X \to X_\alpha$ in the definition of the initial topology, we will now discuss how we can transfer topologies from X_α to X if the f_α are mappings $X_\alpha \to X$. This is the content of the next proposition and the subsequent definition:

Proposition 3.3.1 (Final topology). *Let X be a set and let A be a non-empty (index) set. For each $\alpha \in A$, let (X_α, τ_α) be a topological space and let $f_\alpha : X_\alpha \to X$ be a function.*[4]

Then there exists a finest topology τ on X with respect to which all the functions f_α are continuous; in fact, τ is given by the formula

$$\begin{aligned} \tau &= \Big\{ U \subseteq X : f_\alpha^{-1}(U) \in \tau_\alpha \text{ for all } \alpha \in A \Big\} \\ &= \bigcap_{\alpha \in A} \Big\{ U \subseteq X : f_\alpha^{-1}(U) \in \tau_\alpha \Big\}. \end{aligned}$$

Proof. Let us consider the set $\tau \subseteq 2^X$ that is given by the formula in the proposition. It is immediate to check that τ is a topology on X, and clearly, each function f_α is continuous with respect to τ.

On the other hand, let $\tilde{\tau}$ be another topology on X such that each function f_α is continuous with respect to $\tilde{\tau}$. Then for each $U \in \tilde{\tau}$, all the sets $f_\alpha^{-1}(U)$ (where $\alpha \in A$) are open, so $U \in \tau$. This proves that $\tilde{\tau} \subseteq \tau$, i.e., τ is the finest topology on X such that all the maps f_α are continuous. □

Definition 3.3.2 (Final topology). In the situation of Proposition 3.3.1 the topology τ is called the *final topology* of the family $(f_\alpha)_{\alpha \in A}$.

The final topology has the following property regarding continuous mappings:

Proposition 3.3.3 (Continuity with respect to the final topology). *Let X be a set and let A be a non-empty (index) set. For each $\alpha \in A$, let (X_α, τ_α) be a topological space and let $f_\alpha : X_\alpha \to X$ be a function. We endow X with the final topology of the family $(f_\alpha)_{\alpha \in A}$.*

Let (Y, τ_Y) be a topological space. A mapping $g : X \to Y$ is continuous if and only if $g \circ f_\alpha : X_\alpha \to Y$ is continuous for each $\alpha \in A$.

Proof. "$\Rightarrow$" This implication is obvious since the composition of two continuous maps is always continuous.

"$\Leftarrow$" Let $V \subseteq Y$ be open. For each $\alpha \in A$ the mapping $g \circ f_\alpha : X_\alpha \to Y$ is continuous, so the set

$$f_\alpha^{-1}\Big(g^{-1}(V)\Big) = (g \circ f_\alpha)^{-1}(V)$$

is open in X_α. This proves that $g^{-1}(V)$ is open in X, so g is indeed continuous. □

Let us consider two examples of final topologies:

[4] Note that the domain and the co-domain of the mappings f_α are now swapped compared to the definition of the initial topology.

Example 3.3.4 (Quotient topology). Let (X,τ) be a topological space and let $\sim$ be an equivalence relation on X. Let $X/{\sim}$ denote the set of all equivalence classes of $\sim$ and let $q : X \to X/{\sim}$ denote the so-called *quotient mapping* which maps each point $x \in X$ to its equivalence class.

Then the final topology of q is called the *quotient topology* of $\sim$.

In this situation, Proposition 3.3.3 means the following: If (Y,τ_Y) is another topological space and $h : X/{\sim} \to Y$ is a mapping, then h is continuous if and only if $h \circ q : X \to Y$ is continuous.

Example 3.3.5 (Disjoint union of two topological spaces). Let X be a set which is the disjoint union of two subsets X_1 and X_2. Moreover, let τ_1 and τ_2 be topologies on X_1 and X_2, respectively, and let $j_1 : X_1 \to X$ and $j_2 : X_2 \to X$ denote the canonical embeddings.

Endow X with the final topology τ of the family (j_1, j_2). Then we have

$$\tau = \Big\{U_1 \cup U_2 : \, U_1 \in \tau_1 \text{ and } U_2 \in \tau_2.\Big\}$$

Proof. "$\supseteq$" If $U_1 \in \tau_1$ and $U_2 \in \tau_2$, then $j_1^{-1}(U_1 \cup U_2) = U_1 \in \tau_1$ and $j_2^{-1}(U_1 \cup U_2) = U_2 \in \tau_2$, so we indeed have $U_1 \cup U_2 \in \tau$.

"$\subseteq$" Let $U \in \tau$. Then we can write U as $U = U_1 \cup U_2$, where the sets U_1 and U_2 are chosen as

$$U_1 := U \cap X_1 = j_1^{-1}(U) \in \tau_1 \quad \text{and} \quad U_2 := U \cap X_2 = j_2^{-1}(U) \in \tau_2;$$

this proves the claim. □

3.4 Addenda: More on subbases

For a given subbasis $\mathcal{S}$ on a set X one can concretely describe a basis that generates the same topology:

Proposition 3.4.1 (Constructing a basis from a subbasis). *Let X be a set and let $\mathcal{S} \subseteq 2^X$ be a subbasis. Then*

$$\mathcal{B} := \{B_1 \cap \cdots \cap B_n : \, n \in \mathbb{N}, \, B_1, \ldots, B_n \in \mathcal{S}\}$$

is a basis on X that generates the same topology as $\mathcal{S}$.

Exercise 3.4.2 (Proof of Proposition 3.4.1). Prove Proposition 3.4.1.

Another nice observation about subbases is that Propositions 3.1.7 and 3.1.8 remain true if we replace bases with subbases:

Proposition 3.4.3 (Convergence via subbases). *Let (X,τ) be a topological space, and let $\mathcal{S} \subseteq 2^X$ be a subbasis of τ. A net $(x_i)_{i \in I}$ converges to a point $x \in X$ if and only if for each $B \in \mathcal{S}$ that contains x, the net $(x_i)_{i \in I}$ is eventually in B.*

Proof. This follows from the characterisation of convergence in terms of basis sets in Proposition 3.1.7 and from the description of a basis $\mathcal{B}$ that generates the same topology as $\mathcal{S}$ in Proposition 3.4.1. □

Proposition 3.4.4 (Continuity via subbases). *Let (X_1, τ_1) and (X_2, τ_2) be topological spaces, and let $\mathcal{S}_2 \subseteq 2^{X_2}$ be a subbasis of τ_2. Consider a function $f : X_1 \to X_2$.*

(a) *Let $x \in X_1$. The following assertions are equivalent:*

  (i) *The function f is continuous at x.*

  (ii) *For each $B_2 \in \mathcal{S}_2$ that contains $f(x)$ there exists a neighbourhood $N_1 \subseteq X_1$ of x such that $f(N_1) \subseteq B_2$.*

(b) *The function f is continuous if and only if $f^{-1}(B_2)$ is open in X_1 for each $B_2 \in \mathcal{S}_2$.*

Proof. Similarly as in the proof of the previous proposition, this follows from the characterisation of continuity in terms of basis sets in Proposition 3.1.8 and from the description of a basis $\mathcal{B}$ that generates the same topology as $\mathcal{S}$ in Proposition 3.4.1. □

Note that the previous proposition does not contain an analogue of Proposition 3.1.8(a)(iii).

4

Compactness

Opening Questions.

(a) What is a subset $S \subseteq \mathbb{R}^d$ called compact? When is a subset S of a metric space (X, d) called compact?

(b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Does f always attain a maximum on $[0,1]$? If yes, why? If no, why not?

(c) Consider a net $(x_j)_{j\in J}$ in a topological space (X, τ). Does $(x_j)_{j\in J}$ always have a convergent subnet?

4.1 Compact spaces and compact sets: basics

In this chapter we discuss a most important topological concept – namely *compactness*. You certainly know several things about compact sets in metric spaces from your Analysis courses; in particular, compact subsets C of metric spaces can be characterised by the so-called *Heine–Borel property* which says that every open cover of C contains a finite sub-cover.

In the general setting of topological spaces, one takes this is property as the definition of compactness:

Definition 4.1.1 (Compact and relatively compact sets). Let (X, τ) be a topological space and let $S \subseteq X$.

(a) The set S is called *compact* iff for every set $\mathcal{V} \subseteq 2^X$ of open sets[1] in X such that $\cup\mathcal{V} \supseteq S$, there exists a finite set $\mathcal{F} \subseteq \mathcal{V}$ such that $\cup\mathcal{F} \supseteq S$.

(b) The set S is called *relatively compact* iff its closure $\overline{S}$ is compact.

Example 4.1.2 (Compactness in metric spaces). Let (M, d) be a metric space which we endow, as usual, with the induced topology. As is typically taught in an Analysis course, a subset $S \subseteq M$ is compact (in the sense of Definition 4.1.1) if and only if each sequence in S has a subsequence that converges to a point in S.

A proof of this result, along with more information about compactness and relative compactness in metric spaces, can be found in Section 6.3 of this manuscript.

[1] In other words, $\mathcal{V} \subseteq \tau$.

A similar characterisation of compactness as in metric spaces is also true in general topological spaces. However – as usual – we have to replace sequences with nets; see Theorem 4.2.3 below for details.

Let us mentioned a few more examples of compact sets:

Example 4.1.3 (Finite sets are compact). Let (X, τ) be a topological space. Then every finite set $S \subseteq X$ is compact.

Example 4.1.4 (Compactness with respect to the co-finite topology). Endow a set X with the co-finite topology τ. Then every subset $S \subseteq X$ is compact.

Proof. We may assume that S is non-empty.

Let $\mathcal{V} \subseteq 2^X$ be a set of open subsets of X such that $\cup\mathcal{V} \supseteq S$. Then there exists a non-empty set $U \in \mathcal{V}$. The complement U^c is finite, and for each x in the finite set $S \setminus U$ we can find a set $U_x \in \mathcal{V}$ such that $x \in U_x$.

Hence, S is covered by the finite union $U \cup \bigcup_{x \in S \setminus U} U_x$. □

Example 4.1.5 (Compactness with respect to the co-countable topology). Endow a set X with the co-countable topology τ. Then a subset $S \subseteq X$ is compact if and only if S is finite.

Proof. "$\Leftarrow$" As mentioned in Example 4.1.3, every finite set in every topological space is compact.

"$\Rightarrow$" Assume that S is infinite. Then S has a countable infinite subset C, which we can write as $C = \{c_1, c_2, c_3, \dots\}$ for pairwise distinct elements $c_1, c_2, \dots$.

For each $n \in \mathbb{N}$, we now define a set

$$U_n := X \setminus \{c_n, c_{n+1}, c_{n+2}, \dots\} \subseteq X.$$

Then each U_n is co-countable and thus open. Moreover, $\bigcup_{n \in \mathbb{N}} U_n = X \supseteq S$. However, finitely many of the sets U_n do not cover C, and thus do not cover S. Hence, S is not compact. □

One important property of compactness is that it is preserved by continuous maps:

Proposition 4.1.6 (Continuous images of compact sets). *Let (X_1, τ_1) and (X_2, τ_2) be topological spaces and let $f : X_1 \to X_2$ be continuous. If $S_1 \subseteq X_1$ is compact, then so is $f(S_1)$.*

Proof. Let $(V_\alpha)_{\alpha \in A}$ be a family of open subsets of X_2 such that $\bigcup_{\alpha \in A} V_\alpha \supseteq f(S_1)$. Then all the sets $f^{-1}(V_\alpha)$ are open in X_1, and we have

$$\bigcup_{\alpha \in A} f^{-1}(V_\alpha) = f^{-1}\Big(\bigcup_{\alpha \in A} V_\alpha\Big) \supseteq f^{-1}\big(f(S_1)\big) \supseteq S_1.$$

Since S_1 is compact, there exists a finite subset $F \subseteq A$ such that $\bigcup_{\alpha \in F} f^{-1}(V_\alpha) \supseteq S_1$. Hence,

$$\bigcup_{\alpha \in F} V_\alpha \supseteq \bigcup_{\alpha \in F} f\big(f^{-1}(V_\alpha)\big) \supseteq f\Big(\bigcup_{\alpha \in F} f^{-1}(V_\alpha)\Big) \supseteq f(S_1).$$

This proves that $f(S_1)$ is compact. □

Next we note that the property of compactness does not depend on the choice of the surrounding space:

Proposition 4.1.7 (Compactness is an intrinsic property). *Let (Y, τ) be a topological space, and let $S \subseteq X \subseteq Y$. Endow X with the subspace topology $\tau|_X$. Then S is compact in (Y, τ) if and only if S is compact in $(X, \tau|_X)$.*

In particular, S is compact in (Y, τ) if and only if S is compact in $(S, \tau|_S)$.

Proof. "$\Leftarrow$" Let $j : X \to Y$ denote the inclusion mapping. Then j is continuous. Since S is compact in $(X, \tau|_X)$, it follows from Proposition 4.1.6 that $S = j(S)$ is compact in (Y, τ_Y).

"$\Rightarrow$" Let $(V_\alpha)_{\alpha \in A}$ be a family of open sets in $(X, \tau|_X)$ that covers S. By Proposition 3.2.14(a), for each $\alpha \in A$ there exists an open set U_α in (Y, τ) such that $V_\alpha = U_\alpha \cap X$.

Since the U_α also cover S, finitely many of them – say $U_{\alpha_1}, \dots, U_{\alpha_n}$ – suffices to covers S. Hence,

$$\bigcup_{k=1}^n V_{\alpha_k} = \bigcup_{k=1}^n U_\alpha \cap X = \Big(\bigcup_{k=1}^n U_\alpha\Big) \cap X \supseteq S \cap X = S,$$

which proves that S is compact in $(X, \tau|_X)$. □

The previous proposition is the reason why some authors only define compactness for entire topological spaces. Anyway, if (X, τ) is a topological space and the entire set X itself is compact in this space, then one often says that "(X, τ) a compact topological space", or that "(X, τ) is compact".[2]

Let us briefly mention that, in contrast to compactness, relative compactness is *not* an intrinsic property:

Example 4.1.8 (Relative compactness is not an intrinsic property). Endow $\mathbb{R}$ with the Euclidean topology on $(0, 1)$ with the subspace topology. Then

[2]We should, however, mention that the notion "compact topological space" is used in a somewhat ambiguous way in the literature: some authors define it in them same as we do in this manuscript, while some other authors would call such spaces *quasi-compact* rather than *compact*, and reserve the notion *compact topological space* for spaces which are, in addition, Hausdorff. One reason for this is that the property "compactness" is more well-behaved in Hausdorff spaces, as follows, for instance, from Proposition 4.1.9(a).

$(0,1)$ is relatively compact in $\mathbb{R}$ (since its closure in $\mathbb{R}$ is the compact set $[0,1]$), but not relatively compact in $(0,1)$ (since its closure in $(0,1)$ is the set $(0,1)$, which is not compact).

It might be a bit surprising at first glance that compact subsets of a topological space need not be closed, in general. But there are very simple examples to demonstrate this: for instance, if we endow $\mathbb{N}$ with the co-finite topology, then every subsets of $\mathbb{N}$ is compact (Example 4.1.4). However, only the finite sets and $\mathbb{N}$ itself are closed.

The following proposition gives more information about the question when compact sets are closed:

Proposition 4.1.9 (Closedness of compact sets). *Let (X,τ) be a topological space. The following implications hold:*

(a) *If (X,τ) is Hausdorff, then every compact subset of X is closed.*

(b) *If every compact subset of X is closed, then (X,τ) is $\mathbf{T}_1$.*

Proof. (a) Let our space (X,τ) be Hausdorff and let $S \subseteq X$ be compact. Let $x \in S^c$. It suffices to show that there exists an open set V in X such that $x \in V \subseteq S^c$.

For each $s \in S$ we have $s \neq x$, so – due to the Hausdorffness – there exist disjoint open sets U_s and V_s in X such that $s \in U_s$ and $x \in V_s$. The sets U_s cover S, so the compactness of S implies that finitely many of the, say $U_{s_1}, \dots, U_{s_n}$, suffices to cover S. Hence, the set $V := V_{s_1} \cap \dots \cap V_{s_n}$ is open, contains x and does not intersect S.

(b) Since every finite subset of X is compact, it follows from the assumptions that every finite subset of X – and hence, in particular, every singleton in X – is closed. Thus, (X,τ) is $\mathbf{T}_1$ (Proposition 1.5.3). □

The converses of both implications in the previous proposition fail, in general:

(a) The space $\mathbb{R}$ with the co-countable topology is not Hausdorff, but all its compact subsets are finite (Example 4.1.5) and thus closed.

(b) The space $\mathbb{N}$ with the co-finite topology is $\mathbf{T}_1$, but, as discussed before the preceding proposition, there are non-closed compact sets in $\mathbb{N}$.

Next, we discuss under which conditions subsets of compact sets are compact, too:

Proposition 4.1.10 (Subsets of compact sets). *Let (X,τ) be a topological space and let $S \subseteq X$ be compact. Let $R \subseteq S$ and assume that R is closed in X or, more generally, closed in S with respect to the subspace topology $\tau|_S$.*

Then R is compact, too.

Note that if (X,τ) is Hausdorff in the previous proposition, then a compact subset S of X is closed (Proposition 4.1.9(a)), which means that $R \subseteq S$ is then closed in X if and only if it is closed in S with respect to the subspace topology $\tau|_S$.[3]

Proof of Proposition 4.1.10. The assumption implies in any case that R is closed in S with respect to the subspace topology $\tau|_S$. Since compactness does not depend on the surrounding space (Proposition 4.1.7), we may this assume that $S = X$ and that R is closed in X.

If $\mathcal{V} \subseteq 2^X$ is a set of open sets in X that covers R, the set $\mathcal{V} \cup \{R^c\}$ consists of open sets only and covers X. As $X = S$ is compact, it can be covered by finitely many sets from $\mathcal{V} \cup \{R^c\}$, so R can be covered by finitely many sets from $\mathcal{V}$. □

Corollary 4.1.11 (Subsets of relatively compact sets). *Let (X,τ) be a topological space and let $R \subseteq S \subseteq X$. If S is relatively compact, then so is R.*

Proof. The set $\overline{R}$ is closed and a subset of the compact set $\overline{S}$, so Proposition 4.1.10 shows that $\overline{R}$ is itself compact. □

If (X,τ) is not Hausdorff, one has to be a bit careful when applying the previous proposition since a compact set need not be relatively compact in a non-Hausdorff space.[4]

We close this section with the following analogue of Proposition 4.1.6 for relatively compact sets:

Proposition 4.1.12 (Continuous images of relatively compact sets). *Consider topological space (X_1,τ_1) and (X_2,τ_2) and let $f : X_1 \to X_2$ be continuous. Assume that (X_2,τ_2) is Hausdorff. If $S_1 \subseteq X_1$ is relatively compact, then so is $f(S_1)$.*

Proof. We have $f(S_1) \subseteq f(\overline{S_1})$, and the latter set is compact in X_2 according to Proposition 4.1.6. Since (X_2,τ_2) is Hausdorff, it thus follows that $f(\overline{S_1})$ is closed (Proposition 4.1.9(a)), and thus relatively compact. Hence, $f(S_1)$ is relatively compact by Corollary 4.1.11. □

4.2 Characterisations of compact sets

In this section, we give various characterisations of the compactness of a given set.

We start with a simple reformulation of the definition of compactness by means of de Morgan's law:

[3] You should check whether you can prove this claimed equivalence in detail.

[4] While, in a Hausdorff space, every compact set is closed by Proposition 4.1.9(a) and thus relatively compact.

Proposition 4.2.1 (Compactness via the finite intersection property). *Let (X,τ) be a topological space and let $S \subseteq X$. The following assertions are equivalent:*

(i) *The set S is compact.*

(ii) *Whenever $\mathcal{D} \subseteq 2^X$ is a non-empty set of closed sets such that $(\cap\mathcal{F}) \cap S \neq \emptyset$ for each non-empty finite $\mathcal{F} \subseteq \mathcal{D}$, then $(\cap\mathcal{D}) \cap S \neq \emptyset$.*

Proof. "(i) $\Rightarrow$ (ii)" Assume that $(\cap\mathcal{D}) \cap S$ is empty. Then the complements of the sets in $\mathcal{D}$ cover S, so by the compactness of S there exist sets $D_1, \dots, D_n \in \mathcal{D}$ such that ${D_1}^{\mathrm{c}} \cup \cdots \cup {D_n}^{\mathrm{c}} \supseteq S$. Consequently, $D_1 \cap \cdots \cap D_n \subseteq S^{\mathrm{c}}$, which shows that (ii) fails.

"(ii) $\Rightarrow$ (i)" As for the previous implication, this is just an application of de Morgan's law. $\square$

A simple consequence of the previous proposition is the following: if S is a compact set in a topological space and we have a sequence $C_1 \supseteq C_2 \supseteq C_3 \dots$ of non-empty closed subsets of S, then the intersection $\bigcap_{n\in\mathbb{N}} C_n$ is non-empty. In fact the same is true if we consider the more general case of a chain $\mathcal{C}$ of non-empty closed subsets of S. What is, maybe, a bit more surprising at first glance is the fact that this "chain property" can also be used to *characterize* compactness:

Theorem 4.2.2 (Compactness via monotone intersections). *Let (X,τ) be a topological space and let $S \subseteq X$. The following assertions are equivalent:*

(i) *The set S is compact.*

(ii) *Whenever $\mathcal{C} \subseteq 2^X$ is a non-empty chain*[5] *of closed sets such that $C \cap S \neq \emptyset$ for each $C \in \mathcal{C}$, then $(\cap\mathcal{C}) \cap S \neq \emptyset$.*

Proof. "(i) $\Rightarrow$ (ii)" This implication is a consequence of Proposition 4.2.1. Indeed, let $\mathcal{C} \subseteq 2^X$ be a non-empty chain of closed sets such that each set in $\mathcal{C}$ intersects S. We apply Proposition 4.2.1(ii) to $\mathcal{D} := \mathcal{C}$: if $\mathcal{F} \subseteq \mathcal{C}$ is finite and non-empty, then $(\cap F) \cap S = C \cap S \neq \emptyset$ for the smallest $C \in \mathcal{F}$. Hence, $(\cap\mathcal{C}) \cap S \neq \emptyset$ by the compactness of S and by Proposition 4.2.1.

"(ii) $\Rightarrow$ (i)" This implication is more involved. We pose it as a bonus problem on Problem Sheet 7. $\square$

The final result in this section is a characterization of compactness by means of net convergence; it is reminiscent of the characterisation of compact sets in metric spaces by means of convergent sequences (compare Example 4.1.2 above as well as Section 6.3 below).

[5] With respect to set inclusion.

Theorem 4.2.3 (Characterisation of compact sets via net convergence). *Let (X, τ) be a topological space and let $S \subseteq X$. The following assertions are equivalent:*

(i) *The set S is compact.*

(ii) *Every net in S has a subnet that converges to a point in S.*

(iii) *Every universal net in S converges to a point in S.*

Proof. "(i) $\Rightarrow$ (iii)" Let $(x_i)_{i\in I}$ be a universal net in S. Assume towards a contradiction that this net does not converge to any point $x \in S$. Then, by the definition of convergence, we can find for each point $x \in S$ a neighbourhood $N_x \subseteq X$ such that $(x_i)_{i\in I}$ is not eventually in N_x. By replacing N_x with $(N_x)^{\mathrm{o}}$ if necessary, we may assume that each N_x is open. Since $\bigcup_{x\in S} N_x \supseteq \bigcup_{x\in S}\{x\} = S$, it follows from the compactness of X that we can find points $y_1, \dots, y_n$ such that $N_{y_1} \cup \dots \cup N_{y_n} = S$. This contradicts that fact that $(x_i)_{i\in I}$, which is a universal net in S, is not eventually in N_{y_k} for any $k \in \{1, \dots, n\}$ (see Problem 14(a) on Problem Sheet 4).

"(iii) $\Rightarrow$ (ii)" This is clear since every net has a universal subnet (Proposition 2.6.8).

"(ii) $\Rightarrow$ (i)" We use the characterisation of compactness by means of the finite intersection property that we gave in Proposition 4.2.1:

Let $\mathcal{D} \subseteq 2^X$ be a non-empty set of closed sets such that $(\cap\mathcal{F}) \cap S \neq \emptyset$ for each finite subset $\mathcal{F} \subseteq \mathcal{D}$. Let I denote the set of all non-empty finite subsets of $\mathcal{D}$, which is directed by set inclusion. For each $\mathcal{F} \in I$ we choose an element $x_{\mathcal{F}} \in (\cap\mathcal{F}) \cap S$. Then $(x_{\mathcal{F}})_{\mathcal{F}\in I}$ is a net in S which has, by assumption, a subnet $(y_j)_{j\in J}$ that converges to a point $y \in S$. For every set $D \in \mathcal{D}$, the net $(x_{\mathcal{F}})_{\mathcal{F}\in I}$ is eventually in D (namely for $\mathcal{F} \supseteq \{D\}$), so $(y_j)_{j\in J}$ is eventually in D, too. Since D is closed, this implies that $y \in D$. Hence, $y \in \cap\mathcal{D}$, so $(\cap\mathcal{D}) \cap S$ is non-empty. □

Let us briefly mention that we could also prove the equivalence "(i) $\Leftrightarrow$ (ii)" in the theorem above without employing universal nets:

Remark 4.2.4. The implication "(i) $\Rightarrow$ (ii)" in Theorem 4.2.3 can also be shown without employing universal nets:

Indeed, let assume that S is compact and let $(x_i)_{i\in I}$ be a net in S. We have to prove that this net has an accumulation point in S. According to Theorem 2.2.11, it thus suffices to show that the set

$$\bigcap_{i_0\in I} \overline{\{x_i : i \geq i_0\}}$$

is non-empty and intersects S. To this end, let

$$\mathcal{D} = \left\{\overline{\{x_i : i \geq i_0\}} : i_0 \in I\right\}.$$

Then $\mathcal{D}$ is a set of closed subsets of X, and for each finite subset $\mathcal{F} \subseteq \mathcal{D}$ the directedness of I implies that one the element x_i of our net is in each $\cap\mathcal{F}$. Hence, $(\cap\mathcal{F}) \cap S \neq \emptyset$. The compactness of S then implies, by Proposition 4.2.1, that $(\cap\mathcal{D}) \cap S \neq \emptyset$, as claimed.

4.3 Products of compact spaces: Tychonoff's theorem

The main result of this section is a famous theorem of Tychonoff which says that the product space of arbitrarily many compact spaces is again compact:

Theorem 4.3.1 (Tychonoff: Products of compact spaces are compact). *Let A be a non-empty set, and for each $\alpha \in A$, let (X_α, τ_α) be a topological space. We endow the product space $X := \prod_{\alpha \in A} X_\alpha$ with the product topology.*

If each space X_α is compact, then so is X.

Proof. Let $\big((x_{\alpha,i})_{\alpha\in A}\big)_{i\in I}$ be a universal net in X. According to Theorem 4.2.3 it suffices to prove that this net converges in X.

For each $\alpha \in A$, the net $(x_{\alpha,i})_{i\in I}$ in X_α is universal (Proposition 2.6.10); thus, it converges to a point $x_\alpha \in X_\alpha$. But this implies that $\big((x_{\alpha,i})_{\alpha\in A}\big)_{i\in I}$ converges to $(x_\alpha)_{\alpha\in A}$ in X (Example 3.2.10(a)). □

The above proof seems to be very simple; the reason for this is that we have a priori invested quite some work in establishing the theory of universal subnets.[6] Let us explain in a bit more detail how universal nets proved useful in the proof of Tychonoff's theorem:

In Theorem 4.2.3 we gave two different characterisations of compactness of a set by means of nets: In Theorem 4.2.3(ii), compactness is characterised by the fact that each net has a convergent subnet; and in Theorem 4.2.3(iii), compactness is characterised by the fact that each universal net converges. The first of these two characterisations does not employ universal nets; however, let us now discuss what happens if we try to employ this characterisation in the proof of Theorem 4.3.1:

If $\big((x_{\alpha,i})_{\alpha\in A}\big)_{i\in I}$ is an arbitrary net in X then we can, for each index $\alpha \in A$, find a convergent subnet of the net $(x_{\alpha,i})_{i\in I}$ in X_α. However, the choice of this subnet – more precisely, of its index set and of the corresponding co-final mapping – may depend on α! Thus, these convergent subnets of the single "component nets" $(x_{\alpha,i})_{i\in I}$ (for $\alpha \in A$) do not provide us with an immediate way to get the desired convergent subnet of $\big((x_{\alpha,i})_{\alpha\in A}\big)_{i\in I}$ itself. The use of universal nets resolves this issue since it makes it unnecessary to choose any subnets.

[6] So this is a typical example where developing the right terminology and an appropriate theoretical framework does, in a way, "trivialise" a deep theorem.

4.4 Addenda: A closed graph theorem and a continuous inverse theorem

In this section we prove two theorems. The first one relates continuity of a mapping to closedness of its graph.

Recall that, tor two sets X, Y and a mapping $f : X \to Y$, the *graph* of f is the set

$$\mathrm{Gr}(f) := \Big\{\big(x, f(x)\big) : x \in X\Big\} \subseteq X \times Y.$$

We note that a point $(x, y) \in X \times Y$ is in $\mathrm{Gr}(f)$ if and only if $y = f(x)$.

Theorem 4.4.1 (Closed graph theorem). *Let (X, τ_X) and (Y, τ_Y) be topological spaces and let $f : X \to Y$ be a mapping. Endow $X \times Y$ with the product topology.*

(a) *If (Y, τ_Y) is Hausdorff and f is continuous, then the graph $\mathrm{Gr}(f)$ is closed in $X \times Y$.*

(b) *If (Y, τ_Y) is compact and the graph $\mathrm{Gr}(f)$ is closed in $X \times Y$, then f is continuous.*

Proof. (a) Let $\big((x_i, f(x_i))\big)_{i \in I}$ be a net in $\mathrm{Gr}(f)$ that converges to a point $(x, y) \in X \times Y$.

Then $(x_i)_{i \in I}$ converges to x in X and thus, as f is continuous, the net $\big(f(x_i)\big)_{i \in I}$ in Y converges to $f(x)$. But this net converges also to y; as (Y, τ_Y) is Hausdorff, this implies $y = f(y)$, so $(x, y) \in \mathrm{Gr}(f)$.

(b) Let $x \in X$; in order to show that f is continuous at x it suffices, by Theorem 2.6.12, to show that for each universal net $(x_i)_{i \in I}$ in X that converges to x, the net $\big(f(x_i)\big)_{i \in I}$ in Y converges to $f(x)$; so let $(x_i)_{i \in I}$ be such a universal net in X.

Then $\big(f(x_i)\big)_{i \in I}$ is a universal net in Y (Proposition 2.6.10) and thus it converges to a point $y \in Y$ since (Y, τ_Y) is compact. Hence, we have $(x_i, f(x_i)) \xrightarrow{i} (x, y)$ and thus, as $\mathrm{Gr}(f)$ is closed, we have $(x, y) \in \mathrm{Gr}(f)$ and therefore $y = f(x)$. This proves that $\big(f(x_i)\big)_{i \in I}$ does indeed converge to $f(x)$. $\square$

The second theorem says that continuous bijective mappings are automatically homeomorphisms under sufficient compactness and separation conditions on the underlying spaces.

Theorem 4.4.2 (Continuous inverse theorem). *Let (X, τ_X) and (Y, τ_Y) topological spaces; assume that (X, τ_X) is compact and that (Y, τ_Y) is Hausdorff.*

If $f : X \to Y$ is continuous and bijective, then f is a homeomorphism (and hence, both spaces (X, τ_X) and (Y, τ_Y) are compact and Hausdorff).

For the proof, we use the following proposition:

Proposition 4.4.3 (Images of closed sets). *Let (X, τ_X) and (Y, τ_Y) topological spaces; assume that (X, τ_X) is compact and that (Y, τ_Y) is Hausdorff.*

If $f : X \to X$ is continuous, then $f(C)$ is closed for each closed set $C \subseteq X$.

Proof. Let $C \subseteq X$ be closed. Since X is compact, this implies that C is compact, too (Proposition 4.1.10). Hence, $f(C)$ is compact in Y (Proposition 4.1.6) and hence, as (Y, τ_Y) is Hausdorff, the set $f(C)$ is closed (Proposition 4.1.9(a)). □

Proof of Theorem 4.4.2. We have to prove that f^{-1} is continuous, which is equivalent to saying that f maps open sets to open sets.

So let $U \subseteq X$ be open. Then $(f(U))^{\mathrm{c}} = f(U^{\mathrm{c}})$ is closed in Y according to Proposition 4.4.3, and hence, $f(U)$ is open. □

A nice consequence of Theorem 4.4.2 is the following comparison result for topologies:

Corollary 4.4.4 (Compact topologies are minimal among Hausdorff topologies). *Let (X, τ) be a compact topological space. If $\tilde{\tau}$ is Hausdorff topology on X that is coarser than τ, then $\tilde{\tau} = \tau$.*

Proof. Since $\tilde{\tau}$ is coarser than τ, the identity

$$\mathrm{id} : (X, \tau) \to (X, \tilde{\tau})$$

is continuous; by Theorem 4.4.2 it is thus even a homeomorphism, so $\tilde{\tau} = \tau$. □

It is instructive to note that the above corollary fails if we do note assume $\tilde{\tau}$ to be Hausdorff: for instance, endow the interval $[0,1]$ with the Euclidean topology τ; then $([0,1], \tau)$ is a compact space. If $\tilde{\tau}$ denotes the indiscrete topology on $[0,1]$, then $\tilde{\tau}$ is coarser than τ, but not equal to τ. Corollary 4.4.4 is not applicable in this case since $\tilde{\tau}$ is not Hausdorff.

We close this section by a simple application of Theorem 4.4.2: we explicitly describe the quotient space of $[0,1]$ that is obtained by identifying the points 0 and 1:

Example 4.4.5 (Identifying the end points of the unit interval). Endow the interval $[0,1]$ with the Euclidean topology and let $\sim$ be the equivalence relation on $[0,1]$ given by

$$x \sim y \qquad :\Leftrightarrow \qquad \big(x, y \in (0,1) \text{ and } x = y\big) \quad \text{or} \quad x, y \in \{0,1\}.$$

We claim the that quotient space $[0,1]/\sim$ (with the quotient topology) is homeomorphic to the complex unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, which we endow with the relative topology of the Euclidean topology on $\mathbb{C}$.

To see this, consider the function $f : [0,1] \to \mathbb{T}$ which is given by $f(x) = \exp(2\pi i x)$ for each $x \in [0,1]$. Then f is continuous, and for $x, y \in [0,1]$ we have $x \sim y$ iff $f(x) = f(y)$.

Moreover, the mapping f induces a mapping $\tilde{f} : [0,1]/{\sim} \to \mathbb{T}$ such that $f = \tilde{f} \circ q$, where $q : [0,1] \to [0,1]/{\sim}$ is the quotient mapping. The mapping $\tilde{f}$ is continuous since f is so (Example 3.3.4). Moreover, $\tilde{f}$ is bijective since f is surjective.

Finally we note that $[0,1]/{\sim}$ is compact since it is the image of the compact set $[0,1]$ under the continuous mapping q. Hence, $\tilde{f}$ is a continuous bijection from the compact space $[0,1]/{\sim}$ to the Hausdorff space $\mathbb{T}$, so $\tilde{f}$ is a homeomorphism according to Theorem 4.4.2.

4.5 Addenda: Locally compact spaces

A concept slightly more general than that of a compact space is that of a *locally compact space*:

Definition 4.5.1 (Locally compact spaces). A topological space (X, τ) is called *locally compact* if each point $x \in X$ has a compact neighbourhood.

Examples 4.5.2 (Some examples of locally compact spaces).

(a) Every compact topological space is locally compact.

(b) The space $\mathbb{R}^d$ with the Euclidean topology is locally compact.

(c) Let (X, τ) be a locally compact Hausdorff space. Then every closed and every open subset of X is locally compact (with respect to the relative topology).

In particular, every open subset of a compact Hausdorff space is locally compact.

Proof. Assertions (a) and (b) are clear. For a proof of (c) we refer, for instance, to [Mun00, Corollary 29.3 on p. 185]. □

Locally compact spaces occur quite often and many results that hold for compact spaces can, in some way, be extended to locally compact spaces. For more information about the topic we refer, for instance, to [Mun00, Section 29 in Chapter 3].

5 Countability axioms and separability

Opening Questions.

(a) Is there a basis of the Euclidean topology on $\mathbb{R}^d$ that has only countably many elements?

(b) In Analysis you learned how closed sets in metric spaces and continuity of functions between metric spaces can be characterised by means of sequences. Why do we have to consider nets instead of sequences when we are working with general topological spaces?

(c) Consider a metric space (M, d). Can we always find a countable subset $S \subseteq M$ whose closure equals M?

5.1 First countable spaces

We have seen on various occasions that *nets* are, in the setting of general topological space, the right concept to replace *sequences*, which play, for instance, an important role in the theory of metric spaces.

However, there is a class of topological spaces on which sequences are fine for most purposes, namely *first countable spaces*.

In order to define them, we need to the concept of a *neighbourhood basis* of a point:

Definition 5.1.1 (Neighbourhood basis). Let (X, τ) be a topological space and let $x \in X$. A set $\mathcal{B} \subseteq 2^X$ is called a *neighbourhood basis* iff it consists of neighbourhoods of x,[1] and for each neighbourhood N of x there exists a set $B \in \mathcal{B}$ such that $B \subseteq N$.

Here is an exercise in order to make you familiar with the notion *neighbourhood basis*:

Exercise 5.1.2 (Convergence and continuity via neighbourhood basis). Let (X, τ) be a topological space, let $x \in X$, and assume that $\mathcal{B}$ is a neighbourhood basis of x.

(a) Prove that a net $(x_i)_{i \in I}$ in X converges to x if and only if, for each $B \in \mathcal{B}$, the net is eventually in B.

[1] I.e., if it is a subset of the neighbourhood filter of X.

(b) Let $(\tilde{X}, \tilde{\tau})$ be another topological space and let $f : X \to \tilde{X}$. Prove that f is continuous at x if and only if, for each neighbourhood $\tilde{N} \subseteq \tilde{X}$ of $f(x)$, there exists a set $B \in \mathcal{B}$ such that $f(B) \subseteq \tilde{N}$.

Now we can define what we mean be a first countable space:

Definition 5.1.3 (First countable topological spaces). A topological space (X, τ) is said to be *first countable* iff every point $x \in X$ has a neighbourhood basis that consists of at most countably many sets.

Let us discuss a few examples first:

Example 5.1.4 (Metric spaces are first countable). Every metric space (M, d) is first countable.[2] Indeed, let $x \in M$. Then it is easy to see that

$$\left\{\mathrm{B}_{<\frac{1}{n}}(x) : n \in \mathbb{N}\right\}$$

is a neighbourhood basis of x; clearly, this neighbourhood basis is at most countable.

Example 5.1.5 (The Sorgenfrey topology is first countable). The real line endowed with the Sorgenfrey topology (that was introcuded in Problem 6 on Problem Sheet 2) is first countable.

Indeed, let $x \in \mathbb{R}$. Then

$$\left\{\left[x, x + \frac{1}{n}\right) : n \in \mathbb{N}\right\}$$

is a neighbourhood basis of x.

Example 5.1.6 (The co-finite and the co-countable topology). (a) The natural numbers $\mathbb{N}$ endowed with the co-finite topology are first countable.

Indeed, let $x \in \mathbb{N}$. Then

$$\left\{\{n, n+1, n+2, \dots\} \cup \{x\} : n \in \mathbb{N}\right\}$$

is a neighbourhood basis of x.

(b) Let X be an uncountable set, and let τ denote the co-countable topology on X. Then (X, τ) is not first countable; in fact, no point in X has an at most countable neighbourhood basis.

To see this, let $x \in X$, and assume towards a contradiction that $\mathcal{B}$ is an at most countable neighbourhood basis of x. Then we have

$$\{x\} \subseteq \bigcap_{B \in \mathcal{B}} B \subseteq \bigcap_{y \in X \setminus \{x\}} X \setminus \{y\} = \{x\},$$

[2] With respect to the topology induced by d.

so the intersection of all sets in $\mathcal{B}$ equals the singleton $\{x\}$. However, this cannot be true since

$$\bigcap_{B\in\mathcal{B}} B = \left(\bigcup_{B\in\mathcal{B}} B^{\mathrm{c}}\right)^{\mathrm{c}}$$

and since $\bigcup_{B\in\mathcal{B}} B^{\mathrm{c}}$ is at most countable.[3]

In the following series of results we show why, in first countable spaces, it often suffices to work with sequences rather than with nets. In order to prove these results, we need the following simple auxiliary observation:

Lemma 5.1.7 (Existence of a decreasing neighbourhood basis). *Let (X,τ) be a topological space, let $x \in X$, and assume that x has a neighbourhood basis which is at most countable. Then there exists a sequence of neighboorhoods*

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$$

of x such that $\{B_n : n \in \mathbb{N}\}$ is a neighbourhood basis of x.

Proof. Let $C_1, C_2, \dots$ be an enumeration of the elements of $\mathcal{B}$ (where some sets may occur multiple times within the enumeration in case that $\mathcal{B}$ is finite). We set $B_1 := C_1$, and define the sets B_n for $n \geq 2$ recursively by setting

$$B_n := B_{n-1} \cap C_n \qquad \text{for each } n \geq 2.$$

Then each B_n is a neighbourhood of x since the intersection of two neighbourhoods is always a neighbourhood. Moreover, the sequence $B_1, B_2, \dots$ is clearly decreasing with respect to set inclusion, and the set $\{B_n : n \in \mathbb{N}\}$ is a neighbourhood basis of x, for if N is a neighbourhood of x, then there exists an index n such that $C_n \subseteq N$, which implies $B_n \subseteq N$. $\square$

The following result is reminiscent of Theorem 2.1.16, but we now replace nets with sequences.

Theorem 5.1.8 (Closures via sequences in first countable spaces). *Let (X,τ) be a topological space and let $S \subseteq X$. For a fixed point $x \in X$, consider the following assertions:*

(i) *The point x is an element of the closure $\overline{S}$.*

(ii) *There exists a sequence $(x_n)_{n\in\mathbb{N}}$ in S that converges to x.*

[3] By the way, the same argument shows that an uncountable set X endowed with the co-finite topology is not first countable, either.

We always have "(ii) $\Rightarrow$ (i)*"; if the space* (X, τ) *is first countable, then we also have "*(i) $\Rightarrow$ (ii)*".*

Proof. "(ii) $\Rightarrow$ (i)" This is a special case of the same implication in Theorem 2.1.16.

"(i) $\Rightarrow$ (ii)" Now assume that (X, τ) is first countable and let $x \in \overline{S}$. According to Lemma 5.1.7 we can find a neighbourhood basis $\mathcal{B}$ of x that consists of sets

$$B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots .$$

For each $n \in \mathbb{N}$ the set B_n intersects S (Proposition 1.3.4), so we can find an element $x_n \in B_n \cap S$. The sequence $(x_n)_{n\in\mathbb{N}}$ is located in S, and for each $n_0 \in \mathbb{N}$ we have $x_n \in B_{n_0}$ whenever $n \geq n_0$. Thus, the sequence converges to x according to Exercise 5.1.2(a). □

Similarly as in Corollary 2.1.17, the above theorem yields the following description of closed sets:

Corollary 5.1.9 (Closed sets via sequences in first countable spaces). *Let* (X, τ) *be a topological space and let* $C \subseteq X$*. Consider the following assertions:*

(i) *The set* C *is closed.*

(ii) *For every sequence* $(x_n)_{n\in\mathbb{N}}$ *in* C *that converges to a point* $x \in X$*, we also have* $x \in C$*.*

We always have "(i) $\Rightarrow$ (ii)*"; if the space* (X, τ) *is first countable, then we also have "*(ii) $\Rightarrow$ (i)*".*

Proof. "(i) $\Rightarrow$ (ii)" This is a special case of the same implication in Corollary 2.1.17.

"(ii) $\Rightarrow$ (i)" Now assume that (X, τ) is first countable and let $x \in \overline{C}$; we need to show that $x \in C$. According to Theorem 5.1.8 there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in C that converges to x. Hence, assertion (ii) implies that $x \in C$. □

Next, we also describe continuity in first countable spaces by means of convergent sequences. Recall that, in general spaces, continuity can be described by means of convergent nets (Theorem 2.3.7).

Theorem 5.1.10 (Continuity via sequences in first countable spaces). *Let* (X_1, τ_1) *and* (X_2, τ_2) *be topological spaces and let* $f : X_1 \to X_2$ *be a function. For a fixed point* $x \in X_1$*, consider the following assertions:*

(i) *The function* f *is continuous at* x*.*

(ii) *For each sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X_1 *that converges to* x*, the sequence* $\big(f(x_n)\big)_{n\in\mathbb{N}}$ *in* X_2 *converges to* $f(x)$*.*

We always have "(i) $\Rightarrow$ (ii)*"; if the space* (X_1, τ_1) *is first countable, then we also have "*(ii) $\Rightarrow$ (i)*"*

Proof. We pose the proof as an exercise on one of the problem sheets. □

In a general topological space, the accumulation points of a net $(x_i)_{i\in I}$ are defined to be the limits of the subnets of $(x_i)_{i\in I}$ (Definition 2.2.10). Even if $(x_i)_{i\in I}$ is a sequence, we have to consider subnets of this sequence – rather than only subsequences – in order to get all accumulation points. However, the situation is simpler in first countable spaces, as the following theorem shows:

Theorem 5.1.11 (Accumulation points of sequences). *Let* $(x_n)_{n\in\mathbb{N}}$ *be a sequence in a topological space* (X, τ) *and let* $x \in X$*. Consider the following assertions:*

(i) *The point* x *is an accumulation point of the sequence* $(x_n)_{n\in\mathbb{N}}$

(ii) *There exists a subsequence of* $(x_n)_{n\in\mathbb{N}}$ *that converges to* x*.*

We always have "(ii) $\Rightarrow$ (i)*"; if this space* (X, τ) *is first countable, then we also have "*(i) $\Rightarrow$ (ii)*".*

The proof is not particularly difficult, but due to time constraints we do not present it in the lecture; instead, we postpone it to the addenda in Section 5.4.

A nice corollary of the previous theorem is the following necessary condition for a set in a first countable space to be compact:

Corollary 5.1.12 (Compactness in first countable spaces). *Let* (X, τ) *be a topological space which is first countable. If* $S \subseteq X$ *is compact, then every sequence in* S *has a subsequence that converges to a point in* S*.*[4]

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S. It follows from Theorem 4.2.3(ii) that this sequence has a subnet that converges to a point in S, i.e., the sequence has an accumulation point x in S. But according to Theorem 5.1.11 this implies that $(x_n)_{n\in\mathbb{N}}$ even has a subsequence that converges to x. □

The converse implication in Corollary 5.1.12 does not hold, in general; for a counterexample, we refer to Example 5.4.1 in the addenda in Section 5.4.

[4] Sets S with this property are often called *sequentially compact*.

5.2 Second countable spaces

In this section we discuss another type of countability axiom, namely so-called *second countability*:

Definition 5.2.1 (Second countable topological space). A topological space (X,τ) is called *second countable* if there exists a basis $\mathcal{B}$ of τ that is at most countable.

Let us first note that each second countable space is also first countable:

Proposition 5.2.2 (Every second countable space is first countable). *If a topological space (X,τ) is second countable, then it is also first countable.*

Proof. Let $\mathcal{B}$ be a an at most countable basis of τ and let $x \in X$. Then it follows from the description of open sets by means of basis sets that we gave in Corollary 3.1.4(ii), that the at most countable set

$$\left\{B \in \mathcal{B} : x \in B\right\}$$

is a neighbourhood basis of x. □

The next example presents a space which is second countable; the subsequent example then gives a space which is first countable, but not second countable (which shows, in conjunction with Proposition 5.2.2, that the concept of second countability is stronger than first countability).

Example 5.2.3 (The real line with the Euclidean topology is second countable). Let τ denote the Euclidean topology on $\mathbb{R}$. Then $(\mathbb{R},\tau)$ is second countable since the countable set

$$\left\{(a,b) : a,b \in \mathbb{Q}\right\}$$

is a basis of τ.

Example 5.2.4 (The Sorgenfrey topology is not second countable). Let τ be the Sorgenfrey topology on $\mathbb{R}$. Then $(\mathbb{R},\tau)$ is not second countable.

Indeed, let $\mathcal{B}$ be a basis of τ; we have to show that $\mathcal{B}$ is uncountable. For each $x \in \mathbb{R}$ the set $[x,x+1)$ is open, so – by the description of open sets by means of basis sets in Corollary 3.1.4(ii) – there exists a basis set $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq [x,x+1)$. This shows that $x = \min B_x$, so the mapping

$$\mathbb{R} \ni x \mapsto B_x \in \mathcal{B}$$

is injective. Hence, $\mathcal{B}$ is indeed uncountable.

Subsets of second countable spaces have the following property, which will probably remind you of the definition of compactness:

Proposition 5.2.5 (Open covers of second countable spaces). *Let (X,τ) be a second countable topological space and let $S \subseteq X$.*

If $\mathcal{V} \subseteq \tau$ covers S, then there exists an at most countable subset $\mathcal{C} \subseteq \mathcal{V}$ that covers S.[5]

Proof. Let $\mathcal{B}$ be an at most countable basis of X and assume that $\mathcal{V} \subseteq \tau$ is such that $\cup\mathcal{V} \supseteq S$.

Now, let $\tilde{\mathcal{B}} \subseteq \mathcal{B}$ denote the set of those $B \in \mathcal{B}$ which are contained in at least one set from $\mathcal{V}$; for each $B \in \tilde{\mathcal{B}}$ we can then choose a set $V_B \in \mathcal{V}$ such that $B \subseteq V_B$.

We claim that the at most countable set $\{V_B : B \in \tilde{\mathcal{B}}\}$ covers S. Indeed, let $x \in S$. Then we can find a set $V \in \mathcal{V}$ that contains x. Hence, by the description of open sets by means of basis sets in Corollary 3.1.4(ii), there exists a set $B_0 \in \mathcal{B}$ such that $x \in B_0 \subseteq V$. Clearly, the set B_0 is in $\tilde{\mathcal{B}}$, so we conclude that

$$x \in B_0 \subseteq V_{B_0} \subseteq \bigcup_{B \in \tilde{\mathcal{B}}} V_B;$$

this proves the claim. □

In second countable spaces, the converse implication of Corollary 5.1.12 is also true:

Theorem 5.2.6 (Compactness vs. sequential compactness). *Let (X,τ) be a topological space which is second countable. For each set $S \subseteq X$ the following assertions are equivalent:*

(i) *The set S is compact.*

(ii) *The set S is* sequentially compact, *i.e., each sequence in S has a subsequence that converges to a point in S.*

Proof. "(i) ⇒ (ii)" As shown in Corollary 5.1.12, this implication holds even in first countable spaces.

"(ii) ⇒ (i)" Let $\mathcal{V} \subseteq \tau$ be a cover of S and assume towards a contradiction that no finite subset of $\mathcal{V}$ covers S.

The second countability of (X,τ) implies, as shown in Proposition 5.2.5, that we can find a countable subset $\mathcal{C} \subseteq \mathcal{V}$ that covers S; let $V_1, V_2, V_3, \dots$ be an enumeration of the elements of $\mathcal{C}$. Since no finite subset of $\mathcal{C}$ covers S we can find, for each $n \in \mathbb{N}$, a point $x_n \in S$ that is not in $V_1 \cup \dots \cup V_n$.

[5]In particular, this is true for $S = X$, a property which is sometimes described by saying that (X,τ) is a *Lindelöf space*.

By assumption, the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ that converges to a point $x \in S$. Since $\cup\mathcal{C} \supseteq S$, there exists an index $k \in \mathbb{N}$ such that $x \in V_k$. The set V_k is open and thus a neighbourhood of x, so the sequence $(x_{n_j})_{j\in\mathbb{N}}$ is eventually in V_k.

So we can find an index j_1 such that $x_{n_{j_1}} \in V_k$ and $n_{j_1} \geq k$. This is a contradiction since $x_{n_{j_1}}$ is not in the set

$$V_1 \cup \cdots \cup V_{n_{j_1}} \supseteq V_k.$$

Hence, S can indeed be covered by a finite subset of $\mathcal{V}$. $\square$

5.3 Separable spaces

In this section we discuss one more notion of "smallness" of a topological space that is defined in terms of a certain countability condition. As a preparation, let us clarify what it means for a set to be *dense* in another set.

Definition 5.3.1 (Denseness). Let (X,τ) be a topological space and let $R \subseteq S \subseteq X$. The set R is said to be *dense* in S iff $\overline{R} \supseteq S$.

In particular, the set $R \subseteq X$ is dense in X iff $\overline{R} = X$.

Here is the notion of "smallness" that we mentioned at the beginning of the section:

Definition 5.3.2 (Separable topological spaces). A topological space (X,τ) is called *separable*[6] iff there exists an at most countable set $R \subseteq X$ which is dense in X.

In the following theorem and the subsequent example we clarify the relation between separability and second countability.

Theorem 5.3.3 (Separability vs. second countability). (a) *If a topological space (X,τ) is second countable, then it is separable.*

(b) *A metric space (M,d) is second countable if and only if it is separable.*

For a proof of the theorem, we refer again to the Addenda in Section 5.4. The following example shows, among other things, that the converse of Theorem 5.3.3(a) does not hold, in general.

Example 5.3.4 (The Sorgenfrey topology is separable, but not second countable and not metrizable). Endow $\mathbb{R}$ with the Sorgenfrey topology τ. Then:

[6] This word sounds very similar to the word "separation" in the notion *separation axiom*, but this is a mere coincidence. There is no relation between separability and separations axioms.

(a) We know from Problem 6(e) on Problem Sheet 2 that $\mathbb{Q}$ is dense in $\mathbb{R}$, so $(\mathbb{R}, \tau)$ is separable.

(b) We know from Example 5.2.4 that $(\mathbb{R}, \tau)$ is not second countable; in particular, the converse of Theorem 5.3.3(a) does not hold, in general.

(c) In particular, it follows from Theorem 5.3.3(b) that there is no metric on $\mathbb{R}$ that induces that Sorgenfrey topology.[7]

Finally we note that it is easy to find metric spaces which are not separable (and hence not second countable according to Theorem 5.3.3(b)):

Example 5.3.5 (A non-separable metric space). Let M be an uncountable and let d denote the discrete metric on M. The induced topology is the discrete topology, so it follows that each subset S of M is closed. Hence, there is no at most countable subset of M whose closure equals M, i.e., M is not separable.

5.4 Addenda: Skipped proofs and examples

In this section we present several contents that we skipped in the lecture, namely a proof of Theorem 5.1.11, a counterexample to the converse implication in Corollary 5.1.12, and a proof of Theorem 5.3.3. Let us start with the proof of Theorem 5.1.11:

Proof of Theorem 5.1.11. "(ii) $\Rightarrow$ (i)" This is an immediate consequence of the definition of the notion *accumulation point* (Definition 2.2.10).

"(i) $\Rightarrow$ (ii)" Let x be an accumulation point of the sequence $(x_n)_{n\in\mathbb{N}}$. According to the description of accumulation points in Theorem 2.2.11 this means that x is an element of the set

$$\bigcap_{k\in\mathbb{N}} \overline{\{x_n : n \geq k\}}.$$

We now use Lemma 5.1.7: its says that we can find a neighbourhood basis $\mathcal{B}$ of x that consists of sets $B_1 \supseteq B_2 \supseteq \dots$.

For all $j, k \in \mathbb{N}$ the neighbourhood B_j intersects the set $\{x_n : n \geq k\}$ since x is in the closure of the latter set. So we can recursively define a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ in the following way:

First we choose an index $n_1 \geq 1$ such that x_{n_1} is in B_1. And as soon as we have chosen n_j for some $j \in \mathbb{N}$, we choose an index $n_{j+1} \geq n_j + 1$ such that $x_{n_{j+1}} \in B_{j+1}$.

The subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ does indeed converge to x: for each $j_0 \in \mathbb{N}$ and each $j \geq j_0$ we have $x_{n_j} \in B_j \subseteq B_{j_0}$, so the convergence to x follows from Exercise 5.1.2(a). $\square$

[7] We could rephrase this by saying that "the Sorgenfrey topology on $\mathbb{R}$ is not *metrizable*."

Next we give an example of a first countable but non-compact space (X,τ) such that each sequence in X has a convergent subsequence. This shows that the converse implication in Corollary 5.1.12 does not hold, in general (for $S = X$ in the corollary).

Example 5.4.1. (Sequential compactness vs. compactness) Let Ω denote the first uncountable ordinal number, which we identify with the set of all ordinals numbers that are smaller than Ω. Let τ denote the order topology on Ω. Then the following assertions are satisfied:

(a) The space (Ω,τ) is first countable.

(b) The set Ω is not compact.

(c) The set Ω is *sequentially compact*, i.e., every sequence in Ω has a subsequence that converges to a point in Ω.

Proof. (a) Let $x \in \Omega$. As Ω is not bounded above, there exists a point $y \in \Omega$ such that $y > x$. Moreover, the interval $[0,y]$ is at most countable, so the set

$$\Big\{(a,b) : a < x < b \le y\Big\}$$

is an at most countable neighbourhood basis of x if $x \neq 0$, and the set

$$\Big\{[0,b) : 0 < b \le y\Big\}$$

is an at most countable neighbourhood basis of x if $x = 0$.

(b) The set Ω is not compact since $(x)_{x\in\Omega}$ is a net in Ω that has no convergent subnet.

(c) We first note that Ω is conditionally order complete (see Problem 20(b) on Problem Sheet 5 for a definition of this notion) since it is well-ordered.

Now, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in Ω. Clearly, the sequence is bounded below by 0, and it is also bounded above since every countable subset of Ω is bounded above. We can find a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ which is decreasing or increasing. This subsequence converges to its infimum or supremum, respectively. □

In the following we present a proof of Theorem 5.3.3:

Proof of Theorem 5.3.3. (a) Let $\mathcal{B}$ be an at most countable basis of τ; we may assume that $\emptyset \notin \mathcal{B}$. For each $B \in \mathcal{B}$, we can thus chose an element $x_B \in B$. Then the set

$$C := \{x_B : B \in \mathcal{B}\}$$

is at most countable, and we claim that it is dense in X.

Indeed, let $x \in X$; we have to show that x is in the closure of C. So let $N \subseteq X$ be a neighbourhood of x. Then there exists a set $B \in \mathcal{B}$ such that $x \in B \subseteq N^{\circ}$ (Corollary 3.1.4(ii)). Since the element x_B of C satisfies

$$x_B \in B \subseteq N^{\circ} \subseteq N,$$

we conclude that the neighbourhood N of x intersects C. Thus, $x \in \overline{C}$.

(b) Let (M, d) be a metric space.

"$\Rightarrow$" We have already proved this implication for general topological spaces in (a).

"$\Leftarrow$" Let $C \subseteq M$ be at most countable and dense. We consider the at most countable set

$$\mathcal{B} := \left\{ \mathrm{B}_{<\frac{1}{n}}(c) : c \in C,\ n \in \mathbb{N} \right\}$$

of open balls and claim that it is a basis of τ. To this end, it suffices to prove – see Corollary 3.1.4(iii) – that each open set $U \subseteq X$ is the union of all sets in $\mathcal{B}$ that are subsets of U. So let $U \subseteq X$ be open and fix $x \in U$. We have to find an element $c \in C$ and a number $n \in \mathbb{N}$ such that $x \in \mathrm{B}_{<\frac{1}{n}}(c) \subseteq U$.

By the definition of the topology induced by the metric d (see Proposition 1.4.10 and Definition 1.4.11) there exists a number $\varepsilon > 0$ such that $\mathrm{B}_{<\varepsilon}(x) \subseteq U$.

Let us now consider the smaller open ball with radius $\varepsilon/2$ and center x: As x is in the closure of C, we know that C intersects $\mathrm{B}_{<\varepsilon/2}(x)$. Hence, there exists $c \in C \cap \mathrm{B}_{<\varepsilon/2}(x)$. Thus we conclude that

$$x \in \mathrm{B}_{<\varepsilon/2}(c) \subseteq \mathrm{B}_{<\varepsilon}(x) \subseteq U,$$

where the first inclusion follows from the triangle inequality. So if we choose $n \in \mathbb{N}$ sufficiently large such that $\frac{1}{n} \leq \frac{\varepsilon}{2}$, we conclude that, indeed,

$$x \in \mathrm{B}_{<\frac{1}{n}}(c) \subseteq U.$$

So we showed that the at most countable set $\mathcal{B}$ is indeed a basis of the topology induced by d. □

5.5 Addenda: Stability with respect to standard constructions

Inheritance by subspaces

First and second countability are inherited by subspaces:

Proposition 5.5.1 (First and second countability of subspaces). *Let (X, τ) be a topological space and let $S \subseteq X$ be endowed with the subspace topology $\tau|_S$.*

(a) *If* (X,τ) *is first countable, then so is* $(S,\tau|_S)$*.*

(b) *If* (X,τ) *is first countable, then so is* $(S,\tau|_S)$*.*

Proof. This follows from the description of the open sets in $(S,\tau|_S)$ in Proposition 3.2.14(a). □

For separability, the situation is more subtle. Let us first note that, in metric spaces, separability is also inherited by subspaces:

Proposition 5.5.2 (Separability of subspaces of metric spaces). *Let* (M,d) *be a metric space and let* $S \subseteq M$ *be endowed with the subspace topology.*[8] *If* (M,d) *is separable, then so is* S*.*

Proof. The subspace topology on S is also induced by a metric (Exercise 3.2.15). Moreover, a metric space is separable if and only if it is second countable (Theorem 5.3.3(b)). Hence, the claim follows from the fact that second countability is inherited by subspaces (Proposition 5.5.1(b)). □

Alternatively, one can also give a direct proof of Proposition 5.5.2.

For general topological spaces, separability is not inherited by subspaces. For instance, endow $\mathbb{R}$ with the Sorgenfrey topology and consider the product space $\mathbb{R}^2$. This space is separable, but the anti-diagonal

$$\{(x,-x) \in \mathbb{R}^2 : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$$

is not separable with respect to the subspace topology. For details, we refer for instance to [SjS70, Section 84 on p. 103].

Inheritance by countable products

Product spaces of at most countably many spaces inherit first and second countability, as well as separability:

Proposition 5.5.3 (Countability properties of product spaces). *Let A be a non-empty and at most countable set, and for each $\alpha \in A$ let (X_α,τ_α) be a topological space. We endow the product $X := \prod_{\alpha\in A} X_\alpha$ with the product topology τ.*

(a) *If each space* (X_α,τ_α) *is first countable, then so is* (X,τ)*.*

(b) *If each space* (X_α,τ_α) *is second countable, then so is* (X,τ)*.*

(c) *If each space* (X_α,τ_α) *is separable, then so is* (X,τ)*.*

[8] Of the topology induced by the metric d.

Proof. You have proved this result in Problem 33 on Problem Sheet 8. $\square$

Uncountable products do not preserve first or second countability or separability, in general. As a counterexample, endow $\{0,1\}$ with the discrete topology and consider the product space $\{0,1\}^{\mathbb{R}}$ (compare Problem 34 on Problem Sheet 8).

6 Metric spaces

Opening Questions.

(a) What is a Cauchy sequence in a metric space? Do you have any idea how we could also define *Cauchy nets* in metric spaces?

(b) How "large" can a compact metric space be?

6.1 Countability and separability in metric spaces

Let us note that we have already seen in the previous chapters that metric spaces satisfy various additional properties compared to general topological spaces. Let us sum up in the following list what we already know:

(a) Every metric space is Hausdorff (Proposition 1.5.2). Hence, limits of nets (and thus, in particular, of sequences) are unique in metric spaces.[1]

(b) We have seen in Theorem 5.3.3(b) that a metric space is second countable if and only if it is separable.

(c) Metric spaces are first countable (Example 5.1.4). Thus, all the results from Section 5.1, which show that various topological properties can – in first countable spaces – be described by means of sequences, apply to metric spaces.

For the sake of completeness, we some up the last mentioned results in the special case of metric spaces in the following theorem; most likely, you already know many (or all) of these results from your Analysis courses:

Theorem 6.1.1. *Let* (M, d) *be a metric space.*

(a) *A point* $x \in M$ *is in the closure of a given set* $S \subseteq M$ *if and only if there exists a sequence in* S *that converges to* x.

(b) *A set* $C \subseteq M$ *is closed if and only if for each sequence* $(x_n)_{n\in\mathbb{N}}$ *in* C *that converges to a point* $x \in M$, *the limit* x *is in* C.

(c) *A mapping* f *from* M *to a topological space* (Y, τ_Y) *is continuous at a point* $x \in M$ *if and only if we have* $f(x_n) \to f(x)$ *for each sequence* $(x_n)_{n\in\mathbb{N}}$ *that converges to* x.

[1] On the other hand, there exist Hausdorff spaces whose topology is not induced by a metric. A simple example is the real line with the Sorgenfrey topology, which is Hausdorff according to Problem 6(d) in Problem Sheet 2, but not induced by a metric according to Example 5.3.4(c)

(d) *A point $x \in M$ is an accumulation point of a sequence $(x_n)_{n\in\mathbb{N}}$ if and only if $(x_n)_{n\in\mathbb{N}}$ has a subsequence that converges to x.*

Recall that all assertions of the previous theorem remain true if we replace the metric space (M,d) with the more general notion of a first countable topological space; in this more general setting, we have proved these assertions in Theorem 5.1.8, Corollary 5.1.9, Theorem 5.1.10 and Theorem 5.1.11.

6.2 Complete metric spaces

A particularly important concept for metric spaces is *completeness* which is, in analysis, typically defined by means of *Cauchy sequences*. Similarly, we can also define *Cauchy nets*:

Definition 6.2.1 (Cauchy nets and Cauchy sequences). Let (M,d) be a metric space.

(a) A net $(x_i)_{i\in I}$ is called a *Cauchy net* iff for each $\varepsilon > 0$ there exists an index $i_0 \in I$ such that $\mathrm{d}(x_i,x_j) < \varepsilon$ for all $i,j \geq i_0$.

(b) A sequence $(x_n)_{n\in\mathbb{N}}$ is called a *Cauchy sequences* iff it is a Cauchy net.

Cauchy nets (and sequences) can be characterised in various simple way:

Exercise 6.2.2 (Characterisations of Cauchy nets). Let $(x_i)_{i\in I}$ be a net in a metric space (M,d). Show that the following assertions are equivalent:

(i) The net $(x_i)_{i\in I}$ is a Cauchy net.

(ii) For each $\varepsilon > 0$ there exist an index $i_0 \in I$ such that $\mathrm{d}(x_i,x_{i_0}) < \varepsilon$ for all $i \geq i_0$.

(iii) The net $\big(\mathrm{d}(x_i,x_j)\big)_{(i,j)\in I\times I}$ – where $I \times I$ is endowed with the product direction[2] – in $[0,\infty)$ converges to 0.[3]

In Analysis courses, a metric space is typically defined to be *complete* iff each Cauchy sequence converges. Given the prevalence of nets in topology it is, on the other hand, tempting to call a metric space complete iff every Cauchy *net* converges. Fortunately, it does not matter which of these two definitions we choose; this is the content of the following proposition:

Proposition 6.2.3 (Net completeness vs. sequential completeness). *For a metric space (M,d), the following assertions are equivalent:*

[2]The product direction was discussed in detail in Proposition 2.1.11

[3]With respect to the Euclidean topology.

(a) *Every Cauchy net in M converges.*

(b) *Every Cauchy sequences in M converges.*

Proof. "(a) $\Rightarrow$ (b)" This implication is obvious since every Cauchy sequence is a Cauchy net.

"(b) $\Rightarrow$ (a)" Assume that every Cauchy sequence in M converges, and let $(x_i)_{i\in I}$ be a Cauchy net in M.

We recursively construct a sequence of indices $(i_n)_{n\in\mathbb{N}}$ in I in the following way:

- We choose $i_1 \in I$ such that $\mathrm{d}(x_i, x_{i_1}) < 1$ for all $i \geq i_1$.
- Assume that i_n has already been chosen. Then we choose $i_{n+1} \in I$ such that $i_{n+1} \geq i_n$ and such that $\mathrm{d}(x_i, x_{i_{n+1}}) < \frac{1}{n+1}$ for all $i \geq i_{n+1}$; note that such an index i_{n+1} exists since $(x_i)_{i\in I}$ is a Cauchy net and since I is directed.

The sequence $(x_{i_n})_{n\in\mathbb{N}}$ is clearly a Cauchy sequence in M and thus, by assumption, it converges to a point $x \in M$.

However, we have to be a bit careful now since $(x_{i_n})_{n\in\mathbb{N}}$ need not be a subnet of $(x_i)_{i\in I}$.[4] Anyway, we can still show that the net $(x_i)_{i\in I}$ converges to x:

So let $\varepsilon > 0$; choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. For all $n \geq k$ we have $\mathrm{d}(x_{i_k}, x_{i_n}) < \varepsilon$. Since the sequence $(x_{i_n})_{n\in\mathbb{N}}$ converges to x and since the metric d is continuous (Example 3.2.6), it follows that

$$\mathrm{d}(x_{i_k}, x) = \lim_{n\to\infty} \mathrm{d}(x_{i_k}, x_{i_n}) \leq \varepsilon.$$

Hence, we conclude for all $i \geq i_k$ that

$$\mathrm{d}(x_i, x) \leq \mathrm{d}(x_i, x_{i_k}) + \mathrm{d}(x_{i_k}, x) < \varepsilon + \varepsilon = 2\varepsilon.$$

This proves, as claimed, that the net $(x_i)_{i\in I}$ converges to x. □

The preceding proposition allows us to define completeness of metric spaces without specifying whether we care for Cauchy nets, or for Cauchy sequences only:

Definition 6.2.4 (Completeness of metric spaces). A metric space (M, d) is called *complete* iff it satisfies the equivalent assertions (a) and (b) of Proposition 6.2.3.

[4] Recall that there are nets, no subnet of which is a sequence; see for instance Problem 11(b) on Problem Sheet 3.

The following exercise can be solved arguments that are quite similar to that in the proof of Proposition 6.2.3:

Exercise 6.2.5 (Cauchy nets with convergent subnets). Let $(x_i)_{i\in I}$ be a Cauchy net in a metric space (M,d) and assume that $(x_i)_{i\in I}$ has a subnet that converges to a point $x\in M$. Show that $(x_i)_{i\in I}$ converges to x, too.

The above exercise in particular shows that a Cauchy sequence converges to a point $x\in M$ if it has a subsequence (or, more generally, a subnet) that converges x.

Closedness of subspaces in complete metric spaces can be characterised in terms of completeness:

Proposition 6.2.6 (Closedness of subsets via completeness). *Let (M,d) be a metric space and let $S\subseteq M$. Consider the following two assertions:*

(a) *The set S is closed in M.*

(b) *The metric space $(S,\mathrm{d}|_{S\times S})$ is complete.*

We always have "(b) $\Rightarrow$ (a)". If the space (M,d) is complete, then the converse implication "(a) $\Rightarrow$ (b)" holds, too.

Proof. "(b) $\Rightarrow$ (a)" Assume that $(S,\mathrm{d}|_{S\times S})$ is complete.

Let $(x_i)_{i\in I}$ be a net in S that converges to a point $x\in M$. Then this net is a Cauchy net in $(S,\mathrm{d}|_{S\times S})$, so, by the completeness of the latter space, it converges to a point $y\in S$. But thus, it also converges to y with respect to the metric d on M; since every metric space is Hausdorff, this implies $x=y$, so $y\in S$. This proves that S is closed in M.

"(a) $\Rightarrow$ (b)" Assume now that (M,d) is complete and that S is closed in M.

Let $(x_i)_{i\in I}$ be a Cauchy net in $(S,\mathrm{d}|_{S\times S})$. Then this net is also Cauchy in (M,d), so it converges to a point $x\in M$. The closedness of S in M implies $x\in S$; hence, $(x_i)_{i\in I}$ does also converge in the space $(S,\mathrm{d}|_{S\times S})$, which shows that $(S,\mathrm{d}|_{S\times S})$ is complete. $\square$

It is important to note the completeness is not a *topological invariant*, i.e., it can happen that a complete metric space is homeomorphic to a non-complete metric space. In fact, it is not difficult to construct a complete metric d_1 and a non-complete metric d_2 on $\mathbb{R}$ which both induce the same topology[5]:

Exercise 6.2.7 (Completeness is not a topological invariant). Let d_1 denote the Euclidean metric on $\mathbb{R}$, i.e., $\mathrm{d}_1(x,y)=\left|y-x\right|$ for all $x,y\in\mathbb{R}$; this metric is well-known to be complete.

[5]Such that the identity mapping is an homeomorphism between $(\mathbb{R},\mathrm{d}_1)$ and $(\mathbb{R},\mathrm{d}_2)$

For all $x, y \in \mathbb{R}$ we now define

$$\mathrm{d}_2(x,y) := \frac{\mathrm{d}_1(x,y)}{1+\mathrm{d}_1(x,y)}.$$

Show that d_2 is a metric on $\mathbb{R}$ which induces the same topology as d_1, and that the metric space $(\mathbb{R}, \mathrm{d}_2)$ is not complete.

The previous example shows, in particular, that a topology can be induced by several metrics, even if these metrics are quite different.

If the topology on a topological space (X, τ) is induced by a complete metric, this has certain topological consequences which are – in contrast to the completeness of the metric itself – topologically invariant. We will discuss an example of such a property in Section 8.3 in the context of one of Baire's category theorems.

If one is primarily interested in a topology τ on such a space X, but would like to have the existence of a complete metric that induces τ – for instance for the reason that was just mentioned –, then it makes sense to talk about *complete metrizability* of τ:

Definition 6.2.8 (Metrizable and completely metrizable spaces). A topological space (X, τ) is said to be...

(a) ...*metrizable* iff there exists a metric d on X which induces the topology τ.

(b) ...*completely metrizable* iff there exists a metric d on X which induces the topology τ and which has the property that (X, d) is complete.

Example 6.2.9 (Polish spaces). A topological space (X, τ) is called a *Polish space* iff it is separable and completely metrizable.[6]

Polish spaces play a significant role in parts of Stochastic Analysis.

6.3 Compactness in metric spaces

The purpose of this section is to give a characterisation of compact sets in metric spaces. Let us first recall the notion of *boundedness* (which will, however, turn out to be too weak for the characterisation of compact sets) in a metric space:

Definition 6.3.1 (Bounded sets in metric spaces). A subset S of a metric space (M, d) is called *bounded* iff it is empty or if there exists a point $x \in M$ and a real number $r \in [0, \infty)$ such that $S \subseteq \mathrm{B}_{\le r}(x)$.

[6] Recall that, for metrizable spaces, separability is equivalent to second countability (Theorem 5.3.3(b)).

Exercise 6.3.2 (The reference point in the definition of boundedness). Let (M,d) be a metric space and let $S \subseteq M$ be non-empty. Show that the following assertions are equivalent:

(i) The set S is bounded.

(ii) For all $x \in M$ there exists a real number $r \in [0,\infty)$ such that $S \subseteq \mathrm{B}_{\le r}(x)$.

(iii) There exists a point $x \in S$ and a real number $r \in [0,\infty)$ such that $S \subseteq \mathrm{B}_{\le r}(x)$.

Let us now turn back to the description of compact sets. As you now, a subset of $\mathbb{R}^d$ is compact if and only if it is closed and bounded.

In more general metric spaces, this is not true. A very simple counterexample is $\mathbb{Q}$ with the Euclidean metric (e.g., the set $[0,1] \cap \mathbb{Q}$ in this space is closed and bounded, but not compact); however, this example is a bit of a red herring since $\mathbb{Q}$ with the Euclidean metric is not complete. However, even for complete metric spaces, a closed and bounded subset need not be compact. Here is a simple counterexample:

Example 6.3.3 (Closedness and boundedness in complete metric spaces does not imply compactness). Let M be an infinite set and let d denote that discrete metric on M. Then the metric spaces (M,d) is complete, the set M is obviously closed in M and the set M is bounded. However, M is not compact.

It turns out the what we actually need to characterise compactness is a stronger notion than boundedness, namely the concept of *total boundedness*:

Definition 6.3.4 (Totally bounded sets in metric spaces). A subset S of a metric space (M,d) is called *totally bounded*[7] iff it is empty or if the following property holds: for each $\varepsilon > 0$ there exist finitely many points $x_1,\dots,x_n \in M$ such that

$$S \subseteq \mathrm{B}_{\le\varepsilon}(x_1) \cup \dots \cup \mathrm{B}_{\le\varepsilon}(x_n).$$

It does actually not matter whether the centers $x_1,\dots,x_n$ in the definition of total boundedness are allowed to be arbitrary points in M or whether they are required to be from S; this is a simple consequence of the triangle inequality. We write it down explicitly in the following proposition, but we omit the details of the (straightforward) proof.

Proposition 6.3.5 (A characterisation of totally bounded sets). *For a non-empty subset S of a metric space (M,d), the following assertions are equivalent:*

[7] Some authors use the terminology *pre-compact* instead of *totally bounded*.

(a) *The set S is totally bounded.*

(b) *For each $\varepsilon > 0$ there exist finitely many points $x_1, \dots, x_n \in S$ such that*

$$S \subseteq \mathrm{B}_{\leq \varepsilon}(x_1) \cup \dots \cup \mathrm{B}_{\leq \varepsilon}(x_n).$$

The point in the second assertion above is that the $x_1, \dots, x_n$ are now elements of S and not only of M. The previous proposition can be rephrased – by taking also the case $S = \emptyset$ into account – in the following way:

Corollary 6.3.6 (Total boundedness is an intrinsic property). *Let (M, d) be a metric space and let $S \subseteq N \subseteq M$. The following assertions are equivalent:*

(i) *The set S is totally bounded in (M, d).*

(ii) *The set S is totally bounded in $(N, \mathrm{d}|_{N \times N})$.*

(iii) *The set S is totally bounded in $(S, \mathrm{d}|_{S \times S})$.*

Let us note that every totally bounded set in a metric is separable with respect to the subspace topology:

Proposition 6.3.7 (Totally bounded sets are separable). *Let (M, d) be a metric space and let $S \subseteq M$. If S is totally bounded, then $(S, \mathrm{d}|_{S \times S})$ is separable, and thus second countable.*

Proof. For each $n \in \mathbb{N}$ we can find, due to the total boundedness of S, points $x_1^{(n)}, \dots, x_{k_n}^{(n)} \in S$ such that the open balls around those points with radius $\frac{1}{n}$ cover S. This implies that the at most countable set

$$\bigcup_{n \in \mathbb{N}} \left\{ x_1^{(n)}, \dots, x_{k_n}^{(n)} \right\}$$

is dense in S; so S is indeed separable.

For metric spaces, we already know that separability is equivalent to second countability (Theorem 5.3.3(b)). □

Another way to characterise totally bounded sets, which already indicates its close relation to compactness, is as follows:

Lemma 6.3.8 (Totally bounded sets and Cauchy sequences). *For a subset S of a metric space (M, d) the following assertions are equivalent:*

(i) *The set S is totally bounded.*

(ii) *Every sequence in S has a subsequence which is Cauchy.*

Proof. We may, and will, assume that S is non-empty.

"(i) $\Rightarrow$ (ii)" Let S be totally bounded and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S. Consider the set $Y := \{x_n : n \in \mathbb{N}\}$ of all elements of the sequence $(x_n)_{n\in\mathbb{N}}$. If Y has only finitely many elements, than x_n has a subsequence that is constant, so there is nothing to prove. So assume now that Y is infinite.

We construct a decreasing (with respect to set inclusion) sequence $(Y_n)_{n\in\mathbb{N}_0}$ of infinite subsets $Y_n \subseteq Y$ as follows. We start with $Y_0 := Y$. If Y_n has already been constructed, we cover S with finitely many balls with radius $\frac{1}{n+1}$ (and with centers in, say, S); this is possible due to the total boundedness of S. At least one of these balls has infinite intersection with Y_n (since Y_n is infinite), and we defined this intersection to be Y_{n+1}.

Since each set Y_n is an infinite subset of $Y = \{x_n : n \in \mathbb{N}\}$, we can now find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $x_{n_k} \in Y_k$ for each $k \in \mathbb{N}$. Since the sets Y_k are decreasing with respect to set inclusion and since for $k \in \mathbb{N}$ each set Y_k is, by construction, contained in a ball of radius $\frac{1}{k}$, it follows that $(x_{n_k})_{k\in\mathbb{N}}$ is a Cauchy sequence.

"(ii) $\Rightarrow$ (i)" Let $\varepsilon > 0$ and assume for a contradiction that S cannot be covered by finitely many balls of radius ε and with centers in S. We then recursively choose a sequence $(x_n)_{n\in\mathbb{N}}$ as follows:

Let x_1 be an arbitrary element of S. If $x_1, \dots, x_n$ have already been chosen, then the union

$$\mathrm{B}_{\le\varepsilon}(x_1) \cup \dots \cup \mathrm{B}_{\le\varepsilon}(x_n)$$

does not contain all of S by assumption. Hence, we can choose a point $x_{n+1} \in S$ which is not in this union. Within the sequence $(x_n)_{n\in\mathbb{N}}$, any two elements with distinct indices have distance more than ε, so not subsequence is a Cauchy sequence. □

Now we can give a characterisation of compactness, both in terms of completeness and total boundedness and in terms of sequential completeness:

Theorem 6.3.9 (Compactness is equivalent to completeness and total boundedness). *For a subset S of a metric space (M, d) the following assertions are equivalent:*

(i) *The set S is compact.*

(ii) *The set S is sequentially compact, i.e., each sequence in S has a subsequence that converges to a point in S.*

(iii) *The set S is totally bounded and the metric space $(S, \mathrm{d}|_{S\times S})$ is complete.*

Recall that we have already noted in Theorem 5.2.6 that a subset of a second countable topological space is compact if and only if it is sequentially compact. However, in the above theorem we did not make any a priori assumption on second countability.

Proof of Theorem 6.3.9. "(i) $\Rightarrow$ (ii)" Since every metric space is first countable, compactness of S implies sequential compactness (Corollary 5.1.12).

"(ii) $\Rightarrow$ (iii)" Since S is sequentially complete, every sequence in S has a subsequence that converges and that is thus, in particular, a Cauchy sequence. As shown in Lemma 6.3.8 this implies that S is totally bounded.

To show completeness of S, let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in S. According to (ii), this sequence has a subsequence that converges to a point $x \in S$. But since $(x_n)_{n\in\mathbb{N}}$ is Cauchy, this already implies that the sequence itself converges to x (Exercise 6.2.5).

"(iii) $\Rightarrow$ (i)" We already know that total boundedness of S implies that $(S, \mathrm{d}|_{S\times S})$ is second-countable (Proposition 6.3.7). Hence, it suffices to show sequential compactness of S rather than compactness (Theorem 5.2.6).

So let $(x_n)_{n\in\mathbb{N}}$ be a sequence in S. By the total boundedness of S, this sequence has a subsequence which is Cauchy (Lemma 6.3.8), and due to the completeness of S, the subsequence converges to a point in S. Thus, S is indeed sequentially compact. □

If the surrounding space is compact, the preceding theorem yields the following corollary:

Corollary 6.3.10 (Characterisation of compact sets in complete spaces). *Let* (M, d) *be a complete metric space. For each set* $S \subseteq M$ *the following assertions are equivalent:*

(i) *The set* S *is compact.*

(ii) *The set* S *is totally bounded and closed.*

Proof. This is simply the equivalence of (i) and (iii) in Theorem 6.3.9 since – by the completeness of (M, d) – completeness of $(S, \mathrm{d}|_{S\times S})$ is the same as closedness of S (Proposition 6.2.6). □

7 Extension of continuous functions

Opening Questions.

(a) Consider a continuous function $f : [0,1] \to \mathbb{R}$. Is there always a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ that extends f?

(b) Consider a continuous function $f : (0,1) \to \mathbb{R}$. Is there always a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ that extends f?

(c) Let $\emptyset \neq A, B \subseteq [-1,1]$ be closed and disjoint. Can you find a continuous function $f : [-1,1] \to \mathbb{R}$ which is 0 on A and 1 on B?

7.1 Normal spaces and Urysohn's lemma

In this subsection we discuss a separation property which is stronger than the properties $\mathbf{T}_0$–$\mathbf{T}_2$ that you already know. Let us start with the following definition:

Definition 7.1.1 (Normal spaces). A topological space (X,τ) is called *normal* iff for each pair of disjoint closed sets $C_0, C_1 \subseteq X$ there exists a pair of disjoint open sets $U_0, U_1 \subseteq X$ such that $C_0 \subseteq U_0$ and $C_1 \subseteq U_1$.

For normal spaces, the separation axioms $\mathbf{T}_1$ and $\mathbf{T}_2$ are equivalent:

Exercise 7.1.2 (Normal $\mathbf{T}_1$-spaces are Hausdorff). Let (X,τ) be a normal topological space. Prove that X is $\mathbf{T}_1$ if and only if it is Hausdorff.

Normal Hausdorff spaces have their own name:

Definition 7.1.3 ($\mathbf{T}_4$-spaces). We say that a topological space (X,τ) satisfies the *separation axiom $\boldsymbol{T}_4$*, or is a *$\boldsymbol{T}_4$-space*, iff it is Hausdorff and normal (equivalently, $\mathbf{T}_1$ and normal).

We note in passing that there exists also a notion of $\mathbf{T}_3$-spaces, but we do not treat them in this lecture.

Here are is an example of a class of $\mathbf{T}_4$-spaces:

Example 7.1.4 (Every compact Hausdorff space is $\mathbf{T}_4$). Every compact Hausdorff space (X,τ) is a $\mathbf{T}_4$-space.

Proof. We have to show that (X,τ) is normal, and to this end, we proceed in two steps:

Step 1: Let $C_0 \subseteq X$ be closed and let $c_1 \in X$ such that $c_1 \notin C_0$. We show that there exists disjoint open sets $U_0, U_1 \subseteq X$ such that $C_0 \subseteq U_0$ and $c_1 \in U_1$:

Since our space is Hausdorff we can, for each $x \in C_0$, find open sets V_x and W_x in X such that $V_x \cap W_x = \emptyset$ as well as $x \in V_x$ and $c_1 \in W_x$. Then $\{V_x : x \in C_0\}$ is an open cover of C_0. Since C_0 is closed and X is compact, C_0 is compact, too. Hence, there exist finitely many points $x_1, \dots, x_n \in C_0$ such that $U_0 := V_{x_1} \cup \dots \cup V_{x_n} \supseteq C_0$.

On the other hand, the set $U_1 := W_{x_1} \cap \dots \cap W_{x_n}$ is open, contains c_1 and does not intersect U_0.

Step 2: Let $C_0, C_1 \subseteq X$ be closed and disjoint. We have to find disjoint open sets U_0, U_1 which contain C_0 and C_1, respectively.

For each $x \in C_1$ we can, according to Step 1, find open sets V_x, W_x in X such that $V_x \cap W_x = \emptyset$ as well as $C_0 \subseteq V_x$ and $x \in W_x$. Now we use that C_1 is compact: since $\{W_x : x \in C_1\}$ is an open cover of C_1, there exist finitely many points $x_1, \dots, x_n \in C_1$ such that $U_1 := W_{x_1} \cup \dots \cup W_{x_n} \supseteq C_1$.

On the other hand, the set $U_0 := V_{x_1} \cap \dots \cap V_{x_n}$ is open, contains C_0 and does not intersect U_1. □

The main result in this section characterises normal spaces by the property that disjoint closed sets can be separated by real-valued continuous functions:

Lemma 7.1.5 (Urysohn's lemma). *For a topological space (X, τ) the following assertions are equivalent:*

(i) *The space (X, τ) is normal.*

(ii) *For every pair of disjoint closed sets C_0 and C_1 in X there exists a continuous function $f : X \to [0,1]$ which is 0 on C_0 and 1 on C_1:*

Proof. "(ii) ⇒ (i)" Let $C_0, C_1 \subseteq X$ be disjoint closed sets. Let $f : X \to [0,1]$ be as in (ii). Then the sets

$$U_0 := f^{-1}\Big([0, \frac{1}{2})\Big) \qquad \text{and} \qquad U_1 := f^{-1}\Big((\frac{1}{2}, 1]\Big)$$

are disjoint and open in X, and they contain C_0 and C_1, respectively.

"(i) ⇒ (ii)" The proof of this implication is more involved. We pose its steps as the parts of a bonus exercise on Problem Sheet 9. □

7.2 Tietze's extension theorem

As a consequence of Urysohn's lemma, we now obtain the following extension theorem for continuous real-valued functions:

Theorem 7.2.1 (Tietze's extension theorem). *For a topological space (X, τ) the following assertions are equivalent:*

(i) *The space (X,τ) is normal.*

(ii) *For every closed set $C \subseteq X$ and every continuous function*[1] *$f : C \to \mathbb{R}$ there exists a continuous function $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}|_C = f$.*

For the proof, we use the following lemma as well as the o bservation from the subsequent exercise.

Lemma 7.2.2 (Approximate extension of a continuous function). *Let (X,τ) be a normal topological space and let $C \subseteq X$ be closed. Let $r \in (0,\infty)$ be a real number and let $f : C \to [-r,r]$ be continuous, where $[-r,r]$ is endowed with the Euclidean topology.*

Then there exists a continuous mapping $h : X \to [-\frac{r}{3},\frac{r}{3}]$ such that $|h(x)-f(x)| \leq \frac{2}{3}r$ for all $x \in C$.

Proof. Consider the disjoint subsets

$$C_0 := f^{-1}([-r,-\frac{r}{3}]) \qquad \text{and} \qquad C_1 := f^{-1}([\frac{r}{3},r])$$

of C. They are closed in C and hence in X. By Urysohn's lemma there exists a continuous function $h : X \to [-\frac{r}{3},\frac{r}{3}]$ such that $h(x) = -\frac{r}{3}$ on C_0 and $h(x) = \frac{r}{3}$ on C_1. Clearly, h has the property claimed in the lemma. ☐

Exercise 7.2.3 (Absolutely convergent series of continuous functions). Let (X,τ) be a topological space and, for each $n \in \mathbb{N}$, let $h_n : X \to \mathbb{R}$ be a continuous function. Assume that $\sum_{n=1}^{\infty} \sup_{x\in X} |h_n(x)| < \infty$. Show that the function

$$h : X \to \mathbb{R}, \qquad x \mapsto \sum_{n=1}^{\infty} h_n(x)$$

is well-defined and continuous.

Now we can prove Tietze's extension theorem:

Proof of Theorem 7.2.1. "(ii) $\Rightarrow$ (i)" We can argue similarly as in the the proof of the implication "(ii) $\Rightarrow$ (i)" of Urysohn's lemma: let C_0 and C_1 be closed and disjoint sets. We set $C := C_0 \cup C_1$ and define a function $f : C \to \mathbb{R}$ by setting

$$f(x) := \begin{cases} 0 & \text{if} \quad x \in C_0, \\ 1 & \text{if} \quad x \in C_1. \end{cases}$$

This function is well-defined since C_0 and C_1 are disjoint, and by using that C_0 and C_1 are closed, it is not difficult to check that f is continuous. Hence,

[1] Continuous with respect to the subspace topology on C.

the disjoint sets $U_0 := f^{-1}\Big((-\infty, \frac{1}{2})\Big)$ and $U_1 := f^{-1}\Big((\frac{1}{2}, \infty)\Big)$ are open, and they contain C_0 and C_1, respectively.

"(i) $\Rightarrow$ (ii)" We first assume that there exists a number $c > 0$ such that the range of f is contained in $[-c, c]$.

We inductively construct a sequence of continuous functions

$$h_n : X \to \Big(\frac{2}{3}\Big)^{n-1}[-\frac{c}{3}, \frac{c}{3}]$$

such that

$$\left| f(x) - \sum_{k=1}^{n} h_k(x) \right| \le c\Big(\frac{2}{3}\Big)^n \qquad \text{for all } x \in C$$

for each $n \in \mathbb{N}$:

- First choose h_1 by simply applying Lemma 7.2.2 (with $r := c$) to the function f.
- Now assume that $h_1, \dots, h_n$ have already been chosen. We set $f_n := f - \sum_{k=1}^n h_k|_C$. By applying Lemma 7.2.2 (with $r := c\big(\frac{2}{3}\big)^n$) to the function f_n, we get a function h_{n+1} from the lemma which has precisely the required properties.

If we now define $\tilde{f} := \sum_{k=1}^\infty h_k$ as in Exercise 7.2.3, then $\tilde{f}$ is a continuous mapping from X to $\mathbb{R}$, and we have $\tilde{f}|_C = f$. Moreover, a straightforward application of the geometric series yields that the range of $\tilde{f}$ is contained in the interval $[-c, c]$.

This proves the theorem in case that the range of f is bounded. The general case can be reduced to this case; we discuss this reduction in a problem on Problem Sheet 10. ☐

7.3 Extension of continuous functions on metric spaces

In this section we discuss a few results about the extension of continuous functions on metric spaces. We split the section into two non-enumerated subsections:

A more concrete proof of Urysohn's lemma on metric spaces

You already now from Problem 38 on Problem Sheet 9 that every metric space (M, d) is normal, so Urysohn's lemma holds on (M, d). The purpose of this subsection is to show that, on metric spaces, one can in fact explicitly construct the function whose existence is claimed by Urysohn's lemma.

As you might expect from Problem 38 on Sheet 9, the explicit construction is closely related to the distance of a point x in the metric space (M, d) to non-empty subsets S of M. Recall from Problem 38 that this distance is defined as

$$\mathrm{d}(x, S) := \inf\Big\{\mathrm{d}(x,s) : s \in S\Big\}.$$

You proved various useful properties of this distance notion in Problem 38.

Let us now discuss how we can use the distance of points to sets to explicitly construct the function from Urysohn's lemma: let $C_0, C_1 \subseteq M$ be closed and disjoint and, without loss of generality, non-empty. In Problem 38(d) we separated the sets C_0 and C_1 by open sets U_0 and U_1 by defining

$$f : M \ni x \mapsto \mathrm{d}(x, C_0) - \mathrm{d}(x, C_1) \in \mathbb{R} \tag{7.3.1}$$

and setting $U_0 := f^{-1}\big((-\infty, 0)\big)$ and $U_1 := f^{-1}\big((0, \infty)\big)$. This worked since the continuous function f is strictly negative on C_0 and strictly positive on C_1.

Hence, the function f is already quite close to the function whose existence is claimed in part (ii) of Urysohn's lemma 7.1.5. However, the function f does not take constant values on C_0 and C_1, respectively. But this can easily be resolved:

Proposition 7.3.1 (Explicit version of Urysohn's lemma on metric spaces). *Let (M, d) be a metric space and let $C_0, C_1 \subseteq M$ be closed, disjoint and non-empty. Then the function*

$$\begin{aligned} f : M &\to \mathbb{R}, \\ x &\mapsto \frac{\mathrm{d}(x, C_0) - \mathrm{d}(x, C_1)}{\mathrm{d}(x, C_0) + \mathrm{d}(x, C_1)} \end{aligned}$$

is well-defined and has the following properties: it is continuous, its range is contained in $[-1, 1]$, and it takes constantly the value -1 on C_0 and constantly the value 1 on C_1.

Note that the function in this proposition immediately yields a function as in part (ii) of Urysohn's lemma 7.1.5 by a simple translation and rescaling argument: if f is the function from Proposition 7.3.1, the function

$$\frac{f+1}{2}$$

maps to $[0, 1]$ and takes the values 0 and 1 on C_0 and C_1, respectively.

Proof of Proposition 7.3.1. Since no point $x \in M$ is both in C_0 and in C_1, it follows that $\mathrm{d}(x, C_0) + \mathrm{d}(x, C_1)$ is always strictly positive; hence, f is well-defined.

Since the distance function from a fixed set is continuous (Problem 38(b) on Sheet 9), we conclude that f is continuous as a composition of continuous functions. Moreover, the definition of f readily implies that, for all $x \in M$,

$$-1 = \frac{-\mathrm{d}(x, C_0) - \mathrm{d}(x, C_1)}{\mathrm{d}(x, C_0) + \mathrm{d}(x, C_1)} \leq f(x) \leq \frac{\mathrm{d}(x, C_0) + \mathrm{d}(x, C_1)}{\mathrm{d}(x, C_0) + \mathrm{d}(x, C_1)} = 1.$$

Finally, we show that f takes the values -1 and 1 and C_0 and C_1, respectively:

- If $x \in C_0$, it follows that $\mathrm{d}(x, C_0) = 0$ (and $\mathrm{d}(x, C_1) > 0$), so

$$f(x) = \frac{-\mathrm{d}(x, C_1)}{\mathrm{d}(x, C_1)} = -1.$$

- If $x \in C_1$, it follows that $\mathrm{d}(x, C_1) = 0$ (and $\mathrm{d}(x, C_0) > 0$), so

$$f(x) = \frac{\mathrm{d}(x, C_0)}{\mathrm{d}(x, C_0)} = 1.$$

Thus, the proposition is proved. □

Extension of functions defined on non-closed subsets

Tietze's extension theorems makes an assertions about the possibility to extend continuous functions that are defined on a closed subset of a normal topological space. In this subsection we briefly discuss a few results in case that a given function is defined on a non-closed subset. Let us begin with the following result which we formulate in the general setting of topological space:

Theorem 7.3.2 (Extension of continuous functions defined on dense sets). *Let (X, τ_X) and (Y, τ_Y) be topological spaces, where (Y, τ_Y) is a $\boldsymbol{T}_4$-space, let $D \subseteq X$ be a dense subset of X and let $f : D \to Y$ be continuous (with respect to the subspace topology on D). Then the following assertions are equivalent:*

(i) *There exists a continuous mapping $\tilde{f} : X \to Y$ such that $\tilde{f}|_D = f$.*

(ii) *For each $x \in X \setminus D$ there exits an element $y_x \in Y$ such that $f(x_i) \xrightarrow{i} y_x$ for each net $(x_i)_{i \in I}$ in D that converges to x.*

Proof. "(i) ⇒ (ii)" This implication is obvious.

"(ii) ⇒ (i)" Of course, we define $\tilde{f}(x) = f(x)$ for each $x \in D$ and $\tilde{f}(x) = y_x$ for each $x \in X \setminus D$. We have to show that $\tilde{f}$ is continuous, so let $x \in X$.

We note that, for every net $(x_i)_{i \in I}$ in D that converges to x, the net $\big(\tilde{f}(x_i)\big)_{i \in I} = \big(f(x_i)\big)_{i \in I}$ converges to $\tilde{f}(x)$; indeed, for $x \in X \setminus D$ this follows from (ii) and for $x \in D$ this follows from the continuity of f.

Now, fix a universal net $(x_i)_{i\in I}$ in X that converges to x; it suffices to prove that $\big(\tilde{f}(x_i)\big)_{i\in I}$ converges to $\tilde{f}(x)$ (Theorem 2.6.12).[2]

So let $B \subseteq Y$ be an open neighbourhood of $\tilde{f}(x)$. Since the net $\big(\tilde{f}(x_i)\big)_{i\in I}$ is universal (Proposition 2.6.10), it follows that it is either eventually in B or eventually in B^{c}; so assume towards a contradiction that it is eventually in B^{c}. After restricting our attention to a tail of the index set I, we may thus assume that the entire net $\big(\tilde{f}(x_i)\big)_{i\in I}$ is contained in B^{c}.

The space (Y, τ_Y) is assumed to be $\mathbf{T}_4$, i.e., it is Hausdorff and normal. Hence, the singleton $\{f(x)\}$ is a closed set which is disjoint from B^{c} and hence, due to the normality, there exist disjoint open sets $U_0, U_1 \subseteq Y$ which contain $f(x)$ and B^{c}, respectively.

In order to obtain a contradiction, we now construct a new net $(w_j)_{j\in J}$ in X, which is actually located in D, which also converges to x, and which has the property that $f(w_j) \in U_1$ for each $j \in J$. This will be a contradiction because, as $(w_j)_{j\in J}$ is in D, we must have $f(w_j) \xrightarrow{j} f(x)$, i.e., $\big(f(w_j)\big)_{j\in J}$ has to be eventually in the set U_0 – but this set is disjoint to U_1.

To construct the net $(w_j)_{j\in J}$, we proceed as follows: let $\mathcal{N}(x) \subseteq 2^X$ denote the neighbourhood filter of x which we direct, us usual, by converse set inclusion; endow $I \times \mathcal{N}(x)$ with the product order and define

$$J := \{(i, N) \in I \times \mathcal{N}(x) : x_i \in N^{\mathrm{o}}\}$$

(which we endow with the direction inherited from the product direction on $I \times \mathcal{N}(x)$). We note that J is non-empty and directed since $(x_i)_{i\in I}$ converges to x. Moreover, for the same reason, we can find for each $N \in \mathcal{N}(x)$ an index $i \in I$ such that $(i, N) \in J$.

Consider a pair $(i, N) \in J$. Since D is dense in X, there exists a net $(v_h)_{h\in H}$ in D that converges to x_i. Hence, $\big(\tilde{f}(v_h)\big)_{h\in H} = \big(f(v_h)\big)_{h\in H}$ converges to $\tilde{f}(x_i)$. Since N is a neighbourhood of x_i and U_1 is a neighbourhood of $\tilde{f}(x_i)$, we can find an index $h \in H$ such that $v_h \in N$ and $f(v_h) \in U_1$. We set $w_{(i,N)} := v_h$.[3]

Since we have $w_{(i,N)} \in N$ but $f(w_{(i,N)}) \in U_1$, it follows, as claimed, that the net $(w_{(i,N)})_{(i,N)\in J}$ converges to x while the net $\big(f(w_{(i,N)})\big)_{(i,N)\in J}$ does not converge to $f(x)$. Hence, we arrived at a contradiction. □

Remark 7.3.3 ($\mathbf{T}_3$-spaces). For the proof of Theorem 7.3.2 we did not really need that (Y, τ_Y) is a $\mathbf{T}_4$-space. What we really needed is that singletons are closed (i.e., that the space is $\mathbf{T}_1$) and that every closed set C can be separated

[2] The difficulty here is, of course, that the net $(x_i)_{i\in I}$ might not be contained in D.

[3] Note that the choice of the net $(v_h)_{h\in H}$ depends on i and the choice of the index h depends on N, so $w_{(i,N)}$ does indeed depend on both i and N.

from each point that is not contained in C by open sets. Spaces with these properties are called ***T***$_3$*-spaces*[4], which is a weaker property than being $\mathbf{T}_4$.

So Theorem 7.3.2 actually remains true if (Y, τ_Y) is only assumed to be $\mathbf{T}_3$ instead of $\mathbf{T}_4$.

The problem with the characterisation result in Theorem 7.3.2 is that it is not clear how to check that assertion (ii) is satisfied for a given a function f.

In order to obtain a condition which is easier to handle, we go back to the setting of metric spaces and consider the following strengthened continuity property of functions:

Definition 7.3.4 (Uniform continuity). Let (M_1, d_1) and (M_2, d_2) be metric spaces. A function $f : M_1 \to M_2$ is called *uniformly continuous* iff the following condition is satisfied: for each $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\mathrm{d}_1(x,y) < \delta \quad \Rightarrow \quad \mathrm{d}_2(f(x), f(y)) < \varepsilon$$

for all $x, y \in M_1$.

What is "uniform" about the property in the previous definition is that δ can be chosen to be the same number for all points $x, y \in M_1$.

Now we can prove an extension result for continuous functions between metric spaces:

Theorem 7.3.5 (Extension of uniformly continuous functions). *Let* (M_1, d_1) *and* (M_2, d_2) *be metric spaces, and assume that* (M_2, d_2) *is complete. Let* $D \subseteq M_1$ *be a dense subset and let* $f : D \to M_2$.

If f *is uniformly continuous (with respect to the metric* $\mathrm{d}_1|_{D\times D}$ *on* D*), then there exists a continuous function* $\tilde{f} : M_1 \to M_2$ *such that* $\tilde{f}|_D = f$.

Proof. We show that condition (ii) in Theorem 7.3.2 is satisfied; so fix $x \in M_1 \setminus D$.

For each $n \in \mathbb{N}$ there exists, by the definition of uniform continuity, a number $\delta_n > 0$ such that

$$\mathrm{d}_1(y,z) < 2\delta_n \quad \Rightarrow \quad \mathrm{d}_2(f(y), f(z)) < \frac{1}{n}$$

for all $y, z \in D$. Clearly, we can – and will – choose the sequence $(\delta_n)_{n\in\mathbb{N}}$ to be decreasing. The function f maps the set $\mathrm{B}_{<\delta_n}(x) \cap D$ into a subset of M_2 of diameter at most $\frac{1}{n}$.

[4] Note that, while we only assumed $\mathbf{T}_3$-spaces to be $\mathbf{T}_1$, it is easy to see that they are in fact automatically $\mathbf{T}_2$.

We note that the set $\mathrm{B}_{<\delta_n}(x) \cap D$ is always non-empty since D is dense in M_1. So each set in the decreasing sequence

$$f\Big(\mathrm{B}_{<\delta_1}(x) \cap D\Big) \supseteq f\Big(\mathrm{B}_{<\delta_2}(x) \cap D\Big) \supseteq \dots$$

is non-empty. If we choose points $p_1, p_2, \dots$ in those sets, respectively, then it follows that $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and thus it converges to a point $p \in M_2$ (since we assumed (M_2, d_2) to be complete).

For each $n \in \mathbb{N}$, the point p is in the closure of $f\Big(\mathrm{B}_{<\delta_n}(x) \cap D\Big)$ and thus has distance at most $\frac{1}{n}$ to each point in this set. This shows that

$$f\Big(\mathrm{B}_{<\delta_n}(x) \cap D\Big) \subseteq \mathrm{B}_{\le\frac{1}{n}}(p)$$

for each $n \in \mathbb{N}$. From this it readily follows that, for each net $(x_i)_{i\in I}$ in D that converges to x, the net $\big(f(x_i)\big)_{i\in I}$ converges to $f(x)$; so condition (ii) in Theorem 7.3.2 is indeed satisfied, which proves the assertion.[5] □

We note that, for real-valued mappings, one can combine Theorem 7.3.5 with Tietze's extension theorem 7.2.1:

Corollary 7.3.6 (Extension of functions defined on non-closed sets). *Let* (M, d) *be a metric space, let* $S \subseteq M$ *and let* $f : S \to \mathbb{R}$ *be uniformly continuous.*[6] *Then there exists a continuous function* $\tilde{f} : M \to \mathbb{R}$ *such that* $\tilde{f}|_S = f$.

Proof. According to Theorem 7.3.5 we can extend f to a continuous function from $\overline{S}$ to $\mathbb{R}$. Since the metric space (M, d) is normal (Problem 38 on Sheet 9) we can then apply Tietze's extension theorem 7.2.1 in order to further extend the function to a continuous function defined on the entire space M. □

7.4 Addenda: $\mathbf{T}_3$- vs. $\mathbf{T}_4$-spaces

There are various further separation axioms – for instance, there is an axiom $\mathbf{T}_{3\frac{1}{2}}$ which is weaker than $\mathbf{T}_4$ but stronger than $\mathbf{T}_3$.

For more information, for further separation axioms and for counterexamples which show that none of these notions are in fact equivalent, we refer to [SjS70] (not however that the book [SjS70] uses slightly different terminology: the authors do not require $\mathbf{T}_3$-spaces, $\mathbf{T}_4$-spaces, and so on, to be necessarily Hausdorff, which changes the logical dependencies between the various types of spaces a bit).

[5] In order to apply Theorem 7.3.2 we use that the metric space (M_2, d_2) is normal (Problem 38 on Sheet 9) and is thus $\mathbf{T}_4$ (since every metric space is Hausdorff).

[6] Where $\mathbb{R}$ is endowed with the Euclidean metric.

8 Baire category theorems

Opening Questions.

(a) Do you know a continuous function $f : [0,1] \to \mathbb{R}$ that is nowhere differentiable?

(b) Do you know an infinite-dimensional Banach space that has a countable basis?

8.1 Meagre and co-meagre sets

In this chapter we discuss subsets of topological spaces which are, in a sense, topologically *small* – at least under appropriate assumptions on the space.

First we need the following terminology:

Definition 8.1.1 (Nowhere dense sets). A subset S of a topological space (X,τ) is called *nowhere dense* iff the closure $\overline{S}$ has empty interior.

Let us first discuss a few examples:

Examples 8.1.2 (Nowhere dense sets). (a) If (X,τ) is a topological $\mathbf{T}_1$-space and no singleton in X is open, then every finite subset of X is nowhere dense.

Indeed, every finite set $S \subseteq X$ is closed and has finite interior. So it suffices to prove that every finite open set is empty. Assume for a contradiction that $U \subseteq X$ is finite, open and non-empty. By intersecting U with finitely many complements of singletons we then obtain an open singleton, which is a contradiction.

(b) Endow $\mathbb{R}$ with the Euclidean topology. If the closure of a set $S \subseteq \mathbb{R}$ is at most countable, then S is nowhere dense. This follows from the fact that each countable set in $\mathbb{R}$ has empty interior.

(c) Endow $\mathbb{R}$ with the Euclidean topology. The countable set $S := [0,1] \cap \mathbb{Q}$ is not nowhere dense since its closure $[0,1]$ has non-empty interior.

Exercise 8.1.3 (Properties of nowhere dense sets). Let (X,τ) be a topological space and let $S \subseteq X$.

(a) Show that if S is nowhere dense, then so is every subset of S.

(b) Show that S has empty interior if and only if its complement S^c is dense in X.

(c) Show that S is nowhere dense if and only if the set $(S^c)^o = \left(\overline{S}\right)^c$ is dense in X.

Nowhere dense sets are, in general, considered as "small" in a topological sense. But even more, it is often reasonable to even consider countable unions of nowhere dense sets as "small". Therefore, we introduce the following terminology for these sets:

Definition 8.1.4 (Meagre and co-meagre sets). Let (X, τ) be a topological space and let $S \subset X$.

(a) The set S is called *meagre*[1] iff there exists a countable collection $(N_k)_{k\in\mathbb{N}}$ of nowhere dense sets $N_k \subseteq X$ such that $S = \bigcup_{k\in\mathbb{N}} N_k$.

(b) The set S is called *co-meagre*[2] iff its complement S^c is meagre.

Here are a few simple properties of meagre and co-meagre sets:

Proposition 8.1.5 (Properties of meagre and co-meagre sets). *Let (X, τ) be a topological space.*

(a) *The union of at most countably meagre sets in X is again meagre.*

(b) *Every subset of a meagre set is itself meagre.*

(c) *A subset $S \subseteq X$ is meagre if and only if it is contained in a union of at most countably many closed sets with empty interior.*

(d) *The intersection of at most countable many co-meagre sets is again co-meagre.*

(e) *Every superset of a co-meagre set is itself co-meagre.*

(f) *A set $S \subseteq X$ is co-meagre if and only if it contains an intersection of at most countably many open and dense subsets of X.*

Proof. All assertions follow immediately from the definitions. □

As mentioned before, we are interested in spaces where meagre sets are, in a sense, sufficiently small. More precisely, we are interested in spaces where the following equivalent assertions are satisfied:

[1] Meagre sets are sometimes called *sets of first category* in the literature; sets which are not meagre are then said to be *of second category*.

[2] Please note that the property *co-meagre* means something completely different from *being of second category*: S is of second category iff it is not meagre; and S is co-meagre iff its complement is meagre.

Arguably, this is more confusing than helpful, so we will completely avoid the terminology *of first and second category* in this manuscript.

Proposition 8.1.6 (Equivalent descriptions of Baire spaces). *For a topological space (X, τ) the following assertions are equivalent:*

(i) *Every meagre set in X has empty interior.*

(ii) *The only subset of X which is both open and meagre is $\emptyset$.*

(iii) *Every union of at most countably many closed sets in X with empty interior has empty interior.*

(iv) *Every co-meagre set in X is dense.*

(v) *The only subset of X which is both closed and co-meagre is X.*

(vi) *Every intersection of at most countably many open dense sets in X is dense.*

Proof. "(i) $\Leftrightarrow$ (iv)" This equivalence follows from de Morgan's law and from Exercise 8.1.3(b).

"(ii) $\Leftrightarrow$ (v)" This equivalence is obvious by taking complements.

"(iii) $\Leftrightarrow$ (vi)" This equivalence also follows from de Morgan's law and from Exercise 8.1.3(b).

"(i) $\Rightarrow$ (ii)" This implication is obvious.

"(ii) $\Rightarrow$ (i)" If $S \subseteq X$ is meagre, then S° is meagre (Proposition 8.1.5(b)) and open, and thus empty by (ii).

"(i) $\Rightarrow$ (iii)" This implications holds since a union of at most countably many closed sets with non-empty interior is meagre.

"(iii) $\Rightarrow$ (i)" This implications follows from the fact that each meagre set is contained in a union of the form described in (iii) (Proposition 8.1.5(c)). $\square$

Definition 8.1.7 (Baire space). A topological space (X, τ) is called a *Baire space* iff it satisfies the equivalent assertions in Proposition 8.1.6.

It follows from Proposition 8.1.6(iii) that every Baire space has the following property:

Corollary 8.1.8 (Baire spaces as countable unions of closed sets). *Let (X, τ) be a Baire space and assume that $X \neq \emptyset$. Let C_n be a closed subset of X for each $n \in \mathbb{N}$. If $\bigcup_{n\in\mathbb{N}} C_n = X$, then at least one of the sets C_n has non-empty interior.*

It is the purpose of the rest of this section to show that several important classes of topological spaces are actually Baire spaces.

8.2 Compact spaces

The following is the main result of this section:

Theorem 8.2.1 (Compact Hausdorff spaces are Baire). *If (X,τ) is a compact Hausdorff space, then it is a Baire space.*

Proof. For each $n \in \mathbb{N}$, let U_n be an open and dense subset of X. We have to show that $\bigcap_{n\in\mathbb{N}} U_n$ is dense in X (Proposition 8.1.6(vi)). So let $x \in X$ and let N be an open neighbourhood of x. It is our goal to prove that N intersects $\bigcap_{n\in\mathbb{N}} U_n$.

To this end, we construct a decreasing sequence of compact closed sets C_n ($n \in \mathbb{N}$) with non-empty interior and such that $C_n \subseteq N \cap U_n$ for each $n \in \mathbb{N}$.

To find C_1, we note that the open set $N \cap U_1$ is non-empty since U_1 is dense. Hence, there exists a closed set $C_1 \subseteq N \cap U_1$ with non-empty interior; this follows from Bonus Problem 39(a) on Problem Sheet 9 (which is applicable here since every compact Hausdorff space is normal according to Example 7.1.4).

Now assume that $C_1 \supseteq \cdots \supseteq C_n$ have already been chosen. Since C_n° is non-empty and U_{n+1} is dense in X, the open set $C_n^{\circ} \cap U_{n+1}$ is non-empty. So, again by Bonus Problem 39(a) on Problem Sheet 9, there exists a closed set $C_{n+1} \subseteq C_n^{\circ} \cap U_{n+1}$ with non-empty interior.

As we have chosen the decreasing sequence $(C_n)_{n\in\mathbb{N}}$ now, we can intersect all these sets C_n; the intersection is non-empty since each C_n is closed and non-empty and since X is compact. Hence, there exists a point $x \in \bigcap_{n\in\mathbb{N}} C_n$. Obviously, x is both in N and in $\bigcap_{n\in\mathbb{N}} U_n$. □

Let us give a simple application of Theorem 8.2.1. You have already proved the assertion of the following example in Problem 40(b) on Sheet 10; but by using Theorem 8.2.1 the proof becomes much simpler.[3]

Example 8.2.2 (Countable compact T_2-spaces contain isolated points). Let (X,τ) be a compact Hausdorff space. If X is non-empty and at most countable, then there exists an open singleton[4] in X.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be an enumeration of X. Each of the sets $\{x_n\}$ is closed and their union equals X. Since (X,τ) is a Baire space according to Theorem 8.2.1, it follow from Corollary 8.1.8 that at least one of the sets $\{x_n\}$, say $\{x_{n_0}\}$, has non-empty interior. Thus, $\{x_{n_0}\}^{\circ} = \{x_{n_0}\}$, so $\{x_{n_0}\}$ is an open singleton. □

In the next section we will briefly discuss an analogous result to Theorem 8.2.1, but for complete metric spaces rather than for compact spaces.

[3] One could even argue that the proof of Theorem 8.2.1 is a version of the argument that we used to solve Problem 40(b).

[4] If a singleton $\{x\}$ in a topological space (X,τ) is open, then x is sometimes called an *isolated point in X*.

8.3 Complete metric spaces

You know from your Analysis courses that a metric space (M, d) is called *complete* iff every Cauchy sequence in M converges.[5]

Our only result in this section is the following theorem:

Theorem 8.3.1 (Complete metric spaces are Baire). *If (M, d) is a complete metric space, then it is a Baire space.*[6]

The proof is quite reminiscent of the proof of Theorem 8.2.1; we discuss the details in an exercise on Problem Sheet 11.

8.4 Addenda: Baire's theorem on locally compact spaces

In the Addenda in Section 4.5 we briefly mentioned locally compact spaces, which are a generalisation of compact spaces. One can actually show a generalisation of Theorem 8.2.1, namely that every locally compact Hausdorff space is a Baire space. This is, for instance, Exercise 3 in [Mun00, p. 299].

[5] You can find more information about complete metric spaces in Section 6.2.

[6] As usual, we endow M with the topology induced by d.

9 Connected sets

Opening Questions.

(a) Let $U \subseteq \mathbb{R}$ be non-empty and open, and let $f : U \to \mathbb{R}$ be a continuously differentiable function with derivative 0. Does it follow that f is constant?

(b) How does one prove the identity theorem for analytic functions?

(c) Consider the territory of several countries on a map or on a globe. Can you find countries whose territory you would describe as *connected*? Can you also find countries whose territory you would describe as *disconnected*?

9.1 Connected sets

In Problem 12 on Problem Sheet 4 we have briefly discussed *clopen* sets, and its relation to the continuity of certain functions. This is closely related to the concept of so-called *connected spaces* and *connected sets*:

Definition 9.1.1 (Connected spaces and sets). Let (X, τ) be a topological space.

(a) The space (X, τ) is called *connected* iff it cannot be written as the union of two non-empty disjoint open sets.

(b) A subset $S \subseteq X$ is called *connected* iff the space $(S, \tau|_S)$ is connected.

In order to give a characterisation of connected spaces we need the following terminology which we have already mentioned in Problem 12 on Problem Sheet 4:

Definition 9.1.2 (Clopen set). Let (X, τ) be a topological space. A subset of X is called *clopen*[1] iff it is both closed and open.

Note that the empty set and the entire space X are always clopen.

Proposition 9.1.3 (A characterisation of connectedness). *Let (X, τ) be a topological space. The following assertions are equivalent:*

(i) *The space (X, τ) is connected.*

[1] In German, one might call clopen sets "*abgeschloffen*"; but this terminology seems to be less common than the word "*clopen*" in English.

(ii) *There are no clopen sets in X except for $\emptyset$ and X.*

(iii) *Every continuous mapping $f : X \to \{0,1\}$ – where $\{0,1\}$ is endowed with the discrete topology – is constant.*

Proof. "(i) $\Rightarrow$ (ii)" If $\emptyset \neq S \subsetneq X$ is clopen, then X is the union of the disjoint non-empty open sets S and S^c, hence not connected.

"(ii) $\Rightarrow$ (i)" If X is not connected, then we can write X as $X = U \cup V$ for two non-empty disjoint open sets U and V. Hence, U is non-empty, not equal to X, and clopen (since $U = V^c$).

"(i) $\Rightarrow$ (iii)" Since f is continuous, the sets $f^{-1}(\{0\})$ and $f^{-1}(\{1\})$ are open; moreover, they are disjoint and cover X, so one of them is empty. Hence, f is constant.

"(iii) $\Rightarrow$ (i)" If (X,τ) is not connected, then we can write X as $X = U \cup V$ for two non-empty disjoint open sets U and V. Define a function $f : X \to \{0,1\}$ by setting its value to 0 on U and to 1 on V. Then f is not constant, but continuous (since every pre-image of f is one of the sets $\emptyset$, U, V and X). $\square$

Next, let us discuss for a few spaces and sets whether they are connected:

Examples 9.1.4 (A few examples of connected and non-connected sets).

(a) The connected subsets of the real line (with the Euclidean topology) are precisely the intervals. This can be shown similarly as in Problem 12(a) on Problem Sheet 4.

(b) The set $\mathbb{N}$ with the co-finite topology is connected since there do not exist two disjoint non-empty open sets in this space.

(c) Let (X,τ) be a topological space. The empty set is connected and every singleton in X is connected.

(d) Let (X,τ) be a topological space and let $C_0, C_1 \subseteq X$ be two non-empty disjoint closed sets. Then their union $C := C_0 \cup C_1$ is not connected.

Indeed, the set $C_0 = C \cap (X \setminus C_1)$ is clopen in $(C, \tau|_C)$, but it is non-empty and not equal to C.

(e) Let (X,τ) be a topological space and let $U_0, U_1 \subseteq X$ be two non-empty disjoint open sets. Then their union $U := U_0 \cup U_1$ is not connected.

This can be seen by a similar argument as before.

We close this section with the following very useful result:

Proposition 9.1.5 (Continuous images of connected sets are connected). *Let (X,τ_X) and (Y,τ_Y) be topological spaces. If (X,τ_X) is connected and $f : X \to Y$ is continuous, then $f(X)$ is connected.*

Proof. Let $S \subseteq f(X)$ be clopen with respect to the subspace topology. Then there exists an open set $U \subseteq Y$ and a closed set $C \subseteq Y$ such that $S = U \cap f(X)$ and $S = C \cap f(X)$.

The set $f^{-1}(S)$ is equal to $f^{-1}(U)$ and thus open; it is also equal to $f^{-1}(C)$ and thus closed. So $f^{-1}(S)$ is clopen and hence – as (X, τ_X) is connected – it is either empty or equal to X. In the first case, $S = \emptyset$ and in the latter case, $S = f(X)$ (to conclude these two facts, one needs to use that $S \subseteq f(X)$). $\square$

On Problem Sheet 12, you will see how Proposition 9.1.5 can be used to find further examples of connected spaces.

9.2 Connected components

In this section we consider so-called *connected components* of topological spaces. In order to analyse them, the following proposition is quite useful:

Proposition 9.2.1 (Unions of connected sets). *Let (X, τ) be a topological space. Let A be a non-empty set and for each $\alpha \in A$, let $S_\alpha \subseteq X$ be connected. If $\bigcap_{\alpha \in A} S_\alpha \neq \emptyset$, then $S := \bigcup_{\alpha \in A} S_\alpha$ is connected.*

Proof. Throughout the proof, fix a point $x_0 \in \bigcap_{\alpha \in A} S_\alpha$ and a set $M \subseteq S$ which is clopen with respect to $\tau|_S$ We assume that M is non-empty, and so we have to prove that $M = S$.

Note that, for each $\alpha \in A$, we have $(\tau|_S)|_{S_\alpha} = \tau|_{S_\alpha}$. Thus, for each α, the set $M \cap S_\alpha$ is clopen in S_α with respect to $\tau|_{S_\alpha}$; so, for each α, we have $M \cap S_\alpha \in \{\emptyset, S_\alpha\}$.

Since M is non-empty, it intersects at least one of the sets S_α, say S_{α_0}. Then $M \cap S_{\alpha_0} = S_{\alpha_0}$, so $x_0 \in M$. Hence, M intersects each of the sets S_α, so $M \cap S_\alpha = S_\alpha$ for each α. This proves that $M = S$. $\square$

Next we define what it means for two points in a topological space to be *connected*:

Definition 9.2.2 (Connected points). Let (X, τ) be a topological space. Two points $x, y \in X$ are said to be *connected* iff there exists a connected subset $S \subseteq X$ that contains both x and y.

Proposition 9.2.3 (Connectedness of points is an equivalence relation). *Let (X, τ) be a a topological space. The connectedness of points in X, as defined in Definition 9.2.2, is an equivalence relation on X.*

Proof. Reflexivity: Reflexivity is clear since every singleton is connected.

Symmetry: Symmetry is obvious by the definition of the connectedness of two points.

Transitivity: Let $x, y, z \in X$ and assume that x and y are connected and that y and z are connected. Then there exist connected sets $S_1, S_2 \subseteq X$ such that $x, y \in S_1$ and $y, z \in S_2$. Since y is located in both S_1 and S_2, it follows from Proposition 9.2.1 that $S_1 \cup S_2$ is connected. As this union contains both x and z, it follows that x and z are connected. □

Now we can finally define what we mean by the *connect components* of a topological space, which we have mentioned at the beginning of the section.

Definition 9.2.4 (Connected components). Let (X, τ) be a topological space. The equivalence classes of the equivalence relation "connectedness" are called the *connected components* of (X, τ).

The *connected component* of a point $x \in X$ is the connected component of (X, τ) that contains x.

Proposition 9.2.5 (Characterisation of connected components). *Let (X, τ) be a topological space and let $x \in X$.*

(a) *The connected component of x is the union of all connected sets in X that contains x.*

(b) *The connected component of x is the largest connected set in X that contains x.*

In particular, every connected component of (X, τ) is connected.

Proof. (a) A point $y \in X$ is in the connected component of x if and only if it is connected to x if and only if there exists a connected set in X that contains both y and x if and only if y is in the union of all connected subsets of X that contains x.

(b) It follows from (a) and from Proposition 9.2.1 that the connected component of x is connected; by again applying (a) we see that it is even the largest connected set in X that contains x. □

Examples 9.2.6 (Examples of connected components). (a) A topological space (X, τ) is connected if and only if X is a connected component of (X, τ).

(b) If a set X is endowed with the discrete topology, then the connected components of X are precisely the singletons.

(c) Consider the set $X = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, endowed with the Euclidean topology. Then the only non-empty connected sets in X are the singletons, so the connected components in X are precisely the singletons. This shows, in particular, the following two things:

- It can happen that the connected components in a topological space are precisely the singletons[2], even if the topology is not discrete.
- The connected components in a topological space need not be open, in general (since the set $\{0\}$ is not open in this example).

[2] A topological space with this property is called *totally disconnected*.

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