Universität Ulm

Elements of Functional Analysis

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Contents

1 Normed spaces and linear operators

Opening Questions. (a) Let $f, g : [0,1] \to \mathbb{R}$ be two functions. How can we measure the "distance" of f and g?

(b) Let $d \in \mathbb{N}$ and $A \in \mathbb{C}^{d \times d}$. Recall that the matrix exponential function $\mathbb{R} \ni t \mapsto e^{tA} \in \mathbb{C}^{d\times d}$ is defined by

$$e^{tA} := \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$$

for every $t \in \mathbb{R}$. Why do we know that the above series is actually convergent?

(c) What would be a reasonable concept of an "infinitely large" matrix?

1.1 Normed spaces and Banach spaces

Definition 1.1.1 (Norms and normed spaces). (i) Let V be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A *norm* on V is a mapping $\|\cdot\| : V \to [0,\infty)$ which fulfils the following axioms:

(N1) $\|\cdot\|$ is *positively definite*, i.e. for each $x \in V$ we have $\|x\| = 0$ if and only if $x = 0$.

(N2) $\|\cdot\|$ is *positively homogenious*, i.e. we have $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in V$ and each $\alpha \in K$.

(N3) $\|\cdot\|$ satisfies the *triangle inequality*, i.e. we have $\|x+y\| \le \|x\| + \|y\|$ for all $x, y \in V$.

(ii) A *normed vector space* – or shorter: a *normed space* – is a pair $(V, \|\cdot\|)$ where V is a vector space over $\mathbb{R}$ or $\mathbb{C}$ and where $\|\cdot\|$ is a norm on V. By abuse of language we sometimes simply say that "V is a normed space", thereby suppressing $\|\cdot\|$ in the notation.

The following proposition shows that every norm on a vector space V induces a canonical metric on V.

Proposition 1.1.2. *Let $(V, \|\cdot\|)$ be a normed vector space. For all $x, y \in V$ we define $\mathrm{d}_{\|\cdot\|}(x,y) := \|x - y\|$. Then $\mathrm{d}_{\|\cdot\|}$ is a metric on V which we call the* metric induced by $\|\cdot\|$.

Proof. Let $x, y \in V$. Then we have $\mathrm{d}_{\|\cdot\|}(x,y) = 0$ if and only if $\|x-y\| = 0$ if and only if $x - y = 0$ if and only if $x = y$. This proves that $\mathrm{d}_{\|\cdot\|}$ is positively definite. Moreover, we have

$$\mathrm{d}_{\|\cdot\|}(x,y) = \|x-y\| = |-1|\,\|y-x\| = \mathrm{d}_{\|\cdot\|}(y,x),$$

so $\mathrm{d}_{\|\cdot\|}$ is symmetric. Finally, choose a third element $z \in V$. Using the triangle inequality for the norm we obtain

$$\mathrm{d}_{\|\cdot\|}(x,z) = \|x-z\| = \|(x-y)+(y-z)\| \leq \|x-y\| + \|y-z\| = \mathrm{d}_{\|\cdot\|}(x,y) + \mathrm{d}_{\|\cdot\|}(y,z),$$

so $\mathrm{d}_{\|\cdot\|}$ fulfils the triangle inequality, too. □

Remarks 1.1.3. (a) From now on we assume tacitly, whenever $(V, \|\cdot\|)$ is a normed vector space, that V be endowed with the metric $\mathrm{d}_{\|\cdot\|}$. Hence, we consider every normed vector space as a metric space and the metric on this space is prescribed by the norm. In particular, it is defined what it means for a subset of V to be *open* or *closed* and what it means for a sequence in V to be *convergent* or to be a *Cauchy sequence*.

(b) It is, however, common not to use the notation $\mathrm{d}_{\|\cdot\|}$ explicitly, i.e. one usually prefers to write $\|x-y\|$ instead of $\mathrm{d}_{\|\cdot\|}(x,y)$ for $x, y \in V$.

Remark 1.1.4. Let $(V, \|\cdot\|)$ be a normed vector space and let $W \subseteq V$ be a vector subspace of V. Let $\|\cdot\|_W : W \to [0,\infty)$ be the restriction of the mapping $\|\cdot\| : V \to [0,\infty)$ to W, i.e. let $\|x\|_W := \|x\|$ for all $x \in W$. Then $\|\cdot\|_W$ is a norm on W and we say that the norm $\|\cdot\|_W$ on W is *induced* by the norm $\|\cdot\|$ on V.

Note that the metric $\mathrm{d}_{\|\cdot\|_W}$ induced by the norm $\|\cdot\|_W$ on W coincides with the restriction of the metric $\mathrm{d}_{\|\cdot\|}$ to $W \times W$, i.e. we have

$$\mathrm{d}_{\|\cdot\|_W} = \mathrm{d}_{\|\cdot\|}|_{W\times W}.$$

From now on we let every vector subspace W of a normed space $(V, \|\cdot\|)$ be endowed with the norm $\|\cdot\|_W$ induced by the norm $\|\cdot\|$ on V. To keep the notation convenient, we abbreviate $\|\cdot\|_W =: \|\cdot\|$, i.e. we use the same symbol to denote the norm on V and its restriction to W.

Proposition 1.1.5. *Let $(V, \|\cdot\|)$ be a normed vector space over the scalar field $\mathbb{K}$. Then each of the following mappings is continuous:*

(a) $+ : V \times V \to V$,

(b) $\cdot : \mathbb{K} \times V \to V$,

(c) $\|\cdot\| : V \to [0,\infty)$.

Proof. (a) Let $\big((x_k, y_k)\big)_{k\in\mathbb{N}}$ be a sequence in $V \times V$ which converges to an element $(x, y) \in V \times V$. Then, according to Proposition A.2.3, $(x_k)_{k\in\mathbb{N}}$ converges to x and $(y_k)_{k\in\mathbb{N}}$ converges to y. Hence, we obtain

$$0 \le \|(x_k + y_k) - (x + y)\| \le \|x_k - x\| + \|y_k - y\| \to 0$$

as $k \to \infty$, which proves that $(x_k + y_k)_{k\in\mathbb{N}}$ converges to $x + y$.

(b) Let $\big((\lambda_k, x_k)\big)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{K} \times V$ which converges to an element $(\lambda, x) \in \mathbb{K} \times V$. Then $\lim_{k\to\infty} \lambda_k = \lambda$ and $\lim_{k\to\infty} x_k = x$ according to Proposition A.2.3. Hence, it follows from Remark A.1.14 that there exists a number $r > 0$ such that $|\lambda_k| < r$ for all $k \in \mathbb{N}$. Thus, we have

$$0 \le \|\lambda_k v_k - \lambda v\| \le \|\lambda_k v_k - \lambda_k v\| + \|\lambda_k v - \lambda v\| = |\lambda_k|\|v_k - v\| + |\lambda_k - \lambda|\|v\| \to 0$$

as $k \to \infty$ since $|\lambda_k| < r$ for all indices k. This proves that $\lambda_k v_k \to \lambda v$ as $k \to \infty$.

(c) The metric $\mathrm{d}_{\|\cdot\|}$ is a continuous mapping from $V \times V$ to $[0, \infty)$ according to Proposition A.2.5. Now, let $(x_k)_{k\in\mathbb{N}}$ be a sequence that converges to a vector $x \in V$. Then the sequence $\big((x_k, 0)\big)_{k\in\mathbb{N}}$ in $V \times V$ converges to $(x, 0)$ and therefore we obtain

$$\|x_k\| = \mathrm{d}_{\|\cdot\|}(x_k, 0) \to \mathrm{d}_{\|\cdot\|}(x, 0) = \|x\|.$$

This proves the assertion. □

Definition 1.1.6 (Banach spaces). Let $(E, \|\cdot\|)$ be a normed vector space. If E is complete with respect to the metric $\mathrm{d}_{\|\cdot\|}$, then we call $(E, \|\cdot\|)$ a *Banach space*.

Example 1.1.7 (The Banach spaces $\mathbb{R}^n$ and $\mathbb{C}^n$). Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $n \in \mathbb{N}$. If $\|\cdot\|$ is an arbitrary norm on the vector space $\mathbb{K}^n$, then $(\mathbb{K}^n, \|\cdot\|)$ is a Banach space.

Proof. This is usually proved in the course *Analysis 2*. □

Examples 1.1.8 (Space of bounded functions). Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, let $S \neq \emptyset$ be an arbitrary set and define

$$\ell^\infty(S; \mathbb{K}) := \{f : S \to \mathbb{R} \mid \exists C \ge 0 \; \forall s \in S : \; |f(s)| \le C\}$$

denote the set of all bounded $\mathbb{K}$-valued functions on S. For all $f, g \in \ell^\infty(S; \mathbb{K})$ and all scalars $\alpha \in \mathbb{K}$ we define $f + g \in \ell^\infty(S; \mathbb{K})$ and $\alpha f \in \ell^\infty(S; \mathbb{K})$ by

$$(f + g)(s) = f(s) + g(s) \quad \text{and} \quad (\alpha f)(s) = \alpha f(s) \qquad \text{for all } s \in S.$$

This renders $\ell^\infty(S; \mathbb{K})$ a vector space over $\mathbb{K}$. Now, let us define $\|f\|_\infty := \sup\big\{|f(s)| \,\big|\, s \in S\big\}$ for all $f \in S$. Then $\|\cdot\|_\infty$ is a norm on $\ell^\infty(S; \mathbb{K})$ and $(\ell^\infty(S; \mathbb{K}), \|\cdot\|_\infty)$ is a Banach space.

Proof. It is easy to check that $\ell^\infty(S;\mathbb{K})$ is a vector subspace of the space of all functions from S to $\mathbb{K}$ and it is known from *Linear Algebra* that the latter space is a vector space over $\mathbb{K}$; hence, $\ell^\infty(S;\mathbb{K})$ is also a vector space over $\mathbb{K}$.

We leave it to the reader to check that $\|\cdot\|_\infty$ is indeed a norm on $\ell^\infty(S;\mathbb{K})$. To prove completeness, let $(f_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in the normed space $(\ell^\infty(S;\mathbb{K}),\|\cdot\|)_\infty$. We first show that, for every $s \in S$, $(f_k(s))_{k\in\mathbb{K}}$ is a Cauchy sequence in $\mathbb{K}$: indeed, for every $\varepsilon > 0$ we can find an index $k_0 \in \mathbb{K}$ such that $\|f_j - f_k\|_\infty < \varepsilon$ for all $j,k \geq k_0$. This implies

$$|f_j(s) - f_k(s)| \leq \sup_{t\in S}|f_j(t) - f_k(t)| = \|f_j - f_k\|_\infty < \varepsilon \tag{1.1}$$

for all $s \in S$ and all $j,k \geq k_0$. Hence, $(f_k(s))_{k\in\mathbb{K}}$ is a Cauchy sequence in $\mathbb{K}$ for each $s \in S$. Since $\mathbb{K}$ is complete, we conclude that $(f_k(s))_{k\in\mathbb{K}}$ converges to an element of K which we name $f(s)$. It suffices to prove that $f \in \ell^\infty(\mathbb{N};\mathbb{K})$ and that $(f_k)_{k\in\mathbb{N}}$ converges to f in $\ell^\infty(\mathbb{N};\mathbb{K})$.

Given $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ as above, it follows from (1.1) that

$$|f(s) - f_k(s)| \leq \varepsilon \tag{1.2}$$

for all $s \in S$ and all $k \geq k_0$. In particular, we obtain for each $s \in S$ the estimate $|f(s)| \leq |f(s) - f_{k_0}(s)| + |f_{k_0}(s)| \leq \varepsilon + \|f_{k_0}\|_\infty$, so $f \in \ell^\infty(\mathbb{N};\mathbb{K})$.

Using again (1.2) we conclude that $\|f - f_k\|_\infty \leq \varepsilon$ for all $k \geq k_0$, so $(f_k)_{k\in\mathbb{K}}$ indeed converges to f in the normed space $(\ell^\infty(S;\mathbb{K}),\|\cdot\|)_\infty$. This proves that the latter space is complete. □

The norm $\|\cdot\|_\infty$ on $\ell^\infty(S;\mathbb{K})$ is called the *infinity norm* or the *supremum norm* on $\ell^\infty(S;\mathbb{K})$. In the literature, it is sometimes also denoted by $\|\cdot\|_\infty =: \|\cdot\|_{\mathrm{sup}}$. From now on we will always assume tacitly that the space $\ell^\infty(S;\mathbb{K})$ is endowed with the infinity norm.

To construct further examples of Banach spaces, the following proposition is quite useful.

Proposition 1.1.9. *Let $(E,\|\cdot\|)$ be a Banach space and let F be a vector subspace of E. Then F is closed if and only if $(F,\|\cdot\|)$ is a Banach space.*

Proof. This is an immediate consequence of Proposition A.1.9 and the discussion in Remark 1.1.4. □

Example 1.1.10 (Space of bounded continuous functions). Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and let (M,d) be a non-empty metric space. Let

$$\mathcal{C}_{\mathrm{b}}(M;\mathbb{K}) := \{f \in \ell^\infty(M;\mathbb{K}) |\, f \text{ is continuous}\}$$

denote the space of all $\mathbb{K}$-valued bounded and continuous functions on M. Then $\mathcal{C}_{\mathrm{b}}(M;\mathbb{K})$ is a closed vector subspace of the Banach space $\ell^\infty(M;\mathbb{K})$. In particular, $(\mathcal{C}_{\mathrm{b}}(M;\mathbb{K}),\|\cdot\|_\infty)$ is a Banach space.

Proof. It is easy to see that $\mathcal{C}_b(M;\mathbb{K})$ is a vector subspace of $\ell^\infty(M;\mathbb{K})$. To show that it is closed, let $(f_k)_{j\in\mathbb{N}}$ be a sequence in $\mathcal{C}_b(M;\mathbb{K})$ which converges to a vector $f \in \ell^\infty(M;\mathbb{K})$. We have to show that f is continuous, so let $\varepsilon > 0$ and $x \in M$.

There exists an index $k_0 \in \mathbb{N}$ such that $\|f_k - f\|_\infty < \varepsilon$ and, as f_{k_0} is continuous, there exists a number $\delta > 0$ such that $|f_{k_0}(y) - f_{k_0}(x)| < \varepsilon$ for all $y \in M$ that fulfil $\mathrm{d}(y,x) < \delta$. Hence, we obtain for all those y

$$\begin{aligned} |f(y) - f(x)| &\le |f(y) - f_{k_0}(y)| + |f_{k_0}(y) - f_{k_0}(x)| + |f_{k_0}(x) + f(x)| \\ &< \|f - f_{k_0}\|_\infty + \varepsilon + \|f - f_{k_0}\|_\infty < 3\varepsilon. \end{aligned}$$

This proves that $f \in \mathcal{C}_b(M;\mathbb{K})$. □

Remark 1.1.11. Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and let (M,d) be a non-empty metric space. If M is compact, then every continuous function $f : M \to \mathbb{K}$ is automatically bounded (as $f(M)$ is a compact of subset of $\mathbb{C}$). In this case, we often use the abbreviation $\mathcal{C}(M;\mathbb{K}) =: \mathcal{C}_b(M;\mathbb{K})$.

Finally, we give an example of a normed vector space which is not a Banach space.

Example 1.1.12. Let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and define

$$\|f\|_1 =: \int_0^2 |f(x)|\,\mathrm{d}x$$

for all $f \in \mathcal{C}([0,2];\mathbb{K})$. Then $\|\cdot\|_1$ is a norm on $\mathcal{C}([0,2];\mathbb{K})$, but $(\mathcal{C}_b([0,2];\mathbb{K}),\|\cdot\|_1)$ is not a Banach space.

Proof. One immediately checks that $\|\cdot\|_1$ is a norm on $(\mathcal{C}_b([0,2];\mathbb{K})$. To see that the space is not complete with respect to this norm, we have to construct a Cauchy sequence which does not converge.

For each $k \in \mathbb{N}$, define $f_k \in (\mathcal{C}_b([0,2];\mathbb{K})$ by

$$f_k(x) = \begin{cases} x^n & \text{if} x \in [0,1], \\ 1 & \text{if } x \in (1,2]. \end{cases}$$

Let us check that $(f_k)_{k\in\mathbb{N}}$ is a Cauchy sequence. Let $\varepsilon > 0$ and choose $k_0 \in \mathbb{N}$ such that $\frac{2}{k_0} < \varepsilon$. For all $j,k \ge k_0$ we thus obtain

$$\begin{aligned} \|f_j - f_k\|_\infty &= \int_0^2 |f_j(x) - f_k(x)|\,\mathrm{d}x = \int_0^1 |f_j(x) - f_k(x)|\,\mathrm{d}x \\ &\le \int_0^1 x^j\,\mathrm{d}x + \int_0^1 x^k\,\mathrm{d}x = \frac{1}{j+1} + \frac{1}{k+1} < \frac{2}{k_0} < \varepsilon. \end{aligned}$$

Hence, $(f_k)_{k\in\mathbb{N}}$ is indeed a Cauchy sequence in the normed space $(\mathcal{C}_\mathrm{b}([0,2];\mathbb{K}), \|\cdot\|_1)$.

It remains to shows that the sequence is not convergent. To this end, first note that, for every $x \in [0,2]$, $f_k(x)$ converges to $\mathbb{1}_{[1,2]}(x)$ as $k \to \infty$. Hence, it follows from the dominated converges theorem that

$$\int_0^2 |f_k(x) - \mathbb{1}_{[1,2]}(x)| \,\mathrm{d}x \to 0$$

as $k \to \infty$. Now assume for a contradiction that $(f_k)_{k\in\mathbb{N}}$ converges to a vector $f \in \mathcal{C}_\mathrm{b}([0,2];\mathbb{K})$ with respect to the norm $\|\cdot\|_\infty$. Then we also have

$$\int_0^2 |f_k(x) - f(x)| \,\mathrm{d}x \to 0$$

as $k \to \infty$ and hence,

$$\int_0^2 |f(x) - \mathbb{1}_{[1,2]}(x) \,\mathrm{d}x \le \int_0^2 |f(x) - f_k(x)| \,\mathrm{d}x + \int_0^2 |f_k(x) - f(x)| \,\mathrm{d}x \to 0$$

as $k \to \infty$. We have thus shows that $\int_0^2 |f(x) - \mathbb{1}_{[1,2]}(x) \,\mathrm{d}x = 0$, so $f = \mathbb{1}_{[1,2]}$ almost everywhere on $[0,2]$. However, as f is continuous in the point 1, there exists a $\delta > 0$ such that $f(x) \neq 0$ for all $x \in [0,2]$ that are closer than δ to 1. Hence, f is distinct from $\mathbb{1}_{[1,2]}$ on a non-empty open subset of $[1,2]$. This is a contradiction. ☐

Definition 1.1.13 (Convergence of series). Let $(V, \|\cdot\|)$ be a normed vector space and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in V.

(i) The *series over* $(x_k)_{k\in\mathbb{N}}$ is defined to be the sequence $\left(\sum_{k=1}^n x_k\right)_{k\in\mathbb{N}}$.

Note that the sequence $(x_k)_{k\in\mathbb{N}}$ is uniquely determined by the series over $(x_k)_{k\in\mathbb{N}}$.

(ii) Let $x \in V$. We write $\sum_{k=1}^\infty x_k = x$ if and only if the series over $(x_k)_{k\in\mathbb{N}}$ converges to x.

(iii) The series over $(x_k)_{k\in\mathbb{N}}$ is said to be *absolutely convergent* if the series over $(\|x_k\|)_{k\in\mathbb{N}}$ is convergent in $\mathbb{R}$, i.e. if the sequence $(\sum_{k=1}^n \|x_k\|)_{n\in\mathbb{N}}$ is convergent in $\mathbb{R}$.

The following proposition is one of the reasons why completeness of normed vector spaces if so important in analysis.

Proposition 1.1.14. *Let $(V, \|\cdot\|)$ be a normed vector space. The following assertions are equivalent:*

(a) *The space $(V, \|\cdot\|)$ is a Banach space.*

(b) *Whenever the series over a given sequence $(x_k)_{k\in\mathbb{N}}$ is absolutely convergent, then this series is also convergent.*

Proof. We leave the proof as an exercise. □

1.2 Linear operators

Recall that a mapping $T: V \to W$ between two vector spaces over the same field $\mathbb{K}$ is called *linear* if

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T v_1 + \lambda_2 T v_2$$

for all $v_1, v_2 \in V$ and all $\lambda_1, \lambda_2 \in \mathbb{K}$.

Proposition 1.2.1. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces over the same scalar field and let $T: V \to W$ be a linear mapping. Then the following assertions are equivalent:*

(a) *The mapping T is continuous.*

(b) *There exists a point $v \in V$ such that T is continuous at v.*

(c) *There exists a number $C \geq 0$ such that $\|Tv\|_W \leq C\|v\|_V$ for all $v \in V$.*

(d) *We have* $\sup\left\{\|Tv\|_W : v \in V, \|v\|_V \leq 1\right\} < \infty$.

(e) *We have* $\sup\left\{\|Tv\|_W : v \in V, \|v\|_V = 1\right\} < \infty$.

(f) *We have* $\sup\left\{\frac{\|Tv\|_W}{\|v\|_V} : v \in V \setminus \{0\}\right\} < \infty$.

(g) *Whenever a sequence $(v_k)_{k\in\mathbb{N}}$ in V is bounded, then so is the sequence $(Tv_k)_{k\in\mathbb{N}}$ in W.*

In the above proposition, all suprema are taken in the ordered set $[0, \infty]$, so we have $\sup\emptyset = 0$. This is important in case that V only consists of the vector 0.

Proof of Proposition 1.2.1. We first prove "(a) ⇒ (g) ⇒ (b) ⇒ (a)":

"(a) ⇒ (g)" If there exists a bounded sequence $(v_k)_{k\in\mathbb{N}}$ in V for which $(Tv_k)_{k\in\mathbb{N}}$ is not bounded in W, then we can find a subsequence $(v_{k_j})_{h\in\mathbb{N}}$ (which is, of course, again bounded) for which we have $0 < \|v_{k_j}\|_W \to \infty$ as $j \to \infty$. Now, define $x_j := v_{k_j}/\|Tv_{k_j}\|_W$. Then $\|x_j\|_V \to 0$ and hence. $x_j \to 0$ as $j \to \infty$, but $\|Tx_j\|_W = 1$ for all $j \in \mathbb{N}$, so Tx_j does not converge to $0 = T0$. Hence, T is not continuous at 0; in particular, it is not continuous.

"(g) ⇒ (b)" Assume that (g) holds. We show that T is continuous at 0. So, let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $V \setminus \{0\}$ which converges to 0. Then the sequence

$(v_k)_{k\in\mathbb{N}}$, given by $v_k := x_k/\|x_k\|_V$, is bounded in V and hence, so is the sequence $(Tv_k)_{k\in\mathbb{N}}$ in W. This implies that $Tx_k = \|x_k\|_V Tv_k \to 0 = T0$ as $k \to \infty$, so T is indeed continuous at 0.

"(b) $\Rightarrow$ (a)" Assume that T is continuous at a point $x \in V$ and let $y \in V$ be another vector. Consider a sequence $(v_k)_{k\in\mathbb{N}}$ in V which converges to y. Then $(v_k - y + x)_{k\in\mathbb{N}}$ converges to x and, as T is linear and continuous at x, we conclude that $(Tv_k - Ty + Tx)_{k\in\mathbb{N}}$ converges to Tx. This implies that

$$(Tv_k)_{k\in\mathbb{N}} = \Big((Tv_k - Ty + Tx) + (Ty - Tx)\Big)_{k\in\mathbb{N}}$$

converges to $Tx + Ty - Tx = Ty$. Thus, T is indeed continuous at y.

We leave it to the reader to prove that the assertions (c), (d), (e) and (f) are equivalent and only deal with the remaining equivalence "(c) $\Leftrightarrow$ (g)".

"(c) $\Rightarrow$ (g)" This is obvious.

"(g) $\Rightarrow$ (c)" If a constant C as in (c) does not exists, then we can find, for every integer $k \in \mathbb{N}$, a vector $x_k \in V$ which fulfils $\|Tx_k\|_W > k\|x_k\|_V$. In particular, we have $x_k \neq 0$ since $Tx_k \neq 0$, so we can define $v_k := x_k/\|x_k\|_V$ for each index $k \in \mathbb{N}$. Hence, the sequence $(v_k)_{k\in\mathbb{N}}$ is bounded in V, but we have $\|Tv_k\|_W > k$ for every $k \in \mathbb{N}$, so the sequence $(Tv_k)_{k\in\mathbb{N}}$ is not bounded in W. □

Remark 1.2.2. In functional analysis, linear maps between two vector spaces are often referred to as *operators*. Moreover, for linear operators between normed vector spaces one uses the notion *bounded* synonymously with *continuous*, i.e. we often speak of *bounded linear operators* instead of *continuous linear operators*.

Definition 1.2.3 (Operator norm). Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces over the same scalar field and let $T : V_1 \to V_2$ be a bounded linear operator. The *operator norm of T* is defined to be the number

$$\|T\| := \inf\Big\{C \geq 0 : \ \|Tv\|_W \leq C\|v\|_W \text{ for all } v \in V\Big\} \in [0, \infty).$$

Proposition 1.2.4. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces over the same scalar field and let $T : V_1 \to V_2$ be a bounded linear operator. Then we have*

$$\begin{aligned}
\|T\| &= \min\Big\{C \geq 0 : \ \|Tv\|_W \leq C\|v\|_W \textit{ for all } v \in V\Big\} \\
&= \sup\Big\{\|Tv\|_W : \ v \in V, \ \|v\|_V \leq 1\Big\} \\
&= \sup\Big\{\|Tv\|_W : \ v \in V, \ \|v\|_V = 1\Big\} \\
&= \sup\Big\{\frac{\|Tv\|_W}{\|v\|_V} : \ v \in V \setminus \{0\}\Big\}.
\end{aligned}$$

Proof. We leave this as an exercise. □

Corollary 1.2.5. *Let $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ and $(X, \|\cdot\|_X)$ be normed vector spaces over the same scalar field and let $T : V \to W$ and $S : W \to X$ be bounded linear operators.*

(a) *For every $v \in V$ we have $\|Tv\|_W \leq \|T\| \|v\|_V$.*

(b) *The linear operator $ST := S \circ T$ is bounded and we have $\|ST\| \leq \|S\| \|T\|$.*

Proof. (a) This follows from the first equality in Proposition 1.2.4.

(b) For each $v \in V$ we have, according to (a),

$$\|STx\|_X \leq \|S\| \|Tx\|_W \leq \|S\| \|T\| \|x\|_V.$$

Hence, it follows from the very definition of $\|ST\|$ that $\|ST\| \leq \|S\| \|T\|$. □

Let V, W be vector spaces over the same field $\mathbb{K}$, let $S, T : V \to W$ be linear mappings and let $\lambda, \mu \in \mathbb{K}$. Recall that the linear mapping $\lambda S + \mu T : V \to W$ is defined by $(\lambda S + \mu T)v = \lambda Sv + \mu Tv$ for all $v \in V$. With this addition and scalar multiplication, the space of all linear mappings form V to W becomes itself of vector space over $\mathbb{K}$.

Definition 1.2.6. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces over the same scalar field. We denote the set of all bounded linear operators from V to W be $\mathcal{L}(V; W)$.

Proposition 1.2.7. *Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed vector spaces over the same scalar field $\mathbb{K}$.*

(a) *The set $\mathcal{L}(V; W)$ is a vector subspace of the space of all linear mappings from V to W.*

(b) *The operator norm $\|\cdot\|$ is a norm on the vector space $\mathcal{L}(V; W)$.*

(c) *If $(W, \|\cdot\|_W)$ is a Banach space, then the normed vector space $(\mathcal{L}(V; W), \|\cdot\|)$ is a Banach space, too.*

Proof. (a) One can immediately see that a linear combination of continuous linear mappings from V to W is again continuous.

(b) Clearly, the operator norm is a mapping from $\mathcal{L}(V; W)$ to $[0, \infty)$. Let us show that it fulfils all axioms of a norm:

Positive definiteness: Let $T \in \mathcal{L}(V; W)$. If $T = 0$, then $\|Tv\|_W = \|0\|_W = 0 \leq 0 \cdot \|v\|_V$ for all $v \in V$, so $\|T\| \leq 0$ which proves that $\|T\| = 0$. If we assume, on the other hand, that $\|T\| = 0$, then we have $0 \leq \|Tv\|_W \leq 0 \cdot \|v\| = 0$ for all $v \in V$, so $T = 0$.

Positive homogeneity: Let $T \in \mathcal{L}(V;W)$ and $\alpha \in \mathbb{K}$. Then it follows from Proposition 1.2.4 that

$$\begin{aligned}\|\alpha T\| &= \sup\left\{\|\alpha Tv\|_W : v \in V, \|v\|_V \leq 1\right\} \\ &= \sup\left\{|\alpha|\|Tv\|_W : v \in V, \|v\|_V \leq 1\right\} \\ &= |\alpha| \sup\left\{\|Tv\|_W : v \in V, \|v\|_V \leq 1\right\} = |\alpha|\|T\|.\end{aligned}$$

Triangle inequality: Let $S, T \in \mathcal{L}(V;W)$. For all $v \in V$ we obtain

$$\|(S+T)v\|_W \leq \|Tv\|_W + \|Tv\|_W \leq \|S\|\|v\|_V + \|T\|\|v\|_V = \big(\|S\| + \|T\|\big)\|v\|_V.$$

It thus follows from the very definition of the operator norm $\|S+T\| \leq \|S\|+\|T\|$.

(c) Assume that W is a Banach space and let $(T_k)_{k\in\mathbb{N}}$ be a Cauchy sequence in $\mathcal{L}(V;W)$. It easily follows that, for every $v \in V$, the sequence $(T_k v)_{k\in\mathbb{N}}$ is a Cauchy sequence in W. Hence, as W is a Banach space, $(T_k v)_{k\in\mathbb{N}}$ converges to a vector in W.

Let us define a mapping $T : V \to W$, $Tv := \lim_{k\to\infty} T_k v$. One readily checks that T is linear. Let us show that T is even a bounded operator. Since $(T_k)_{k\in\mathbb{N}}$ is a Cauchy sequence, it is bounded, i.e. there exists a number $C \geq 0$ such that $\|T_k\| \leq C$. Now, consider a vector $v \in V$. We have

$$\|Tv\| = \lim_{k\to\infty} \|T_k v\| \leq \limsup_{k\to\infty} \|T_k\|\|v\| \leq C\|v\|.$$

Thus, T is bounded. Finally, we have to check that $(T_k)_{k\in\mathbb{N}}$ converges to T with respect to the operator norm. So, let $\varepsilon > 0$. There exists an index k_0 such that $\|T_k - T_j\| < \varepsilon$ for all $j, k \geq k_0$. Hence, we obtain for every $v \in V$ and every $k \geq k_0$

$$\|(T_k - T)v\| = \lim_{j\to\infty} \|T_k v - T_j v\| \leq \limsup_{j\to\infty} \|T_k - T_j\|\|v\| \leq \varepsilon\|v\|.$$

This proves that $\|T_k - T\| \leq \varepsilon$ whenever $k \geq k_0$, so we have indeed $\lim_{k\to\infty} T_k = T$. □

Remark 1.2.8. The converse implication in Proposition 1.2.7(c) is also true. To see this one needs the so-called *Hahn–Banach theorem* which is an important result in functional analysis. However, this result is not part of this course since we are going to focus on the theory of *Hilbert spaces* for which the Hahn–Banach theorem is not needed.

2 Inner products and Hilbert spaces

Opening Questions. (a) You know what the standard scalar product of two vectors $x, y \in \mathbb{C}^d$ is. Is there also a way to define a "standard scalar product" of two functions $f, g : [0,1] \to \mathbb{C}$?

(b) How can we represent a number $\lambda \in \mathbb{R}$ in a computer, given that our computer has only finite storage capacity? How can we represent a function $f : [0,1] \to \mathbb{R}$ in our computer?

(c) Is there a reasonable way to "transpose" a bounded linear operator?

(d) Suppose we are given a real number x_i for every i out of a (possibly infinite) index set I. What is our understanding of the "series" $\sum_{i \in I} x_i$?

(e) Let $x_{m,n}$ be a complex number for all $m, n \in \mathbb{N}$ and let $x \in \mathbb{C}$. What do we mean by $\lim_{m,n\to\infty} x_{m,n} = x$?

2.1 Inner products and orthogonality

Definition 2.1.1. (i) Let V be a vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. An *inner product* on V is a mapping $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ which fulfils the following properties:

(I1) For all $v \in V$ we have $\langle v, v \rangle \in [0, \infty)$; moreover, $\langle v, v \rangle = 0$ implies $v = 0$.

(I2) For all $v, w \in V$ and all $\lambda \in \mathbb{K}$ we have $\langle v, \lambda w \rangle = \lambda \langle v, w \rangle$.

(I3) For all $v, w, x \in V$ we have $\langle x, v + w \rangle = \langle x, v \rangle + \langle x, w \rangle$.

(I4) For all $v, w \in V$ we have $\langle v, w \rangle = \overline{\langle w, v \rangle}$.

(ii) A *pre-Hilbert space* is a pair $(V, \langle \cdot, \cdot \rangle)$ where V is a vector space over $\mathbb{R}$ or $\mathbb{C}$ and $\langle \cdot, \cdot \rangle$ is an inner product on V.

On every pre-Hilbert space $(V, \langle \cdot, \cdot \rangle)$ we define $\|v\| := \sqrt{\langle v, v \rangle}$ for all $v \in V$.

(iii) Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space. Two vectors $v, w \in V$ are said to be *orthogonal* if $\langle v, w \rangle = 0$. We denote this by $v \perp w$.

Proposition 2.1.2. *Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space over the field $\mathbb{K}$. Then the following assertions hold.*

(a) *Let $v \in V$. Then $\langle v, 0 \rangle = \langle 0, v \rangle = 0$. In particular, we have $\langle v, v \rangle = 0$ if and only if $v = 0$.*

(b) *The inner product is* anti-linear *in its first argument, meaning that we have* $\langle \lambda v + \mu w, x\rangle = \overline{\lambda}\langle v, x\rangle + \overline{\mu}\langle w, x\rangle$ *for all vectors* $v, w, x \in V$ *and all scalars* $\lambda, \mu \in \mathbb{K}$.

(c) *For all* $v, w \in V$ *the so-called* Cauchy–Schwarz inequality *holds, meaning that we have* $|\langle v, w\rangle| \leq \|v\|\,\|w\|$.

(d) *For all* $v, w \in V$ *the so-called* parallelogram equality *holds, meaning that we have* $\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2$.

(e) *If* $v, w \in V$ *are orthogonal, then* Pythagoras' Theorem $\|v+w\|^2 = \|v\|^2+\|w\|^2$ *holds.*

Proof. Assertions (b), (d) and (e) are immediate consequences of the definition of an inner product. Since the inner product is linear in the second component, we have $\langle v, 0\rangle = 0$ for all $v \in V$; by (I4) this also implies $\langle 0, v\rangle = 0$ for all $v \in V$. This proves (a), it only remains to prove (c). Let $\mu = \|w\|^2$ and let $\lambda = \langle w, v\rangle$. Then we have

$$\begin{aligned} 0 \leq \|\mu v - \lambda w\|^2 &= \overline{\mu}\mu\langle v, v\rangle - \overline{\mu}\lambda\langle v, w\rangle - \overline{\lambda}\mu\langle w, v\rangle + \overline{\lambda}\lambda\langle w, w\rangle \\ &= \|w\|^4 \cdot \|v\|^2 - \|w\|^2 \cdot |\langle v, w\rangle|^2 - \|w\|^2 \cdot |\langle v, w\rangle|^2 + \|w\|^2 \cdot |\langle v, w\rangle|^2 \\ &= \|w\|^4 \cdot \|v\|^2 - \|w\|^2 \cdot |\langle v, w\rangle|^2, \end{aligned}$$

so $\|w\|^2 \cdot |\langle v, w\rangle|^2 \leq \|w\|^4 \cdot \|v\|^2$. If $w \neq 0$, this implies the assertion. If, on the other hand, $w = 0$, then both sides of the inequality in question are 0. This proves the assertion. □

Remark 2.1.3. A slight modification of the above proof shows that the Cauchy–Schwarz inequality becomes an *equality* if and only if v and w are linearly depended.

Proposition 2.1.4. *Let* $(V, \langle\cdot,\cdot\rangle)$ *be a pre-Hilbert space over the field* $\mathbb{K}$. *Then the mapping* $\|\cdot\| : V \to [0, \infty)$ *is a norm on* V.

Proof. Positive definiteness is immediate from the definition of an inner product, so let $v, w \in V$ and $\alpha \in \mathbb{K}$. We have

$$\|\alpha v\|^2 = \langle \alpha v, \alpha v\rangle = \overline{\alpha}\alpha\langle v, v\rangle = |\alpha|^2 \cdot \|v\|^2,$$

so $\|\alpha v\| = |\alpha|\,\|v\|$. Moreover,

$$\begin{aligned} \|v + w\|^2 = \langle v + w, v + w\rangle &= \langle v, v\rangle + \langle v, w\rangle + \langle w, v\rangle + \langle w, w\rangle \\ &= \|v\|^2 + 2\operatorname{Re}\langle v, w\rangle + \|w\|^2 \leq \|v\|^2 + |\langle v, w\rangle| + \|w\|^2 \\ &\leq \|v\|^2 + 2\|v\|\,\|w\| + \|w\|^2 = \big(\|v\|^2 + \|w\|\big)^2, \end{aligned}$$

so $\|v + w\| \leq \|v\| + \|w\|$. □

We call the norm in the above proposition the *norm induced by the inner product* on V. From now on we tacitly endow every pre-Hilbert with the norm induced by the inner product.

Definition 2.1.5. A *Hilbert space* is a pre-Hilbert space which is complete with respect to the norm induced by the inner product.

Example 2.1.6. Let $n \in \mathbb{N}$ and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Define $\langle v, w \rangle = \sum_{k=1}^{n} \overline{v_k} w_k$ for all $v, w \in \mathbb{K}^n$. Then $(\mathbb{K}^n, \langle \cdot, \cdot \rangle)$ is a Hilbert space.

Example 2.1.7. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and set $V = \mathcal{C}([0,1]; \mathbb{K})$. For all $f, g \in V$ we define

$$\langle f, g \rangle := \int_0^1 \overline{f(x)} \cdot g(x) \, \mathrm{d}x.$$

Then $(V, \langle \cdot, \cdot \rangle$ is a pre-Hilbert space, but not a Hilbert space.

Proof. We readily checks that $\langle \cdot, \cdot \rangle$ fulfils (I2)–(I4). Moreover, we have $\langle f, f \rangle = \int_0^1 |f(x)|^2 \, dxx \geq 0$ for each $f \in V$. Now, let $f \in V$ and assume that $0 = \langle f, f \rangle = \int_0^1 |f(x)|^2 \, \mathrm{d}x$. If $f \neq 0$, then there exist an element $y \in [0,1]$ for which we have $\delta := f(y) > 0$; by the continuity of f we can thus find an open neighbourhood U of y in $[0,1]$ such that $f(x) \geq \delta/2$ for all $x \in U$. Denoting the Lebesgue measure of U by $|U|$, we thus obtain

$$\int_0^1 |f(x)|^2 \, \mathrm{d}x \geq \int_U \delta^2/4 \, \mathrm{d}x = |U|\delta/2 > 0$$

which is a contradiction. Hence, $f = 0$.

We have thus shown that $\langle \cdot, \cdot \rangle$ is an inner product on V. To see that V is not complete, one can proceed quite similarly as in Example 1.1.12. $\square$

Now we discuss an example of a function space which is endowed with a similar inner product as above but which is, in contrast to Example 2.1.7, a Hilbert space. This example is most important for many applications.

Example 2.1.8. Fix $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let (Ω, μ) be a measure space and let $\mathcal{M}(\Omega; \mathbb{K})$ denote the space of all measurable functions from Ω to $\mathbb{K}$. For two functions $f, g \in \mathcal{M}(\Omega; \mathbb{K})$ we write $f \sim g$ if we have $f(x) = g(x)$ for μ-almost all $x \in \Omega$. One immediately checks that $\sim$ is an equivalence relation on $\mathcal{M}(\Omega; \mathbb{K})$.

For each $f \in \mathcal{M}(\Omega; \mathbb{K})$ we denote the equivalence class of f with respect to $\sim$ by $[f]$. Let

$$\mathcal{M}(\Omega; \mathbb{K})/\sim := \{[f] : f \in \mathcal{M}(\Omega; \mathbb{K})\}$$

denote the set of all equivalence classes of the relation $\sim$. Then the mappings

$$+ : \mathcal{M}(\Omega;\mathbb{K})/\sim \times \mathcal{M}(\Omega;\mathbb{K})/\sim \to \mathcal{M}(\Omega;\mathbb{K}), \quad [f]+[g] := [f+g]$$
$$\cdot : \mathbb{K} \times \mathcal{M}(\Omega;\mathbb{K}) \to \mathcal{M}(\Omega;\mathbb{K}) \quad \lambda \cdot [f] := [\lambda f]$$

are well-defined and render $\mathcal{M}(\Omega;\mathbb{K})$ a vector space over $\mathbb{K}$.

For what follows note that, whenever for all $[f] \in \mathcal{M}(\Omega;\mathbb{K})$, the question whether f is integrable does not depend on the choice of the representative f of $[f]$; in case that f is integrable, the value $\int_\Omega f(\omega)\,\mathrm{d}\omega$ does not depend on the choice of the representative f.

We now define

$$L^2(\Omega;\mathbb{K}) := \{[f] \in \mathcal{M}(\Omega;\mathbb{K}) : \; f^2 \text{ is integrable}\}.$$

By a similar reasoning as in the proof of the Cauchy–Schwarz inequality, one can prove that we have $\int_\Omega |f(\omega)\overline{g(\omega)}|\,\mathrm{d}\omega \le \Big(\int_\Omega |f(\omega)|^2\,\mathrm{d}\omega \cdot \int_\Omega |f(\omega)|^2\,\mathrm{d}\omega \Big)^{1/2}$ for all $f \in \mathcal{M}(\Omega;\mathbb{K})$. This implies to things:

(a) The function $(f+g)^2$ is integrable whenever f^2 and g^2 are integrable. Thus, the set $L^2(\Omega;\mathbb{K})$ is a vector subspace of $\mathcal{M}(\Omega;\mathbb{K})/\sim$.

(b) The mapping $\langle\cdot,\cdot\rangle : L^2(\Omega;\mathbb{K})\times L^2(\Omega;\mathbb{K}) \to \mathbb{K}$ which is given by $\langle [f],[g]\rangle = \int_\Omega f(\omega)g(\omega)\,\mathrm{d}\omega$ for all $[f],[g] \in L^2(\Omega;\mathbb{K})$ is well-defined.

Now, one can readily check that $\langle\cdot,\cdot\rangle$ is an inner product on $L^2(\Omega;\mathbb{K})$, meaning that $(L^2(\Omega;\mathbb{K}),\langle\cdot,\cdot\rangle)$ is a pre-Hilbert space. Actually, as proved in the course "Measure Theory", this is even a Hilbert space!

Remark 2.1.9. One can also consider the construction of $L^2(\Omega;\mathbb{K})$ in Example 2.1.8 from a different perspective. To this end, define

$$\mathcal{L}^2(\Omega;\mathbb{K}) := \{f \in \mathcal{M}(\Omega;\mathbb{K}) : \; f^2 \text{ is integrable}\}.$$

Then $\mathcal{L}^2(\Omega;\mathbb{K})$ is a vector subspace of $\mathcal{M}(\Omega;\mathbb{K})$ and $\|f\| := \Big(\int_\Omega |f(\omega)|^2\,\mathrm{d}\omega\Big)^{1/2}$ for each $f \in \mathcal{L}^2(\Omega;\mathbb{K})$ defines a semi-norm on $\mathcal{L}^2(\Omega)$. If one factors out the kernel of this semi-norm, one again arrives at the space $L^2(\Omega;\mathbb{K})$; compare Exercise 2 on Exercise Sheet 1!

2.2 Nets and unconditional convergence of series

Definition 2.2.1. A *directed set* is a pair $(J,\preceq)$ where J is a non-emptyset and $\preceq$ is a relation on J which fulfils the following three properties:

(D1) We have $j \preceq j$ for every $j \in J$, i.e. the relation $\preceq$ is reflexive.

(D2) Whenever $j_1 \preceq j_2$ and $j_2 \preceq j_3$ for elements $j_1, j_2, j_3 \in J$, then we also have $j_1 \preceq j_3$, i.e. the relation $\preceq$ is transitive.

(D3) For all $j_1, j_2 \in J$ there exists $j_3 \in J$ which fulfils $j_1 \preceq j_3$ and $j_2 \preceq j_3$.

By abuse of notation we shall often call J a directed set, thereby suppressing the relation $\preceq$ in the notation. Moreover, if $(J, \preceq)$ is a directed set then, for elements $j_1, j_2 \in J$, we often write $j_2 \succeq j_1$ as an alternative notation for $j_1 \preceq j_2$.

Examples 2.2.2. (a) The natural numbers $\mathbb{N}$, together with their usual order $\leq$, are a directed set.

(b) The non-negative real numbers $[0, \infty)$, together with their usual order $\leq$, are a directed set.

(c) Let I be an arbitrary, non-empty set and let $\mathcal{F}$ denote the set of all finite subsets of I. Then $(\mathcal{F}, \subseteq)$ is a directed set.

(d) Let (M, d) be a metric space and let $x \in M$. Denote the set of all neighbourhoods of x by $\mathcal{U}$. Then $(\mathcal{U}, \supseteq)$ is a directed set.

(e) Endow the set $\mathbb{N} \times \mathbb{N}$ with the relation $\preceq$ defined by $(n_1, n_2) \preceq (m_1, m_2)$ if and only if $n_1 \leq m_1$ and $n_2 \leq m_2$. Then $(\mathbb{N} \times \mathbb{N}, \preceq)$ is a directed set.

Remark 2.2.3. Let $(J, \preceq)$ be a directed set and let $j_1, \dots, j_n \in J$. Then there exists an element $i \in I$ such that $i \succeq j_1$, ..., $i \succeq j_n$. This follows inductively from (D3).

Definition 2.2.4. Let S be a non-empty set. A *net* in S is a family $(x_j)_{j \in J}$ of elements $x_j \in S$, where J is a directed set.

Example 2.2.5. Let S be a non-empty set. Every sequence $(x_n)_{n \in \mathbb{N}}$ is a net (where $\mathbb{N}$ is endowed with its usual order).

Definition 2.2.6. Let (M, d) be a metric space and let $(x_j)_{j \in J}$ be a net in M.

(i) The net $(x_j)_{j \in J}$ is said to *converge* to a element $x \in M$ if for every $\varepsilon > 0$ there exists an index $j_0 \in J$ such that $\mathrm{d}(x_j, x) < \varepsilon$ for all $j \in J$ which fulfil $j \succeq j_0$.

(ii) The net $(x_j)_{j \in J}$ is said to be a *Cauchy net* if for every $\varepsilon > 0$ there exists an index $j_0 \in J$ such that $\mathrm{d}(x_{j_1}, x_{j_2}) < \varepsilon$ for all $j_1, j_2 \in J$ which fulfil $j_1 \succeq j_0$ and $j_2 \preceq j_0$.

We have pointed out in Example 2.2.5 that every sequence is a net. The reader should convince herself/himself that the above definition of convergent nets and Cauchy nets is consistent with the definition of convergent sequences and Cauchy sequences. It is easy to see that a net in a metric space converges to at most one point.

Example 2.2.7. Let (M, d) be a metric space and let $x \in M$. Let $\mathcal{U}$ be the directed set of all neighbourhoods of x (endowed with the relation $\supseteq$). Assume that, for every $U \in \mathcal{U}$, we are given an element $x_U \in U$. Then the net $(x_U)_{U\in\mathcal{U}}$ converges to x.

Proof. Let $\varepsilon > 0$. Choose $U_0 := B_\varepsilon(x) \in \mathcal{U}$. For every $U \in \mathcal{U}$ with $U_0 \supseteq U$ we have $x_u \in U \subseteq U_0 = B_\varepsilon(x)$ and thus $\mathrm{d}(x_U, x) < \varepsilon$. □

Example 2.2.8. Let $x_{m,n}$ be a complex number for all $m, n \in \mathbb{N}$ and let $x \in \mathbb{C}$. We write $\lim_{m,n\to\infty} x_{m,n} = x$ if the net $(x_{m,n})_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ converges to x. Here, $\mathbb{N}\times\mathbb{N}$ is endowed with the same relation $\preceq$ as in Example 2.2.2(e).

It is very easy to check that every convergent net in a metric space is a Cauchy net. The following proposition shows that the converse implication is true if and only if the metric space under consideration is complete.

Proposition 2.2.9. *A metric space (M, d) is complete if and only if every Cauchy net in (M, d) converges.*

Proof. "$\Rightarrow$" Assume that (M, d) is complete and consider a Cauchy net $(x_j)_{j\in J}$ in M. For each $n \in \mathbb{N}$ we can find an index $j_n \in J$ such that $\mathrm{d}(x_i, x_j) < 1/n$ for all $i, j \in J$ which fulfil $i, j \succeq j_n$. It follows from Remark 2.2.3 that we can even choose j_n such that $j_n \succeq j_1, \dots j_n \succeq j_{n-1}$. It readily follows that the sequence $(x_{j_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in M and hence, it converges to an element $x \in M$.

We now show that the net $(x_j)_{j\in J}$ converges to x. Let $\varepsilon > 0$ and choose $m \in \mathbb{N}$ such that $\mathrm{d}(x_{j_n}, x) < \varepsilon/2$ for all $n \ge m$. Now, choose $n \ge m$ such that $1/n < \varepsilon/2$. For each $j \succeq j_n$ we have

$$\mathrm{d}(x_j, x) \le \mathrm{d}(x_j, x_{j_n}) + \mathrm{d}(x_{j_n}, x) < \frac{1}{n} + \frac{\varepsilon}{2} < \varepsilon.$$

Hence, the net $(x_j)_{j\in J}$ converges indeed to x.

"$\Leftarrow$" This implication is obvious since every sequence is a net. □

Proposition 2.2.10. *Let (M, d_M) and (N, d_N) be metric spaces and let $f : M \to N$ be a mapping. Let $x \in M$. The following assertions are equivalent:*

(i) *The mapping f is continuous at x.*

(ii) *For every net $(x_j)_{j\in J}$ in M that converges to x, the net $(f(x_j))_{j\in J}$ in N converges to $f(x)$.*

Proof. "$\Rightarrow$" Let f be continuous at x and consider $(x_j)_{j\in J}$ a net in M that converges to x. Let $\varepsilon > 0$. There exists $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ and there exists an index $j_0 \in J$ such that $x_j \in B_\delta(x)$ for all $j \succeq j_0$. Hence we have $f(x_j) \in B_\varepsilon(f(x))$ for all $j \succeq j_0$. This proves that $(f(x_j))_{j\in J}$ converges to $f(x)$.

"$\Leftarrow$" If (i) is not fulfilled then, according to Proposition A.1.12, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in M that converges to x but for which the sequence $(f(x_n))_{n\in\mathbb{N}}$ in N does not converge to $f(x)$. Since every sequence is a net, it follows that (ii) is not fulfilled. $\square$

Now we can finally define unconditional convergence of series.

Definition 2.2.11. Let $(V, \|\cdot\|)$ be a normed vector space, let I be a non-empty set and consider a family $(x_i)_{i\in I}$ of elements $x_i \in V$.

(i) The *unconditional series over* $(x_i)_{i\in I}$ is defined to be the net $(\sum_{i\in F})_{F\in\mathcal{F}}$, where $\mathcal{F}$ denotes the directed set of all finite subsets of I (endowed with the relation $\subseteq$).

(ii) Let $x \in V$. We write $\sum_{i\in I} x_i = x$ if and only if the unconditional series over $(x_i)_{i\in I}$ converges to x.

By abuse of language we sometimes say that "the series over $(x_i)_{i\in I}$ is unconditionally convergent to x" when we in fact mean that "the unconditional series over $(x_i)_{i\in I}$ converges to x."

Remark 2.2.12. Let $(V, \|\cdot\|)$ be a normed vector space, let $x \in V$ and consider a family $(x_i)_{i\in I}$ of elements $x_i \in V$.

According to the above definitions we have $\sum_{i\in I} x_i = x$ if and only if for every $\varepsilon > 0$ there exists a finite set $J \subseteq I$ such that $\|\sum_{j\in J'} x_j - x\| < \varepsilon$ for every finite set $J' \subseteq I$ that contains J.

The case $I = \mathbb{N}$ is of particular importance and for this case, we also have two further definitions of "series convergence" available which we introduced in Definition 1.1.13. In the following remark we compare all three notions.

Remark 2.2.13. Let $(V, \|\cdot\|)$ be a normed vector space, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in V and consider the following assertions:

(i) The series over $(x_n)_{n\in\mathbb{N}}$ is absolutely convergent.

(ii) The unconditional series over $(x_n)_{n\in\mathbb{N}}$ is convergent.

(iii) The series over $(x_n)_{n\in\mathbb{N}}$ is convergent.

We always have (ii) $\Rightarrow$ (iii). In this case we have $\sum_{n\in\mathbb{N}} x_n = \sum_{n=1}^{\infty} x_n$. In case that $(V, \|\cdot\|)$ is a Banach space the implication (i) $\Rightarrow$ (ii) holds, too. These assertions are not difficult to prove.

On the other hand, one can show that the converse implications are in general false, may $(V, \|\cdot\|)$ be a Banach space or not.

Remark 2.2.14. Let $(V, \|\cdot\|)$ be a normed vector space, let $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in V. It is possible to prove that the following assertions are equivalent:

(i) $\sum_{n\in\mathbb{N}} x_n = x$.

(ii) $\sum_{n=1}^{\infty} x_{\varphi(n)} = x$ for every bijection $\varphi : \mathbb{N} \to \mathbb{N}$.

Proposition 2.2.15. *Let $(V, \|\cdot\|)$ be a normed vector space and consider a family $(x_i)_{i\in I}$ of elements $x_i \in V$. If the unconditional sequence over $(x_i)_{i\in I}$ converges, then x_i is 0 for all but at most countably many indices i.*

Proof. Let $n \in \mathbb{N}$. It suffices to prove that there are only finitely many $i \in I$ for which we have $\|x_i\| \geq \frac{1}{n}$. As usual, let $\mathcal{F}$ denote the directed set of all finite subsets of I, endowed with the relation $\subseteq$. Since the net $(\sum_{i\in F} x_i)_{F\in\mathcal{F}}$ converges, it is a Cauchy net. Hence, there exists a set $F_0 \in \mathcal{F}$ such that

$$\|\sum_{i\in F_1} x_i - \sum_{i\in F_2} x_i\| < \frac{1}{n}$$

for all finite sets $F_1, F_2 \subseteq I$ that contain F_0. Now, let $i_0 \in I \setminus F_0$. If we set $F_2 := F_0$ and $F_1 := F_0 \cup \{i_0\}$ in the above inequality, then we obtain $\|x_{i_0}\| < \frac{1}{n}$. As F_0 is finite and $i_0 \in I \setminus F_0$ was arbitrary, we obtain the assertion. □

2.3 Orthonormal bases

Definition 2.3.1. Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space. A subset $E \subseteq V$ is called an *orthonormal system* in V if we have

$$\langle e, f \rangle = \begin{cases} 1 & \text{if } e = f, \\ 0 & \text{if } e \neq f \end{cases}$$

for all $e, f \in V$.

Theorem 2.3.2. *Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space. For every orthonormal system $E \subseteq V$ the following assertions are equivalent:*

(i) *The linear hull of E is dense in V.*

(ii) *We have $v = \sum_{e\in E} \langle e, v\rangle e$ for all $v \in V$ (this identity is often called the* Fourier expansion *of v).*

(iii) *We have $\langle v, w\rangle = \sum_{e\in E} \langle v, e\rangle\langle e, w\rangle$ for all $v, w \in V$.*

(iv) *We have $\|v\|^2 = \sum_{e\in E} |\langle e, v\rangle|^2$ for all $v \in V$ (this equation is often called* Parseval's identity*).*

Definition 2.3.3. Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space. A orthonormal system $E \subseteq V$ which fulfils the equivalent assertions Theorem 2.3.2 is called an *orthonormal basis* of V.

For the proof of Theorem 2.3.2 we need the following two lemmas and the subsequent proposition.

Lemma 2.3.4. *Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space over the scalar field $\mathbb{K}$. Then the inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$ is continuous.*

Proof. We leave this as an exercise to the reader. □

Lemma 2.3.5. *Let $(V, \|\cdot\|)$ and $(W, \|\cdot\|)$ be normed vector spaces and let $(T_j)_{j\in J}$ be a net of bounded linear operators from V to W. If $\sup_{j\in J}\|T_j\| < \infty$, then the set*

$$\{v \in V : \lim_j T_j v = 0\}$$

is closed in V.

Proof. This is another easy exercise. □

Proposition 2.3.6. *Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space, let $E \subseteq V$ be a orthonormal system in V and let $v \in V$. Then the unconditional series over $(|\langle e, v\rangle|^2)_{e\in E}$ converges and we have* Bessel's inequality

$$\sum_{e\in E} |\langle e, v\rangle|^2 \leq \|v\|^2.$$

Moreover, the following assertions are equivalent:

(i) *Bessel's inequality is an equality.*

(ii) *The vector v is contained in the closure of the linear hull of E.*

(iii) *We have $v = \sum_{e\in E} \langle e, v\rangle e$.*

Proof. As usual, we denote the set of finite subsets of E by $\mathcal{F}$ and we endow $\mathcal{F}$ with the relation $\subseteq$. For every $F \in \mathcal{F}$ we first compute that

$$\begin{aligned} 0 \leq \|v - \sum_{e\in F} \langle e, v\rangle e\|^2 &= \Big\langle v - \sum_{e\in F} \langle e, v\rangle e, v - \sum_{e\in F} \langle e, v\rangle e \Big\rangle \\ &= \|v\|^2 - \sum_{e\in F} |\langle e, v\rangle|^2 - \sum_{e\in F} |\langle e, v\rangle|^2 + \sum_{e\in F}\sum_{f\in F} \langle v, e\rangle \cdot \langle f, v\rangle \cdot \langle e, f\rangle \\ &= \|v\|^2 - \sum_{e\in F} |\langle e, v\rangle|^2, \end{aligned}$$

so $\sum_{e\in F}|\langle e,v\rangle|^2 \le \|v\|^2$. Hence, $s := \sup_{F\in\mathcal{F}} \sum_{e\in F}|\langle e,v\rangle|^2 \le \|v\|^2$. As $|\langle e,v\rangle|^2$ is non-negative for every e, one easily concludes that the unconditional series $\left(\sum_{e\in F}|\langle e,v\rangle|^2\right)_{F\in\mathcal{F}}$ converges to s, so $\sum_{e\in E}|\langle e,v\rangle|^2 = s \le \|v\|^2$, which proves Bessel's inequality.

We have equality in Bessel's inequality if and only if the net $\Big(\|v\|^2 - \sum_{e\in F}|\langle e,v\rangle|^2\Big)_{F\in\mathcal{F}}$ converges to 0 if and only if the net $\Big(\|v-\sum_{e\in F}\langle e,v\rangle e\|\Big)_{F\in\mathcal{F}}$ converges to 0 if and only if the net $(\sum_{e\in F}\langle e,v\rangle e)_{F\in\mathcal{F}}$ converges to v, so the equivalence of (i) and (iii) is proved.

The implication "(iii) $\Rightarrow$ (ii)" is obvious since one can easily check that a closed subset of a metric space is closed with respect to limits of nets. So it remains to prove "(ii) $\Rightarrow$ (iii)".

Let L denote the linear hull of E. First one checks by a direct computation that the equality $w = \sum_{e\in E}\langle e,w\rangle e$ is true for every $w \in L$. For each $F \in \mathcal{F}$ we now define a linear operator $T_F : V \to V$ which is given by

$$T_F w := w - \sum_{e\in F}\langle e,w\rangle e.$$

For every $w \in L$ the net $(T_F w)_{F\in\mathcal{F}}$ converges to 0. Moreover, for each $F \in \mathcal{F}$ the linear operator T_F is bounded with operator norm $\|T_f\| \le 2$. This follows from Bessel's inequality applied to the orthonormal system F.

Hence, $(T_F)_{F\in\mathcal{F}}$ is a net of linear operators which fulfils $\sup_{F\in\mathcal{F}}\|T_F\| < \infty$. It thus follows from Lemma 2.3.5 that the net $(T_F w)_{F\in\mathcal{F}}$ converges to 0 on the closure of L. In particular, $v = \sum_{e\in E}\langle e,v\rangle e$ since v is contained in this closure. □

Proof of Theorem 2.3.2. "(i) $\Rightarrow$ (ii)" This follows from the implication (ii) $\Rightarrow$ (iii) in Proposition 2.3.6.

"(ii) $\Rightarrow$ (iii)" This implication is an immediate consequence of the continuity of the inner product in the second component and of the (anti-)linearity of the inner product.

"(iii) $\Rightarrow$ (iv)" Put $w := v$ in (iii) to obtain (iv).

"(iv) $\Rightarrow$ (i)" This is an immediate consequence of the implication (iii) $\Rightarrow$ (ii) in Proposition 2.3.6. □

Example 2.3.7. Let $\mathbb{K} \in \{\mathbb{R};\mathbb{C}\}$, let $I \neq \emptyset$ be an arbitrary set and define

$$\begin{aligned}&\ell^2(I;\mathbb{K})\\ &\quad := \{x = (x_i)_{i\in I} : \text{ the unconditional series over } (|x_i|^2)_{i\in I} \text{ converges in } \mathbb{R}\}.\end{aligned}$$

One can show that the conditional series over $(x_i y_i)_{i\in I}$ converges for all $x, y \in \ell^2(I;\mathbb{K})$ and that $\langle x,y\rangle =: \sum_{i\in I} x_i y_i$ defines an inner product an H. Moreover, the pre-Hilbert space $(\ell^2(I;\mathbb{K}),\langle\cdot,\cdot\rangle)$ is a Hilbert space. Compare Exercise 9 and the Exercise Sheet 4 where this was proved in the special case $I = \mathbb{N}$.

For each $i \in I$ we denote by $e_i \in \ell^2(I;\mathbb{K})$ the i-th canonical unit vector, i.e. the vector whose components are all 0 except for the i-th component which is 1. Then $\{e_i : i \in I\}$ is an orthonormal basis of $\ell^2(I;\mathbb{K})$.

Remark 2.3.8. The above example shows that an orthonormal basis of a (pre-)Hilbert space is not an algebraic basis, in general. For instance, not every vector in $\ell^2(\mathbb{N};\mathbb{R})$ is a linear combination of $\{e_n : n \in \mathbb{N}\}$.

Theorem 2.3.9 (Fischer-Riesz). *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space over the scalar field $\mathbb{K}$ and let E be an orthogonal basis of H. Then the mapping*

$$J: H \to \ell^2(E;\mathbb{K}), \quad x \mapsto (\langle e, x\rangle)_{e\in E}$$

is linear and bijective and we have $\|Jx\| = \|x\|$ for all $x \in H$.

Proof. It follows from Parseval's identity that the family $(\langle e, x\rangle)_{e\in E}$ is indeed contained in $\ell^2(E;\mathbb{K})$ for all $x \in H$ and that we have $\|Jx\| = \|x\|$ for all $x \in H$. In particular, J is injective. Obviously, J is linear.

To see that J is also surjective, let $\alpha = (\alpha_e)_{e\in E} \in \ell^2(E;\mathbb{K})$. Let us that the conditional series over the family $(\alpha_e e)_{e\in E}$ in H converges in H. Let $\mathcal{F}$ be the set of all finite subsets of E, endowed with the order $\subseteq$. The net $\left(\sum_{e\in F}|\alpha_e|^2\right)_{F\in\mathcal{F}}$ is a Cauchy net since $\alpha \in \ell^2(E;\mathbb{K})$ and for all $F, G \in \mathcal{F}$ we have

$$\|\sum_{e\in F}\alpha_e e - \sum_{e\in G}\alpha_e e\|^2 = \sum_{e\in F\Delta G}|\alpha_e|^2.$$

Hence, the net $\left(\sum_{e\in F}\alpha_e e\right)_{F\in\mathcal{F}}$ is a Cauchy net in H and thus convergent is H is a Hilbert space. Set $x := \sum_{e\in E}\alpha_e e$. Then one readily checks that $Jx = \alpha$, so J is indeed surjective. $\square$

2.4 Optimal approximation

Definition 2.4.1. Let $(V, \|\cdot\|)$ be a normed vector space and let $x \in V$ and $S \subseteq V$. And element $z \in S$ is called a *proximum* of x in S if $\|x - y\| \geq \|x - z\|$ for all $y \in S$.

Recall that a subset C of a real or complex vector space V is called *convex* if $\lambda v + (1-\lambda)w \in C$ for all $v, w \in C$ and all $\lambda \in [0,1]$. Equivalently, we have $\sum_{k=1}^n \lambda_k v_k \in C$ for all $n \in \mathbb{N}$, all $v_1, \dots, v_n \in C$ and all $\lambda_1, \dots, \lambda_n \geq 0$ that fulfils $\sum_{k=1}^n \lambda_k = 1$.

Theorem 2.4.2. *Let $(V, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space, let $x \in V$ and let $C \subseteq V$ be a convex set.*

(a) *There exists at most one proximum of x in C.*

(b) *If* $(V, \langle \cdot, \cdot \rangle)$ *is even a Hilbert space and* C *is closed, then there exists a proximum of* x *in* C.

Proof. (b) Define $s := \inf\{\|x - y\| : y \in C\}$. Then we can find a sequence $(y_n)_{n\in\mathbb{N}}$ in C such that $\lim_{n\to\infty}\|x - y_n\| = s$. Define $z_n := x - y_n$ for each $n \in \mathbb{N}$. We show that $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, so let $\varepsilon > 0$.

For all sufficiently large $n \in \mathbb{N}$, $n \geq n_0$ say, we have $\|z_n\|^2 < s^2 + \frac{\varepsilon^2}{4}$. For all $n, m \geq n_0$ we thus obtain from the parallelogram identity that

$$\begin{aligned} \|z_n - z_m\|^2 &= 2\|z_n\|^2 + 2\|z_m\|^2 - \|z_n + z_m\|^2 \\ &= 2\|z_n\|^2 + 2\|z_m\|^2 - 4\|\frac{z_n + z_m}{2}\|^2 \\ &= 2\|z_n\|^2 + 2\|z_m\|^2 - 4\|x - \frac{y_n + y_m}{2}\|^2 \\ &< 4(s^2 + \frac{\varepsilon^2}{4}) - 4s^2 = \varepsilon^2; \end{aligned}$$

for the inequality between the third and the fourth line we used that $\frac{y_n+y_m}{2} \in C$ since C is convex. Hence, $\|z_n - z_m\| < \varepsilon$ for all $m, n \geq n_0$, so $(z_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. As H is complete, it thus follows that the sequence $(z_n)_{n\in\mathbb{N}}$ converges and hence, so does the sequence $(y_n)_{n\in\mathbb{N}}$. If $y_0 \in H$ denotes the limit of the latter sequence, then we have $\|x - y_0\| = \lim_{n\to\infty}\|x - y_n\| = s$. Moreover, y_0 is contained in C since C is closed. Hence, y_0 is a proximum of x in C.

(a) Define s as in the proof of (b). Let y_1 and y_2 be proxima of x in C and define $z_1 = x - y_1$ and $z_2 = x - y_2$. Then a similar computation as in (b) shows that

$$0 \leq \|z_1 - z_2\|^2 = 2\|z_1\|^2 + 2\|z_2\|^2 - 4\|x - \frac{y_1 + y_2}{2}\|^2 \leq 2s^2 + 2s^2 - 4s^2 = 0.$$

Hence, $z_1 = z_2$ and thus, $y_1 = y_2$. □

The following simple characterisation of proxima in convex sets is often quite useful.

Proposition 2.4.3. *Let* $(V, \langle \cdot, \cdot \rangle)$ *be a pre-Hilbert space, let* $C \subseteq V$ *be convex and let* $x, z \in V$. *Then the following assertions are equivalent:*

(i) *The vector* z *is the proximum of* x *in* C.

(ii) *We have* $z \in C$ *and* $\operatorname{Re}\langle x - z, y - z \rangle \leq 0$ *for all* $y \in C$.

Proof. We leave the proof as an exercise. □

The case where the convex subset C is a vector subspace is of particular interest.

Corollary 2.4.4. *Let $(V, \langle\cdot,\cdot\rangle)$ be a pre-Hilbert space, let $W \subseteq V$ be a vector subspace of V and let $x, z \in V$. Then the following assertions are equivalent:*

(i) *The vector z is the proximum of x in W.*

(ii) *We have $z \in W$ and $x - z$ is orthogonal to all elements of W.*

Proof. It suffices to show that assertion (ii) is equivalent to assertion (ii) in Proposition (ii) (for $C := W$). For both implications we may assume that $z \in W$.

If $x - z$ is orthogonal to every element of W, then we have $\langle x - z, y - z\rangle = 0$ and hence $\operatorname{Re}\langle x - z, y - z\rangle = 0$ for all $y \in W$ since $y - z \in W$ for each such y.

Now assume on the other hand that $\operatorname{Re}\langle x - z, y - z\rangle \leq 0$ for all $y \in W$ and let $\tilde{y} \in W$. Then we have

$$\operatorname{Re}\langle x - z, \tilde{y}\rangle = \operatorname{Re}\langle x - z, (\tilde{y} + z) - z\rangle \leq 0$$

since $\tilde{y} + z \in W$. Since W is a vector subspace, we conclude from this that we also have $\operatorname{Re}\langle x - z, -\tilde{y}\rangle \leq 0$, so actually $\operatorname{Re}\langle x - z, \tilde{y}\rangle = 0$. If the scalar field is real, the proof is finished, so assume that the scalar field is complex.

Then we can find a real number θ such that $e^{i\theta}\langle x - z, \tilde{y}\rangle \in \mathbb{R}$. Since W is a vector subspace of V, the vector $e^{i\theta}\tilde{y}$ is also contained in W and thus, it follows from what we have just proved that

$$0 = \operatorname{Re}\langle x - z, e^{i\theta}\tilde{y}\rangle = \operatorname{Re}\left(e^{i\theta}\langle x - z, \tilde{y}\rangle\right) = e^{i\theta}\langle x - z, \tilde{y}\rangle.$$

Therefore, $\langle x - z, \tilde{y}\rangle = 0$ as claimed. □

Corollary 2.4.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let $G \subseteq H$ be closed vector subspace. The set*

$$G^\perp := \{x \in H : \langle x, y\rangle = 0 \textit{ for all } y \in G\}$$

is a vector subspace of H and we have $H = G \oplus G^\perp$.

Proof. Obviously, $G^\perp$ is a vector subspace of H (this is actually for every subset G of H and not only for closed vector subspaces).

Since 0 is the only vector which is orthogonal to itself, we have $G \cap G^\perp = \{0\}$. Now, let $z \in H$; according to Theorem 2.4.2(b), there exists a proximum y of z in G and according to Corollary 2.4.4, the vector $x := z - y$ is orthogonal to all elements of G. Hence, $z = y + x$ where $y \in G$ and $x \in G^\perp$. □

As a consequence we obtain the following characterisation of orthonormal basis in Hilbert spaces:

Theorem 2.4.6. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let E be a orthonormal system in H. The following assertions are equivalent:*

(i) *The orthonormal system E is an orthonormal basis of H.*

(ii) *The orthonormal system E is maximal, meaning that there exists no orthonormal system $F \subseteq V$ which fulfils $F \supsetneq E$.*

(iii) *No vector in V except for 0 is orthogonal to all elements of E.*

Proof. We first show that "(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii)" holds even if $(H, \langle \cdot, \cdot \rangle)$ is only a pre-Hilbert space.

"(ii) $\Rightarrow$ (iii)" Suppose that E is a maximal orthonormal system. If there was a non-zero vector $v \in H$ that is orthogonal to all elements of E, then $E \cup \{v/\|v\|\}$ would be a orthonormal system that strictly contains E, which contradicts the maximality of E.

"(iii) $\Rightarrow$ (ii)" Now assume on the other hand that no vector in H except for 0 is orthogonal to all elements of E. If F was an orthonormal system in F fulfilling $F \supsetneq E$, then we could find a vector $f \in F \setminus E$ and this vector would be non-zero and orthogonal to each element of E, which contradicts our assumption. Hence, E is maximal.

"(i) $\Rightarrow$ (iii)" Assume that E is an orthonormal basis of H let $v \in H$ be orthogonal to each $e \in E$. According to Theorem 2.3.2(ii) we have

$$v = \sum_{e \in E} \langle e, v \rangle e = \sum_{e \in E} 0 = 0,$$

which proves (iii).

Finally, we show that implication "(iii) $\Rightarrow$ (i)" in case that $(H, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Assume that (iii) holds. According to Theorem 2.3.2(i) we only have to show that the linear hull of E is dense in H. So let L denote the linear hull of E and suppose to the contrary that its closure $\overline{L}$ is a proper subset of H. It is easy to see that $\overline{L}$ is itself a vector subspace of H and hence, we have $H = \overline{L} \oplus \overline{L}^{\perp}$ according to Corollary 2.4.5. In particular, $\overline{L}^{\perp}$ contains a non-zero element v since $\overline{L} \neq H$. The vector v is orthogonal to each element of $\overline{L}$ and thus, in particular, to each element of E. □

It is a consequence of Zorn's Lemma that we can find a maximal orthonormal system in every pre-Hilbert space. In conjuction with the above theorem, this yields that every Hilbert space possesses an orthonormal basis. More generally, we have the following result:

Theorem 2.4.7. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $F \subseteq H$ be an orthonormal system. Then there exists an orthonormal basis E of H which contains F. In particular, there exists an orthonormal basis in H.*

Proof. This is a consequence of Theorem 2.4.6(ii) and Zorn's Lemma. We leave the details as an exercise. □

Remark 2.4.8. (a) One can show that there exists a pre-Hilbert space which does not possess a orthonormal basis.

(b) Using Gram–Schmidt's orthogonalisation algorithm one can, on the other hand, prove that a pre-Hilbert-space $(V, \langle \cdot, \cdot \rangle)$ always possesses a orthonormal basis provided that there exists a countable dense subset of V. This assumption is fulfilled for many concrete examples of pre-Hilbert spaces.

2.5 Duality and the Riesz–Fréchet representation theorem

Definition 2.5.1. Let $(V, \|\cdot\|)$ be a normed vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then we call the operator space $V' := \mathcal{L}(V; \mathbb{K})$ the *dual space* of V. Each operator in V' is called a *bounded linear functional* on V.

Proposition 2.5.2. *Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space over the scalar field $\mathbb{K}$ and let $x \in V$. Then the mapping $\varphi_x : V \to \mathbb{K}$, $y \mapsto \varphi_x(y) := \langle x, y \rangle$ is a bounded linear functional on V and we have $\|\varphi_x\| = \|x\|$.*

Proof. All assertions are clear in case that $x = 0$, so let $x \neq 0$.

It follows from the definition of the inner product that the mapping φ_x is linear. Moreover, we have $|\varphi_x(y)| = |\langle x, y \rangle| \leq \|x\| \cdot \|y\|$ for each $y \in V$ due to the Cauchy–Schwarz inequality. Hence, φ_x is bounded and we have $\|\varphi_x\| \leq \|x\|$. On the other hand, $\|\varphi_x\| \cdot \|x\| \geq |\varphi_x(x)| = |\langle x, x \rangle| = \|x\| \cdot \|x\|$, so $\|\varphi_x\| = \|x\|$. □

On Hilbert spaces, *every* bounded linear functional is of the above form. For applications, this is one of the most important results in Hilbert space theory.

Theorem 2.5.3 (Riesz–Fréchet Representation Theorem). *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $\varphi \in H'$. Then there exists exactly one vector $x \in H$ such that $\varphi(y) = \langle x, y \rangle$ for all $y \in H$. Moreover, we have $\|x\| = \|\varphi\|$.*

In the proof we are going to use the following proposition use easy proof we leave to the reader:

Proposition 2.5.4. *Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space and let $x_1, x_2 \in H$. Then the following assertions are equivalent:*

(i) $x_1 = x_2$.

(ii) $\langle x_1, y \rangle = \langle x_2, y \rangle$ *for all* $y \in H$

(iii) $\langle y, x_1 \rangle = \langle y, x_2 \rangle$ *for all* $y \in H$.

Proof. We clearly have "(i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii)". To show "(ii) $\Rightarrow$ (i)", choose $y = x_1 - x_2$ and compute

$$\|x_1 - x_2\|^2 = \langle x_1 - x_2, y\rangle = \langle x_1, y\rangle - \langle x_2, y\rangle = 0.$$

Hence, $x_1 = x_2$. $\square$

Now we prove the Theorem of Riesz-Fréchet.

Proof of Theorem 2.5.3. The assertion is obvious if $\varphi = 0$, so assume that $\varphi \neq 0$. Define $G := \ker \varphi$. Then G is a closed vector subspace of H and hence, we have $H = G \oplus G^\perp$ according to Corollary 2.4.5.

We have $G \neq H$ as $\varphi \neq 0$ and hence, $G^\perp \neq 0$. Thus, there exists a vector $0 \neq v \in G^\perp$ and we define $x := \frac{\overline{\varphi(v)}}{\|v\|^2} v \in G^\perp$. Note that $\varphi(v) \neq 0$ since $v \notin G$ and hence, $x \neq 0$. Moreover, we have

$$\|x\|^2 = \frac{|\varphi(v)|^2}{\|v\|^2} = \varphi(v).$$

As the restriction of φ to $G^\perp$ is an injective linear mapping from $G^\perp$ to the scalar field, it follows that $G^\perp$ is one-dimensional, so every non-zero element of $G^\perp$ is a multiple of x. Now, let $y \in H$. Then we can decompose y as $y = g + \alpha x$, where $g \in G$ and where α is a scalar. Hence, we obtain

$$\varphi(y) = \varphi(g) + \alpha\varphi(x) = \alpha\varphi(x) = \langle x, g\rangle + \alpha\|x\|^2 = \langle x, g + \alpha x\rangle = \langle x, y\rangle.$$

The proves the existence of x as claimed. Uniqueness follows from Proposition 2.5.4. Finally, we know from Proposition 2.5.2 that $\|\varphi\| = \|x\|$. $\square$

Theorem 2.5.5. *Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. For each operator $T \in \mathcal{L}(H)$ there exists exactly one operator in $\mathcal{L}(H)$, which we denote by T^*, that fulfils $\langle x, Ty\rangle = \langle T^*x, y\rangle$ for all $x, y \in H$.*

Moreover, the following assertions are fulfilled:

(a) *We have $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for each $T \in \mathcal{L}(H)$.*

(b) *We have $(T^*)^* = T$ for each $T \in \mathcal{L}(H)$.*

(c) *We have $\|T^*\| = \|T\|$ for each $T \in \mathcal{L}(H)$.*

(d) *We have $\|T^*T\| = \|TT^*\| = \|T\|^2$ for each $T \in \mathcal{L}(H)$.*

(e) *We have $(\lambda T + \mu S)^* = \overline{\lambda}T^* + \overline{\mu}S^*$ for all $T, S \in \mathcal{L}(H)$ and all scalars λ, μ.*

(f) *We have $(TS)^* = S^*T^*$ for all $T, S \in \mathcal{L}(H)$.*

Definition 2.5.6. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert-space. For every $T \in \mathcal{L}(H)$ the operator $T^* \in \mathcal{L}(H)$ is called the *adjoint operator*, or briefly: the *adjoint*, of T.

Proof of Theorem 2.5.5. For each $x \in H$, consider the mapping ψ_x from H into the scalar field which given by $\psi_x(y) = \langle x, Ty\rangle$ for each $y \in H$. This is a bounded linear functional and hence, due to Theorem 2.5.3, there exists a uniquely determined vector $z_x \in H$ such that $\psi_x(y) = \langle z_x, y\rangle$ for all $y \in H$. We define a mapping $T^* : H \to H$ by $T^*x = z_x$ for all $x \in H$. Then we have indeed

$$\langle x, Ty\rangle = \langle T^*x, y\rangle$$

for all $x, y \in H$. Moreover, T^* is linear, for if x_1, x_2 are vectors in H and if λ_1, λ_2 are scalars, then

$$\begin{aligned}\langle T^*(\alpha_1 x_1 + \alpha_2 x_2), y\rangle = \langle \alpha_1 x_1 + \alpha_2 x_2, Ty\rangle &= \overline{\alpha_1}\langle x_1, Ty\rangle + \overline{\alpha_2}\langle x_2, Ty\rangle \\ &= \overline{\alpha_1}\langle T^*x_1, y\rangle + \overline{\alpha_2}\langle T^*x_2, y\rangle = \langle \alpha_1 T^*x_1 + \alpha_2 T^*x_2, y\rangle\end{aligned}$$

for all $y \in H$. Hence, it follows from Proposition 2.5.4 that $T^*(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T^*x_1 + \alpha_2 T^*x_2$.

Uniqueness of T^* follows from Proposition 2.5.4, too. Clearly, $T^* = 0$ if and only if $T = 0$; hence, in order to show that is T^* is bounded and that $\|T\| = \|T^*\|$, we may assume that $T \neq 0$ and $T^* \neq 0$. In this case we obtain for each $x \in H$ the estimates

$$\|T^*x\|^2 = \langle x, TT^*x\rangle \le \|x\|\|T\|\|T^*x\| \quad \text{and} \quad \|Tx\|^2 = \langle T^*Tx, x\rangle \le \|T^*\|\|Tx\|\|x\|$$

and hence

$$\|T^*x\| \le \|T\|\|x\| \quad \text{and} \quad \|Tx\| \le \|T^*\|\|x\|.$$

This proves that T^* is bounded and that assertion (c) holds. Assertion (a) follows from the symmetry if the inner product and assertion (b) follows from (a) and from the uniqueness of $(T^*)^*$. Assertion (d) can be shown similarly as (d), and assertions (e) and (f) follow immediately from the definition of the adjoint operator and from Proposition 2.5.4. □

2.6 Orthogonal projections

We denote the *image* of a linear mapping T by $\operatorname{im} T$; the identity mapping from a vector space V to itself is denoted by I_V or simply by I. If V is a vector space and $T : V \to V$ is linear, then we define $T^2 := T \circ T$.

Definition 2.6.1. Let $(V, \|\cdot\|)$ be a normed vector space.

(i) A bounded linear operator $P \in \mathcal{L}(V)$ is called a *projection* if $P^2 = P$.

(ii) If $P \in \mathcal{L}(V)$ is a projection, the we call the operator $Q := I - P$ the *complementary projection to P*.

Note the the complementary projection $Q = I - P$ is indeed a projection since $Q^2 = I^2 - IP - PI + P^2 = I - 2P + P = I - P = Q$.

In the following two propositions, use easy proves we leave to the reader, we collect some elementary information about projections.

Proposition 2.6.2. *Let $(V, \|\cdot\|)$ be a normed vector space, let $P \in \mathcal{L}(V)$ be a projection and let $Q := I - P$ denote the complementary projection. Then the following assertions hold.*

(a) *We have* $\ker P = \operatorname{im} Q$ *and* $\ker Q = \operatorname{im} P$.

(b) *The vector subspaces* $\ker P$ *and* $\operatorname{im} P$ *of V are both closed.*

(c) *We have* $V = \ker P \oplus \operatorname{im} P$.

(d) *Let $v \in V$ and decompose v as $v = x + y$, where $x \in \ker P$ and $y \in \operatorname{im} P$. Then $y = Pv$ and $x = Qv$.*

(e) *If $P \neq 0$, then $\|P\| \geq 1$.*

Proposition 2.6.3. *Let $(V, \|\cdot\|)$ be a normed vector space. An operator $P \in \mathcal{L}(V)$ is a projection if and only if $Px = x$ for all $x \in \operatorname{im} P$.*

Definition 2.6.4. Let $(V, \|\cdot\|)$ be a normed vector space. A vector subspace W of V is called *complemented* in V if there exists a projection $P \in \mathcal{L}(V)$ with range W.

It follows from Proposition 2.6.2(b) that a complemented subspace is always closed. On the other hand, a closed subspace of a normed vector space – or even of a Banach space – need not be complemented, in general. However, we shall see below that the situation is much simpler in Hilbert spaces.

Proposition 2.6.5. *Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space and let $P \in \mathcal{L}(V)$ be a projection. Then the following assertions are equivalent:*

(i) *Every element of* $\operatorname{im} P$ *is orthogonal to every element of* $\ker P$.

(ii) $\|P\| \leq 1$.

Proof. Denote the complementary projection of P by $Q = I - P$.

"(i) $\Rightarrow$ (ii)" Assume (i) and let $v \in V$. Then $v = Pv + Qv$ and $Pv \perp Qv$. It thus follows from Pythagoras' Theorem that

$$\|v\|^2 = \|Pv\|^2 + \|Qv\|^2 \geq \|Pv\|^2,$$

so $\|Pv\| \leq \|v\|$, which proves that $\|P\| \leq 1$.

"(ii) $\Rightarrow$ (i)" Suppose the $\|P\| \le 1$ and let $y \in \ker P$, $x \in \operatorname{im} P$. For each scalar α we have $\alpha y \in \ker P$ and hence

$$\|x + \alpha y\| \ge \|P(x + \alpha y)\| = \|x\|.$$

According to Exercise 10 on Exercise Sheet 4, this implies that $x \perp y$. $\square$

Definition 2.6.6. Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert space. A projection $P \in \mathcal{L}(V)$ is called an *orthogonal projection* if the equivalent conditions (i) and (ii) in Proposition 2.6.5 are fulfilled.

Proposition 2.6.7. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $P \in \mathcal{L}(H)$ be a projection. Then the following assertions are equivalent:*

(i) *P is an orthogonal projection.*

(ii) *$P^* = P$.*

(iii) *$P^*P = PP^*$.*

Proof. "(i) $\Rightarrow$ (ii)" Let P be an orthogonal projection and let $x, y \in H$. According to Proposition 2.6.2(c) we can decompose those two vectors as $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in \operatorname{im} P$ and $x_2, y_2 \in \ker P$. Hence, we obtain

$$\langle Px, y \rangle = \langle Px_1 + Px_2, y \rangle = \langle x_1, y \rangle = \langle x_1, y_1 \rangle + \langle x_1, y_2 \rangle = \langle x_1, y_1 \rangle$$

and similarly

$$\langle P^*x, y \rangle = \langle x, Py \rangle = \langle x_1 + x_2, y_1 \rangle = \langle x_1, y_1 \rangle.$$

Hence, $\langle Px, y \rangle = \langle P^*x, y \rangle$ for all $x, y \in H$ which implies, according to Proposition 2.5.4, that $Px = P^*x$ for all $x \in H$. Therefore, $P = P^*$.

"(ii) $\Rightarrow$ (iii)" This implication is obvious.

"(iii) $\Rightarrow$ (i)" Assume that $P^*P = PP^*$. We first note that $\ker P = \ker P^*$. Indeed, if $x \in \ker P$, then we have

$$\|P^*x\|^2 = \langle P^*x, P^*x \rangle = \langle x, PP^*x \rangle = \langle x, P^*Px \rangle = \langle Px, Px \rangle = \|Px\|^2 = 0.$$

Thus, $\ker P \subseteq \ker P^*$. A similar computation yields the converse inclusion, so $\ker P = \ker P^*$.

Now, let $x \in \ker P$, $y \in \operatorname{im} P$. Then we can find a vector $z \in H$ such that $y = Pz$ and hence,

$$\langle x, y \rangle = \langle x, Pz \rangle = \langle P^*x, z \rangle = \langle 0, z \rangle = 0.$$

Hence, all elements of $\ker P$ are orthogonal to all elements of $\operatorname{im} P$. $\square$

It is worthwhile pointing out that in the proof of implication "(iii) $\Rightarrow$ (i)" above we did not use that P is a projection, i.e. we showed that following result: Whenever $T \in \mathcal{L}(H)$ fulfils the inequality $T^*T = TT^*$, then all elements of $\ker T$ are orthogonal to all elements of $\operatorname{im} T$.

Theorem 2.6.8. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and let $G \subseteq H$ be a closed vector subspace.*

(a) *There exists exactly one orthogonal projection $P \in \mathcal{L}(H)$ with range G; in particular, G is complemented. Moreover, we have $\ker P = G^\perp$.*

(b) *For every $x \in H$ the vector Px is the proximum of x in G.*

(c) *Let F be an orthonormal basis of G (which is a Hilbert space with respect to the inner product inherited from H). Then we have*

$$Px = \sum_{e \in F} \langle e, x \rangle e$$

for all $x \in H$.

Proof. (a) According to Corollary 2.4.5 we have $H = G \oplus G^\perp$. Hence, for each $z \in H$ we can find uniquely determined vectors $x_z \in G$ and $y_z \in G^\perp$ such that $z = x_z + y_z$. We define a mapping $P : H \to H$ by $Pz = x_z$ for all $z \in H$. Then one readily checks that P is linear and that $P^2 = P$. Moreover, we have $\|Pz\|^2 \leq \|x_z\|^2 + \|y_z\|^2 = \|z\|^2$ for all $z \in H$, so $\|P\| \leq 1$, meaning that the projection P is in fact an orthogonal projection. It is easy to see that $\operatorname{im} P = G$.

Next we note that every orthogonal projection $Q \in \mathcal{L}(H)$ with range G has kernel $\ker Q = G^\perp$. We clearly have $\ker Q \subseteq G^\perp$. On the other hand, we have $\operatorname{im} Q \oplus \ker Q = H = G \oplus G^\perp = \operatorname{im} Q \oplus G^\perp$. In conjuction with the inclusion $\ker Q \subseteq G^\perp$ this immediately implies $\ker Q = G^\perp$.

In particular, we obtain $\ker P = G^\perp$. If $Q \in \mathcal{L}(H)$ is a second orthogonal projection with range G, then its follows from what we have just shown that $\ker Q = G^\perp = \ker P$; hence, the ranges of P and Q coincide. It is, however, easy to see that to projections (no matter whether orthogonal or not) coincide if and only if both the ranges and their kernels coincide. Hence, $P = Q$, so we proved the uniqueness of P.

(b) Let $x \in H$ and $g \in G$. The vector $x - Px$ is contained in the kernel of P and hence, its is orthogonal to all elements of G. Since $Px - g$ is contained in G, it is thus orthogonal to $x - Px$ and hence we obtain

$$\|x - g\|^2 = \|(x - Px) + (Px - g)\|^2 = \|x - Px\|^2 + \|Px - g\|^2 \geq \|x - Px\|^2.$$

Thus, Px is indeed the proximum of x in G.

(c) First note that $Pe = e$ for every $e \in E$. For each$g \in G$ we have the Fourier expansion

$$g = \sum_{e \in F} \langle e, g \rangle e.$$

Now, let $x \in H$. Then $Px \in G$ and hence,

$$Px = \sum_{e \in F} \langle e, Px \rangle e = \sum_{e \in F} \langle e, x \rangle e$$

since $\langle e, Px \rangle = \langle P^* e, x \rangle = \langle Pe, x \rangle = \langle e, x \rangle$ for every $e \in E$. □

Elliptic PDEs on the interval 3

Opening Questions. (a) Is there a way to "differentiate" the non-differentiable function $(-1,1) \ni x \mapsto |x| \in \mathbb{R}$? If so, how should its derivative look like?

(b) Consider the two differential equations

$$\begin{cases} x''(t) = x(t) & \text{for } t \in [0,1], \\ x(0) = 2, \\ x'(0) = 3 \end{cases} \qquad \text{and} \qquad \begin{cases} x''(t) = x(t) & \text{for } t \in [0,1], \\ x(0) = 2, \\ x(1) = 3. \end{cases}$$

What is the difference between both problems? Do both of them possess a solution?

3.1 Distributions and Sobolev spaces in one dimension

In this section we present a generalisation of the derivative of a function from classical analysis. The first step towards this generalisations is to consider function from a different perspective: given, for instance, a function $f : (0,1) \to \mathbb{R}$ we won't, for the moment, pay too much attention any more on the question what f does with single values in $(0,1)$; instead we will consider how f "acts" – by means of integration – on a class of very smooth "test functions".

Definition 3.1.1. Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A mapping $\varphi : I \to \mathbb{K}$ is called a *test function* if φ is infinitely often differentiable and if the exists a compact set $K \subseteq I$ such that $\varphi(x) = 0$ for all $x \in I \setminus K$.

The set of all test function from I to $\mathbb{K}$ is denoted by $\mathcal{D}(I;\mathbb{K})$.

Note that a test function is automatically infinitely often continuously differentiable.

Example 3.1.2. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be given by

$$\varphi(x) = \begin{cases} e^{\frac{1}{x^2-1}} & \text{if } x \in (-1,1), \\ 0 & \text{if } x \in \mathbb{R} \setminus (-1,1). \end{cases}$$

Then $\varphi(x) = 0$ for all $x \in \mathbb{R} \setminus [-1,1]$ and one can show that φ is infinitely often differentiable (the only points for this is not completely obvious are -1 and 1). Hence, $\varphi \in \mathcal{D}(\mathbb{R};\mathbb{R})$.

In fact, one can show that exist a lot of test functions on each open interval $I \subseteq \mathbb{R}$ – sufficiently many to distinguish any two elements of $L^2(I;\mathbb{K})$ by integrating them against all test functions on I. To make this precise, we first note the following observation:

Remark 3.1.3. Let $I \subseteq \mathbb{R}$ by an open interval and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $[f] \in L^2(I;\mathbb{K})$ and let $\varphi \in \mathcal{D}(I;\mathbb{K})$. Then the function $f\varphi : I \to \mathbb{R}$ is integrable and the integral $\int_I f(x)\varphi(x)\,\mathrm{d}$ does not depend on the choice of the representative f of $[f]$.

Proof. There exists a compact set $K \subseteq I$ such that $\varphi(x) = 0$ for all $x \in I \setminus K$. Moreover, since φ is continuous, we can find a number $C \geq 0$ such that $|\varphi(x)| \leq C$ for all $x \in K$. Hence, $\int_I |\varphi(x)|^2\,\mathrm{d} \leq \int_K C^2\,\mathrm{d} < \infty$, i.e. $[\varphi] \in L^2(I;\mathbb{K})$. Hence, $f\varphi$ is integrable and the $\int_I f(x)\varphi(x)\,\mathrm{d}$ does not depend on the choice of the representative of $[f]$ (compare Example 2.1.8). □

Theorem 3.1.4. *Let $I \subseteq \mathbb{R}$ be an open interval an let $\mathbb{K} \in \{\mathbb{R};\mathbb{C}\}$. Let $[f],[g] \in L^2(I;\mathbb{K})$ and assume that*

$$\int_I f(x)\varphi(x)\,\mathrm{d} = \int_I g(x)\varphi(x)\,\mathrm{d}$$

for all $\varphi \in \mathcal{D}(I;\mathbb{K})$. Then $[f] = [g]$.

Proof. This is a special case of the so-called *Fundamental Lemma of the Calculus of Variations* which can be shown by measure theoretic methods and for which we refer to the literature. □

Definition 3.1.5. Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A *distribution* on I is a linear mapping $\tau : \mathcal{D}(I;\mathbb{K}) \to \mathbb{K}$ with the following property: For each compact set $K \subseteq I$ there exist a constant $C \geq 0$ and an integer $n \in \mathbb{N}_0$ such that

$$|\tau(\varphi)| \leq C \sum_{k=0}^{n} \sup_{x \in K} |\varphi^{(k)}(x)|$$

for all $\varphi \in \mathcal{D}(I;\mathbb{K})$ that fulfil $\varphi(x) = 0$ for every $x \in I \setminus K$.

The set of all distributions on I is a vector space (with the canonical addition and scalar multiplication) which we denote by $\mathcal{D}'(I;\mathbb{K})$.

The estimate in the above definition can be understood as some kind of continuity condition. However, in order to understand this more precisely, we would need the theory of topological vector spaces.

Example 3.1.6. Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbb{K} \in \{\mathbb{R};\mathbb{C}\}$. If $f : I \to \mathbb{K}$ is a continuous function, then the mapping $\tau_f : \mathcal{D}(I;\mathbb{R}) \to \mathbb{K}$ which is given by

$$\tau_f(\varphi) = \int_I f\varphi\,\mathrm{d} \qquad \text{for all } \varphi \in \mathcal{D}(I;\mathbb{K})$$

is a distribution.

First note, that the integral $\int_I f\varphi \, \mathrm{d}$ is indeed well-defined since $f\varphi$ is continuous and zero outside of some compact set. Moreover, if $K \subseteq I$ is compact and $C := |K| \sup_{x\in K}|f(x)|$ (where $|K|$ denotes the Lebesgue measure of K), then we have

$$|\tau_f(\varphi)| \le C \sup_{x\in K}|\varphi(x)|$$

for all $\varphi \in \mathcal{D}(I;\mathbb{K})$. Thus, we have indeed $\tau_f \in \mathcal{D}'(I;\mathbb{K})$.

Finally, we point out that, if $\int_I f_1\varphi \, \mathrm{d} = \int_I f_2\varphi \, \mathrm{d}$ for two continuous functions $f_1, f_2 : I \to \mathbb{K}$ and all $\varphi \in \mathcal{D}(I;\mathbb{K})$, then one can conclude – similarly as in Remark 3.1.3 – from the Fundamental Lemma of the Calculus of Variations that $f_1 = f_2$. Hence, the mapping $f \mapsto \tau_f$ from the vector space of continuous $\mathbb{K}$-valued functions on I to $\mathcal{D}'(I;\mathbb{K})$ is injective, i.e. we can – and shall – identify each continuous function $f : I \to \mathbb{K}$ with the distribution τ_f.

Example 3.1.7. Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. For each $[f] \in L^2(I;\mathbb{K})$ we can define a linear mapping $\tau_{[f]} : \mathcal{D}(I;\mathbb{R}) \to \mathbb{K}$ by means of the formula

$$\tau_{[f]}(\varphi) = \int_I f\varphi \, \mathrm{d} \qquad \text{for all } \varphi \in \mathcal{D}(I;\mathbb{K})$$

(compare Remark 3.1.3). It is easy to check that τ_f is in fact a distribution on I. Moreover, the mapping $L^2(I;\mathbb{K}) \to \mathcal{D}'(I;\mathbb{K})$ is injective (see again Remark 3.1.3) and hence, we may – and shall – identify each $[f] \in L^2(I;\mathbb{K})$ with the distribution $\tau_{[f]}$.

For a distribution $\tau \in \mathcal{D}'(I;\mathbb{K})$ we write $\tau \in L^2(I;\mathbb{K})$ if there exists a vector $[f] \in L^2(I;\mathbb{K})$ such that $\tau = \tau_{[f]}$.

Examples 3.1.8. Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Let $x_0 \in I$.

(a) Define a mapping $\delta_{x_0} : \mathcal{D}(I;\mathbb{K}) \to \mathbb{K}$ by $\delta_{x_0}(\varphi) = \varphi$ for all $\varphi \in \mathcal{D}(I;\mathbb{K})$. Then δ_{x_0} is a distribution on I, the so-called *Dirac delta distribution at* x_0.

(b) More generally, let $k \in \mathbb{N}_0$ and let $\tau : \mathcal{D}(I;\mathbb{K}) \to \mathbb{K}$ be given by $\tau(\varphi) = \varphi^{(k)}(x_0)$ for all $\varphi \in \mathcal{D}(I;\mathbb{K})$. Then τ is a distribution on I.

The main point about distributions is that we can differentiate them:

Definition 3.1.9. Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$. Let $\tau \in \mathcal{D}'(I;\mathbb{K})$.

(i) The mapping

$$\tau' : \mathcal{D}(I;\mathbb{K}) \to \mathbb{K}, \quad \varphi \mapsto \tau'(\varphi) := -\tau(\varphi'),$$

which can be easily checked to be a distribution, too, is called the *derivative* of τ.

(ii) We define $\tau^{(0)} := \tau$ and $\tau^{(k+1)} := (\tau^{(k)})'$ for each $k \in \mathbb{N}_0$. The distribution $\tau^{(k)}$ is called the *k-th derivative of* τ.

(iii) If $f : I \to \mathbb{K}$ is a continuous function (respectively, then we call the distribution $(\tau_f)'$ the *distributional derivative of* f.

(iv) If $[f] \in L^2(I;\mathbb{K})$, then we call the distribution $(\tau_{[f]})'$ the *distributional derivative of* $[f]$.

By the above definition, we have found a way to "differentiate" many functions (respectively, there associated distributions) which are not differentiable in the classical sense. Yet, to things remain to be clarified:

- We have to show that the definition of a derivative in Definition 3.1.9 is *consistent* with the classical notion of a derivative. This is the content of Proposition 3.1.10 below.
- It it not clear at all whether the distributional derivative of a function is again a function (more precisely: a distribution induced by a function in the sense of Example 3.1.6 or Example 3.1.7). This matter will be discussed in Examples 3.1.11 and it gives rise to the definition of weak derivatives and Sobolev spaces below.

Proposition 3.1.10. *Let $I \subseteq \mathbb{R}$ be an open interval, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $k \in \mathbb{N}_0$. Let $f : I \to \mathbb{K}$ be a k-times continuously differentiable function. Then the k-th order distributional derivative of f coincides with the k-the order classical derivative of f; more precisely, we have*

$$\tau_{f^{(k)}} = (\tau_f)^{(k)}$$

(using the notation from Example 3.1.6).

Hence, if we identify continuous $\mathbb{K}$-valued functions on I with the corresponding distributions, then the distributional derivative of a C^1-function is the same as the classical derivative. The proof of Proposition 3.1.10 relies on a simple partial integration (this also explains why we introduced the strange minus-sign in Definition 3.1.9). We leave the details of the proof as an exercise.

Examples 3.1.11. Let $I = (-1,1)$ and let $f : (-1,1) \to \mathbb{R}$, $f(x) = |x|$ for all $x \in (-1,1)$. We use the notation from Examples 3.1.6 and 3.1.7.

(a) Let $g : (-1,1) \to \mathbb{R}$ be given by

$$g(x) := \operatorname{sgn} x := \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

Then $(\tau_f)' = \tau_{[g]}$.

(b) We have $(\tau_f)^{(2)} = 2\delta_0$.

(c) There exists no continuous function $h : I \to \mathbb{R}$ such that $\tau_h = \delta_0$. Moreover, there exists no equivalence class $[h] \in L^2(I, \mathbb{K})$ such that $\tau_{[h]} = \delta_0$.

We leave the proofs of the assertions in the above examples as an exercise.

Definition 3.1.12. Let $I \subseteq \mathbb{R}$ be an open interval, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $k \in \mathbb{N}_0$.

(i) A vector $[f] \in L^2(I;\mathbb{K})$ is called *k-times weakly differentiable* if, for each $j \in \{0, \dots, k\}$ the j-the distributional derivative of $[f]$ can be represented by an element of $L^2(I;\mathbb{K})$, meaning more precisely that there exists a vector $[g_j] \in L^2(I;\mathbb{K})$ such that $(\tau_{[g]})^{(j)} = \tau_{[g_j]}$.

In this case, $[g_j]$ is uniquely determined (see Theorem 3.1.4 and Example 3.1.7) and called the *j-th weak derivative of* $[f]$. We denote it by $[g_j] =: [f]^{(j)}$. For $j = 1$ we also write $[f]' := [f]^{(1)}$ and for $j = 2$ we write $[f]'' := [f]^{(2)}$.

(ii) The set of all k-times weakly differentiable elements of $L^2(I;\mathbb{K})$ (which can easily be checked to be a vector subspace of $L^2(I;\mathbb{K})$) is denoted by $H^k(I;\mathbb{K})$ and is called the *Sobolev space of order k on I*.

Theorem 3.1.13. *Let $I \subseteq \mathbb{R}$ be an open interval, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $k \in \mathbb{N}_0$. For all $f, g \in H^k(I;\mathbb{K})$ we define*

$$\langle f, g \rangle_{H^k} := \sum_{j=0}^{k} \langle [f]^{(j)}, [g]^{(j)} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(I;\mathbb{K})$. Then $\langle \cdot, \cdot \rangle_{H^k}$ is an inner product on $H^k(I;\mathbb{K})$ and $(H^k(I;\mathbb{K}); \langle \cdot, \cdot \rangle_{H^k})$ is a Hilbert space.

Proof. It is straightforward to check that $\langle \cdot, \cdot \rangle_{H^k}$ is an inner product. We leave the proof of the completeness as an exercise. $\square$

One can prove the following representation results of Sobolev functions in one dimension. For the proof we refer to the literature.

Proposition 3.1.14. *Let $I \subseteq \mathbb{R}$ be an open interval, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $[\hat{u}] \in H^1(I;\mathbb{K})$. Then there exists a uniquely determined continuous function $u : \overline{I} \to \mathbb{K}$ such that $[\hat{u}] = [u|_I]$. Moreover, if $\hat{u}'$ denotes any representative of $[\hat{u}]'$ and if $x_0 \in \overline{I}$ is an arbitrary point, then we have*

$$u(x) = u(x_0) + \int_{x_0}^{x} \hat{u}'(y)\, \mathrm{d}y$$

for all $x \in \overline{I}$.

Proof. For this proof we refer to the literature. □

Definition 3.1.15. In the situation of the above proposition, the function $u : \overline{I} \to \mathbb{K}$ is called the *continuous representative of* $[\hat{u}]$.

Corollary 3.1.16. *Let $I \subseteq \mathbb{R}$ be an open interval, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and let $[\hat{u}] \in H^k(I;\mathbb{K})$ for some $k \in \mathbb{N}$. If there exists a continuous function $f : \overline{I} \to \mathbb{K}$ such that $[\hat{u}]^{(k)} = [f|_I]$, then there exists exactly one k-times continuously differentiable function $u : \overline{I} \to \mathbb{K}$ for which we have $[\hat{u}] = [u|_I]$; moreover, $u^{(k)} = f$.*

Proof. This follows by induction from Proposition 3.1.14. □

Definition 3.1.17. Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. By $H_0^1(I;\mathbb{K})$ we denote the space of all vectors $[\hat{u}] \in H^1(I;\mathbb{K})$ whose continuous representative is 0 on the boundary of I.

Remark 3.1.18. It is not difficult to see that $H_0^1(I;\mathbb{K})$ is a vector subspace of $H^1(I;\mathbb{K})$. Moreover, one can conclude from Proposition 3.1.14 that $H_0^1(I;\mathbb{K})$ is even closed in the Hilbert space $(H^1(I;\mathbb{K}), \langle \cdot, \cdot \rangle_{H^1})$.

We leave the proof of the following important theorem and of the subsequent corollary as an exercise.

Theorem 3.1.19 (Poicaré inequality). *Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If I is bounded, then there exists a constant $C > 0$, depending only on the Lebesgue measure of I, such that*

$$\langle [u], [u] \rangle \le C \langle [u]', [u]' \rangle$$

for all $[u] \in H_0^1(I;\mathbb{K})$. Here, $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $L^2(I;\mathbb{K})$.

Corollary 3.1.20. *Let $I \subseteq \mathbb{R}$ be an open interval and let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. If I is bounded, then the mapping $\langle \cdot, \cdot \rangle_{H_0^1} : H_0^1(I,\mathbb{K}) \times H_0^1(I;\mathbb{K}) \to \mathbb{K}$ given by*

$$\langle [u], [v] \rangle_{H_0^1} := \langle [u]', [v]' \rangle$$

(where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $L^2(I;\mathbb{K})$) is an inner product on $H_0^1(I;\mathbb{K})$ which renders $H_0^1(I;\mathbb{K})$ a Hilbert space.

3.2 Elliptic boundary value problems on the interval

Theorem 3.2.1. *Let $x_1, x_2 \in \mathbb{R}$ such that $x_1 < x_2$ and let $f : [x_1, x_2] \to \mathbb{R}$ be a continuous function. Then there exists a function $u : [x_1, x_2] \to \mathbb{R}$ which is twice continuously differentiable which solves the following* boundary value problem*:*

$$\begin{cases} u'' = f \\ u(x_1) = u(x_2) = 0. \end{cases}$$

Proof. Abbreviate $I \coloneqq (x_1, x_2)$ and define a mapping $\varphi : V \to \mathbb{R}$ given by $\varphi([v]) = -\int_I f(x)v(x)\,\mathrm{d}x = -\langle [f], [v]\rangle_{L^2}$ for all $[v] \in V$. This mapping is obviously linear. Moreover, if $C > 0$ denotes the constant from Poincaré's inequality, then we have

$$|\varphi([v])| \le \|[f]\|_{L^2} \cdot \|[v]\|_{L^2} \le \|[f]\|_{L^2} \cdot C\|[v]\|_{H_0^1}$$

for all $[v] \in V$, so φ is a continuous linear functional on the Hilbert space $(H_0^1(I;\mathbb{R}), \langle \cdot, \cdot \rangle_{H_0^1})$. Therefore, due to the representation theorem of Riesz–Fréchet we can find a vector $[\hat{u}] \in V$ such that $\varphi([v]) = \langle [\hat{u}], [v]\rangle_{H_0^1}$ for all $[v] \in V$. Now, denote by $\hat{u}'$ a representative of $[\hat{u}]'$ and let $v \in \mathcal{D}(I;\mathbb{R})$ a test function. Then we have

$$\begin{aligned}
\tau_{[f]}(v) &= \int_I f(x)v(x)\,\mathrm{d}x = -\varphi([v]) = -\langle [\hat{u}], [v]\rangle_{H_0^1} \\
&= -\int_I \hat{u}'(x)v'(x)\,\mathrm{d}x = \tau'_{[\hat{u}']}(v) = \tau'_{[\hat{u}]'}(v).
\end{aligned}$$

Thus, the distributional derivative of the L^2-vector $[\hat{u}]'$ is given by $\tau_{[f]}$. Since $[f]$ is also an L^2-vector, we conclude that $[\hat{u}]'$ is in $H^1(I;\mathbb{R})$ (so $[\hat{u}] \in H^2(I;\mathbb{R})$) and that $[\hat{u}]'' = [f]$.

Up to now we have only used that $[f] \in L^2(I;\mathbb{R})$. Now the continuity of f comes into play. Since $[\hat{u}] \in H^1$, there exists a continuous functions $u : [x_1, x_2] \to I$ such that $u(x_1) = u(x_2) = 0$ and $[\hat{u}] = [u|_I]$. Since $[u]'' = [f]$ and since f is continuous, it follows from Corollary 3.1.16 that u is twice continuously differentiable and that the second classical derivative u'' of u coincides with f. □

Remarks 3.2.2. (a) One can also prove uniqueness of the function u in the above theorem.

(b) The above theorem can also be proved by more elementary methods (more precisely: by integrating twice an choosing appropriate integration constants).

Yet, the advantage of the approach presented here is that it still works in higher dimensions. However, we should point out that things are more complicated in higher dimensions: for instance, Proposition 3.1.14 does not longer holds if the interval I is replaced with an open subset of $\mathbb{R}^n$. As a consequence, the solution u of a boundary value problem might no longer be twice continuously differentiable up to the boundary, but only in the interior of the domain we are working on.

4 Spectral theory

Opening Questions. (a) If $A \in \mathbb{C}^{d\times d}$ is an invertible matrix and $B \in \mathbb{C}^{d\times d}$ is "small", does it follows that $A+B$ is invertible, too?

(b) Why is a matrix $A \in \mathbb{C}^{d\times d}$ invertible of 0 is not an eigenvalue of A?

(c) Can you write down the geometric series for matrices? Can you write it down for bounded linear operators?

(d) How can we "diagonalise" a linear operator?

4.1 Intertibility and spectrum of linear operators

Definition 4.1.1. Let $(E, \|\cdot\|)$ be a Banach space. An operator $T \in \mathcal{L}(E)$ is called *invertible* if it is bijective and if its inverse operator $T^{-1} : E \to E$ is continuous, too.

Remark 4.1.2. The condition in the above definition that T^{-1} be continuous is actually redundant. This is a consequence of the so-called *open mapping theorem* in functional analysis, which is not part of this course. The theorem is usually taught in the course "Functional Analysis". You can also find it in your favourite functional analysis book.

Recall that we denote the identity operator on a vector space V by I_V, or simply by I.

Proposition 4.1.3. *Let $(E, \|\cdot\|)$ be a Banach space and let $T \in \mathcal{L}(E)$. Then the following assertions are equivalent:*

(i) *T is invertible.*

(ii) *There exists an operator $S \in \mathcal{L}(E)$ such that $TS = ST = I$.*

If the equivalent assertions (i) *and* (ii) *are fulfilled, then $S = T^{-1}$.*

Proof. We leave the proof as an exercise. □

Proposition 4.1.4. *Let $(E, \|\cdot\|)$ be a Banach space and let $T \in \mathcal{L}(E)$.*

(a) *If the series over the sequence $(T^n)_{n\in\mathbb{N}_0}$ is convergent, then the operator $I-T$ is invertible and we have*

$$(I-T)^{-1} = \sum_{n=0}^{\infty} T^n.$$

(b) *If there exist real numbers* $C \geq 0$ *and* $\rho \in [0,1)$ *such that* $\|T^n\| \leq C\rho^n$ *for all* $n \in \mathbb{N}$*, then the operator* $I - T$ *is invertible.*

(c) *If* $\|T\| < 1$*, then the operator* $1 - T$ *is invertible.*

Proof. (a) Assume that the series over the sequence $(T^n)_{n\in\mathbb{N}_0}$ is absolutely convergent and define

$$S := \sum_{n=0}^{\infty} T^n.$$

Then we have $(I-T)S = \sum_{n=0}^{\infty} T^n - \sum_{n=1}^{\infty} T^n = I$ and, similarly, $S(I-T) = I$. Hence, $I-T$ is invertible and S is its inverse operator according to Proposition 4.1.3.

(b) This is a consequence of assertion (b).

(c) Let $\rho := \|T\| < 1$. Then $\|T^n\| \leq \|T\|^n = \rho^n$ for each $n \in \mathbb{N}$, so the assertion follows from (b). □

If $(E, \|\cdot\|)$ is a Banach space and $S, T \in \mathcal{L}(E)$ are both invertible, then ST is clearly invertible, too and $(ST)^{-1} = T^{-1}S^{-1}$.

Corollary 4.1.5. *Let* $(E, \|\cdot\|)$ *be a Banach space and let* $T \in \mathcal{L}(E)$ *be invertible. Let* $S \in \mathcal{L}(E)$ *and assume that* $\|S - T\| < \frac{1}{\|T^{-1}\|}$*. Then the operator* S *is invertible, too. In particular, the set of all invertible operators is open in* $\mathcal{L}(E)$*.*

Proof. Note that

$$S = T + S - T = T(I - T^{-1}(T - S)).$$

The operator T is invertible by assumption and the operator $I - T^{-1}(T-S)$ is invertible according to Proposition 4.1.4(c) since $\|T^{-1}(T-S)\| \leq \|T^{-1}\| \cdot \|S-T\| < 1$. Hence, S is invertible, too. □

Definition 4.1.6. Let $(E, \|\cdot\|)$ be a complex Banach space and let $T \in \mathcal{L}(E)$.

(i) A complex number λ is called a *spectral value* of T if the operator $\lambda I - T$ is not invertible. The set $\sigma(T)$ of all spectral values of T is called the *spectrum* of T.

(ii) A complex number λ is called an *eigenvalue* of T if there exists a vector $v \in E \setminus \{0\}$ such that $Tv = \lambda v$.

In this case, the vector v is called a *eigenvector* of T for the eigenvalue λ, and the non-zero vector subspace $\{w \in E : Tw = \lambda w\} = \ker(\lambda I - T)$ is called the *eigenspace* of T for the eigenvalue λ.

Note that $\lambda \in \mathbb{C}$ is an eigenvalue of T if and only if the operator $\lambda I - T$ is not injective. In particular, every eigenvalue of T is a spectral value of T.

Proposition 4.1.7. *Let $(E, \|\cdot\|)$ be a complex Banach space and let $T \in \mathcal{L}(E)$.*

(a) *Every $\lambda \in \sigma(T)$ fulfils $|\lambda| \le \|T\|$.*

(b) *Assume that $\mu \in \mathbb{C}$ is not a spectral value of T. If λ is any complex number that fulfils $|\lambda - \mu| < \frac{1}{\|(\mu I - T)^{-1}\|}$, than λ is not a spectral value of T, either.*

(c) *The spectrum $\sigma(T)$ is a compact subset of T.*

Proof. (a) Let $\lambda \in \mathbb{C}$ fulfil $|\lambda\| > \|T\|$. Then the operator $\frac{T}{\lambda}$ has norm < 1 and hence, $I - \frac{T}{\lambda}$ is invertible. Thus, $\lambda I - T = \lambda(I - \frac{T}{\lambda})$ is invertible, too.

(b) Let μ and λ be as in the assertion. We have to prove that $\lambda I - T$ is invertible. Since $\|(\lambda - IT) - (\mu I - T)\| = |\lambda - \mu| < \frac{1}{\|(\mu I - T)^{-1}\|}$, this follows from Corollary 4.1.5.

(c) According to (a) the spectrum of T is bounded and according to (b), $\mathbb{C} \setminus \sigma(T)$ is open, so $\sigma(T)$ is closed. □

The following corollary is an immediate consequence of Proposition 4.1.7(b):

Corollary 4.1.8. *Let $(E, \|\cdot\|)$ be a complex Banach space, let $T \in \mathcal{L}(E)$ and let $\mu \in \mathbb{C} \setminus \sigma(T)$. Then*

$$\|(\mu I - T)^{-1}\| \ge \frac{1}{\operatorname{dist}(\mu, \sigma(T))},$$

where $\operatorname{dist}(\mu, \sigma(T)) := \inf\{|\mu - \lambda| : \ \lambda \in \sigma(T)\}$ *denotes the distance of μ to the spectrum of T.*

Remark 4.1.9. The series representation of $(I - T)^{-1}$ in Proposition 4.1.4 and a glance at the proof of Corollary 4.1.5 yield a bit more precise information in Proposition 4.1.7.

In assertion (a) of the latter proposition we obtain for every $\lambda \in \mathbb{C}$ fulfilling $|\lambda| > \|T\|$ the representation

$$(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}},$$

and in assertion (b) of the proposition we obtain the representation

$$(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} (\mu - \lambda)^n \Big((\mu I - T)^{-1} \Big)^{n+1}$$

whenever $|\lambda - \mu| < \frac{1}{\|(\mu I - T)^{-1}\|}$.

Proposition 4.1.10. *Let* $(E, \|\cdot\|)$ *be a complex Banach space and let* $T \in \mathcal{L}(E)$*. If* $E \neq \{0\}$*, then the spectrum of* T *is non-empty.*

Proof. The proof requires a bit of complex analysis and the Hahn–Banach theorem which is a fundamental result in functional analysis that follows from Zorn's Lemma. It can be found for instance in *Winfried Kaballo, Grundkurs Funktionalanalysis (Springer 2011)*, Satz 4.3.

Here, we only give the proof in case that E is a Hilbert space. So, let E be a Hilbert space, i.e. let the norm on E be induced by an inner product $\langle \cdot, \cdot \rangle$. We may assume throughout that $T \neq 0$. Assume for a contradiction that $\sigma(T) = \emptyset$ and let $x \in E \setminus \{0\}$. Then $y := (0 \cdot I - T)^{-1} x \neq 0$ since $(0 \cdot I - T)^{-1}$ is bijective and thus, in particular, injective. Now, consider the mapping

$$f : \mathbb{C} \to \mathbb{C}, \qquad \lambda \mapsto \langle y, (\lambda \cdot I - T)^{-1} x \rangle.$$

It follows from the second series expansion in Remark 4.1.9 that this mapping is analytic. Moreover, we can conclude from the same proposition that we have $f(\lambda) = \langle y, \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}} x \rangle$ whenever $|\lambda| > \|T\|$ and hence,

$$|f(\lambda)| \leq \|y\| \cdot \|x\| \cdot \sum_{n=0}^{\infty} \frac{\|T\|^n}{c^{n+1} \|T\|^{n+1}} = \|y\| \cdot \|x\| \cdot \frac{1}{\|T\|} \tag{4.1}$$

whenever c is a real number strictly larger than 1 and $|\lambda| \geq c\|T\|$. Since the mapping f is continuous, it is bounded and the closed ball with radius $2\|T\|$ and thus, it follows from estimate (4.1) (for $c = 2$) that f is bounded on all of $\mathbb{C}$. Yet, according to Liouville's Theorem, a bounded entire function is constant, so $f(\lambda) = f(0)$ for all $\lambda \in \mathbb{C}$. Finally, we use the estimate (4.1) one again and let $c \to \infty$. This shows that $\lim_{|\lambda|} f(\lambda) = 0$ and hence, $\|y\|^2 = \langle y, y \rangle = f(0) = 0$. This is a contradiction. □

4.2 Compact operators

Let (M, d) be a metric space and let $K \subseteq M$. Recall that K is called *compact* if the following equivalent conditions are fulfilled:

(i) For every familiy of opensets $(U_i)_{i \in I}$ in M which fulfils $\bigcup_{i \in I} U_i \supseteq K$ we can find a finite subset $\tilde{I}$ of the index set I for which we still have $\bigcup_{i \in \tilde{I}} U_i \supseteq K$ (for short: *every open cover of K possess a finite sub-cover*).

(ii) Every sequence in K has a subsequence that converges to an element of K.

(iii) Every net in K has a subnet that converges to an element of K.

(iv) The set K is closed and *totally bounded*, the latter assertion meaning that, every every $\varepsilon > 0$, we can find finitely many open balls of radius ε in K whose union contains K.

The set K is called *relatively compact* if its closure in M is compact. Every compact subset of M is bounded, and so is every relatively compact subset of M.

Recall that, for every $x \in M$ and every $r > 0$ the set $\overline{B}_r(x)$ denotes the closed ball in M with center x and radius r.

Definition 4.2.1. Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be Banach spaces over the same scalar field and let $T : E \to F$ be linear. The mapping T is called *compact* if the set $T\big(\overline{B}_1(0)\big)$ is relatively compact in F.

Proposition 4.2.2. *Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be Banach spaces over the same scalar field and let $T : E \to F$ be linear and compact. Then T is continuous.*

Proof. The proof follows easily from what was said above. We leave the details as an exercise. □

We are now going to consider the case $E = F$ and $\mathbb{K} = \mathbb{C}$ in which case we can study the spectrum of linear operators. The spectrum of a compact operator has many surprising and useful properties. Here, we are only going to prove the following auxiliary result. Instead of studying the spectrum of compact operators on Banach spaces in more details, we will focus on the spectral theory of so-called *normal* compact operators on Hilbert spaces in the final subsection, which is of particular importance.

Lemma 4.2.3. *Let $(E, \|\cdot\|)$ be a complex Banach space and let $T \in \mathcal{L}(E)$ be compact. If a complex number $\lambda \neq 0$ is contained in the topological boundary of the spectrum of T (i.e. if $\lambda \in \partial\sigma(T)$), then λ is an eigenvalue of T.*

Obviously, every point in $\partial\sigma(T)$ is a spectral value of T (since $\sigma(T)$ is closed). The point in the above lemma is that such a number is even an eigenvalue of T.

Proof of Lemma 4.2.3. Let $\lambda \in \partial\sigma(T)$. Then there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $\mathbb{C} \setminus \sigma(T)$ which converges to λ. Let us define $R_n := (\mu_n I - T)^{-1}$ for each index n. Since $\lambda \in \sigma(T)$, we have $\operatorname{dist}(\mu_n, \sigma(T)) \to 0$ and hence, it follows from Corollary 4.1.8 that $\|R_n\| \to \infty$ as $n \to \infty$.

We can find vectors $y_n \in E$ such that $\|y_n\| = 1$ and $\|R_n y\| \geq \frac{1}{2}\|R_n\|$ for all indices n and hence, $0 < \|R_n y_n\| \to \infty$. Consider the vectors $x_n := \frac{R_n y_n}{\|R_n y_n\|}$. A short computation shows that

$$(\lambda I - T)x_n = (\lambda - \mu_n)x_n + (\mu_n I - T)\frac{R_n y_n}{\|R_n y_n\|} = (\lambda - \mu_n)x + \frac{y_n}{\|R_n y_n\|} \to 0$$

as $n \to \infty$. Hence, we have found a sequence $(x_n)_{n\in\mathbb{N}}$ of normalised vectors such that $(\lambda - T)x_n$ converges to 0. Such a sequence is called a *approximate eigenvector* of T and λ is then called an *approximate eigenvalue* of T.

Now we use that T is compact and that $\lambda \neq 0$. This implies that there exists a subsequence (Tx_{n_k}) of (Tx_n) which converges to a vector $y \in E$. Since $\lambda x_{n_k} - Tx_{n_k}$ converges to 0, it follows that (x_{n_k}) converges to $\frac{1}{\lambda}y =: x$. Clearly, $(\lambda I - T)x = \lim_{k\to\infty}(\lambda I - T)x_{n_k} = 0$, so λ is indeed an eigenvalue of T. □

It is worthwhile noting that we can actually prove much more about the spectrum of compact operators. In fact, the following result is true:

Theorem 4.2.4. *Let $(E, \|\cdot\|)$ be a complex Banach space and let $T \in \mathcal{L}(E)$ be compact. Then the following assertions hold:*

(a) *For every $\varepsilon > 0$, there exists at most finitely many spectral value of T of modulus larger than ε. In particular, 0 is the only possible accumulation point of $\sigma(T)$.*

(b) *If E is infinite-dimensional, then 0 is a spectral value of T.*

(c) *Every spectral value of T, except for possibly 0, is an eigenvalue of T.*

For the proof we refer to the literature, for instance to Chapter 11 (and in particular Theorem 11.14) of *Winfried Kaballo, Grundkurs Funktionalanalysis (Springer 2011)*. We are only going to prove this result in an important special case below. On the other hand, in this special case we obtain much more detailed information then in the above theorem.

4.3 Normal and self-adjoint operators

Definition 4.3.1. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space und let $T \in \mathcal{L}(H)$.

(i) The operator T is called *self-adjoint* if $T^* = T$.

(ii) The operator T is called *normal* if $T^*T = TT^*$.

Proposition 4.3.2. *Let $(H, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and let $T \in \mathcal{L}(H)$ be self-adjoint.*

(a) *All spectral values of T are real; in particular, all eigenvalues of T are real.*

(b) *If λ_1 and λ_2 are two different eigenvalues of T with eigenvectors x_1 and x_2, respectively, then $x_1 \perp x_2$.*

Proof. We leave the proof as an exercise. □

Remark 4.3.3. If $(H, \langle \cdot, \cdot \rangle)$ is a complex Hilbert space and $T \in \mathcal{L}(H)$ is only normal (but not necessarily self-adjoint), then it is still not difficult to show that following:

(a) If $\lambda \in \mathbb{C}$ is an eigenvector of T with eigenvector x, then $\overline{\lambda}$ is an eigenvalue of T^* with eigenvector x.

(b) If λ_1 and λ_2 are two different eigenvalues of T with eigenvectors x_1 and x_2, respectively, then $x_1 \perp x_2$.

Lemma 4.3.4. *Let $(H, \langle \cdot, \cdot \rangle)$ be a non-zero complex Hilbert space and let $T \in \mathcal{L}(H)$ be self-adjoint and compact. Then T has an eigenvalue.*

Proof. We know from Proposition 4.1.10 that the spectrum of T is non-empty since $H \neq \{0\}$. If $\sigma(T)$ contains a non-zero number, then so does its boundary $\partial\sigma(T)$ and hence, T has an eigenvalue according to Lemma 4.2.3. So assume that $\sigma(T) = 0$. Then the so-called *spectral radius*

$$r(T) := \max\{|\lambda| : \lambda \in \sigma(T)\}$$

is 0. Since T is self-adjoint, we know from the exercise sheet 13 that $\|T\| = r(T) = 0$, so $T = 0$. Again since $H \neq \{0\}$, this implies that 0 is an eigenvalue of T. □

Theorem 4.3.5 (Spectral Theorem for Compact Self-Adjoint Operators). *Let $(H, \langle \cdot, \cdot \rangle)$ be complex Hilbert space and let $T \in \mathcal{L}(H)$ be compact and self-adjoint. For each eigenvalue λ of T, choose an orthonormal basis E_λ of $\ker(\lambda I - T)$ and define*

$$E := \bigcup_{\substack{\lambda \text{ is an} \\ \text{eigenvalue of } T}} E_\lambda$$

Then E is an orthonormal basis of H. In particular, H possess an orthonormal basis that consists of eigenvectors of T.

Proof. It follows from Proposition (b) that E is an orthonormal system in H. Let G be the closure of the linear span of E; we have to show that $G = H$. According to Corollary 2.4.5 have have $H = G \oplus G^\perp$, so it suffices to show that $G^\perp = \{0\}$.

Note that $T\big(\operatorname{span}(E)\big) \subseteq \operatorname{span}(E)$ and hence, $TG \subseteq G$. This implies that we also have $TG^\perp \subseteq G^\perp$; indeed, if $x \perp y$ for all $y \in G$, then $\langle Tx, y \rangle = \langle x, Ty \rangle = 0$ for all $y \in G$ since $Ty \in G$ for each such y.

Now, let $S \in \mathcal{L}(G^\perp)$ denote the restriction of T to $G^\perp$ (which is itsself a Hilbert space with respect to the inner product induced by H). We have

$$\langle x, S^* y \rangle = \langle Sx, y \rangle = \langle Tx, y \rangle = \langle x, Ty \rangle = \langle x, Sy \rangle$$

for all $x, y \in G^{\perp}$, so $S^* = S$, meaning that S is self-adjoint. Moreover, S has no eigenvalues for if λ was an eigenvalue of S with an eigenvector $x \in G^{\perp}$, then x would also be an eigenvector of T for the eigenvalues λ which would imply $x = \overline{\operatorname{span}(E_\lambda)} \subseteq G$ – a contradiction.

Finally, it is easy to see that S is compact since T is so. Hence, S is a compact self-adjoint operator on $G^{\perp}$ without eigenvalues. According to Proposition 4.3.4 this can only be true of $G^{\perp} = \{0\}$. □

Let us mention one more general result about eigenvalues of compact operators:

Remark 4.3.6. Let $(E, \|\cdot\|)$ be a Banach space, let $T \in \mathcal{L}(E)$ be compact and let λ be an eigenvalue of T. If $\lambda \neq 0$, then the associated eigenspace $F := \ker(\lambda I - T)$ is finite dimensional. Indeed, let $\overline{B}_1^F(0)$ denote the closed unit ball in F and let $\overline{B}_1^E(0)$ denote the closed unit ball in E. We have $TF \subseteq F$ and hence $T\overline{B}_1^F(0) \subseteq F \cap T\overline{B}_1^E(0)$; the latter set is compact in F.

However, we clearly have $T\overline{B}_1^F(0) = \lambda \overline{B}_1^F(0)$ and since $\lambda \neq 0$, this implies that the closed unit ball of F is relatively compact and hence compact. Thus, we only have to show the general result that a Banach space with compact unit ball is finite dimensional. For general Banach spaces this result relies on what is usually called the *Riesz Lemma*; since we focus on Hilbert space theory, this lemma is not part of the this cours.

Here, we only prove that a Hilbert space with compact unit ball is finite dimensional. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space with compact unit ball and assume for a contradiction that $\dim H = \infty$. Choose an orthonormal basis E of H. Since $\dim H = \infty$, E cannot be finite, so we can find a sequence $(e_n)_{n \in \mathbb{N}}$ of pairwise distinct elements in E. This sequence is contained in the unit ball of H, so we can choose a subsequence $(e_{n_k})_{k \in \mathbb{N}}$ which converges in H. In particular, $(e_{n_k})_{k \in \mathbb{N}}$ is a Cauchy sequence. Yet, we have $\|e_{n_k} - e_{n_j}\|^2 = 2$ according to Pythagoras' Theorem whenever $k \neq j$. This is a contradiction.

Examples 4.3.7. Let $[f] \in L^2((0, \pi); \mathbb{C})$. Similarly as in the proof of Theorem 3.2.1 one can show that there exists a vector $[u_g] \in H^2((0, \pi); \mathbb{C}) \cap H_0^1((0, \pi); \mathbb{C})$ which fulfils $u_g'' = f$ and similarly as in in Exercise 29(b) on Exercise Sheet 12 one can show that $[u_g]$ is uniquely determined.

Since $[u_g]$ is an element of $H^2((0, \pi); \mathbb{C}) \cap H_0^1((0, \pi); \mathbb{C})$, it is, of course, an element of $L^2((0, \pi); \mathbb{C})$, too. Define a mapping $T : L^2((0, \pi); \mathbb{C}) \to L^2((0, \pi); \mathbb{C})$ by $T[f] = [u_g]$ for each $[f] \in L^2((0, \pi); \mathbb{C})$. Then T is linear; moreover, one can proof that T is self-adjoint and compact (where the compactness follows from essentially from a so-called *Sobolev embedding theorem* which belongs to the content of each course on partial differential equations).

Hence, Theorem 4.3.5 can be applied to the operator T. We leave it as an exercise to show that the set of all eigenvalues of T is given by $\{\frac{1}{k^2} : k \in \mathbb{N}\}$ and that, for each $k \in \mathbb{N}$, the eigenspace $\ker(\frac{1}{k^2} - T)$ is spanned by the equivalence class of the mapping $s_k : (0, \pi) \to \mathbb{C}$ which is given by $s_k(x) := \sin(kx)$ for all $x \in (0, \pi)$. Hence, it follows from Theorem 4.3.5 that the set of eigenvectors $\{[s_k] : k \in \mathbb{N}\}$ is an orthonormal basis of $L^2((0,1); \mathbb{C})$.

Appendices

A

Metric spaces

A.1 Basics

Definition A.1.1 (Metrics and metric spaces). (i) Let M be an arbitrary set. A *metric* on M is a mapping $\mathrm{d} : M \times M \to [0, \infty)$ which fulfils the following axioms:

(M1) d is *positively definite*, i.e. for all $x, y \in M$ we have $\mathrm{d}(x, y) = 0$ if and only if $x = y$.

(M2) d is *symmetric*, i.e. we have $\mathrm{d}(x, y) = \mathrm{d}(y, x)$ for all $x, y \in M$.

(M3) d satisfies the *triangle inequality*, i.e. we have $\mathrm{d}(x, z) \le \mathrm{d}(x, y) + \mathrm{d}(y, z)$ for all $x, y, z \in M$.

(ii) A *metric space* is a pair (M, d) where M is a set and where d on M. By abuse of language we sometimes simply say that "M is a metric space", thereby suppressing d in the notation.

Example A.1.2 (Euclidean metric). Let $n \in \mathbb{N}$ and let $M \subseteq \mathbb{R}^n$. Define $\mathrm{d}(x, y) = \|x - y\|_2$ for all $x, y \in \mathbb{R}^n$, where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^n$. Then (M, d) is a metric space.

The above example is a special case of the following situation:

Remark A.1.3. Let (M, d) be a metric space and let $S \subseteq M$. Then $(S, \mathrm{d}|_{S \times S})$ is a metric space, too.

Let us give a further example of a metric space before we proceed with the theory:

Example A.1.4 (SNCF metric). Let M be a finite set with $n+1$ elements ($n \ge 1$) which we denote by $p, v_1, \dots, v_n$. Define a mapping $\mathrm{d} : M \times M \to [0, \infty)$ by

$$\mathrm{d}(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \text{ and } p \in \{x, y\} \\ 2 & \text{if } x \neq y \text{ and } p \notin \{x, y\}. \end{cases}$$

for all $x, y \in M$. Then (M, d) is a metric space and we call d a *SNCF metric* on M.

Definition A.1.5 (Balls, open and closed sets). Let (M, d) be a metric space.

(i) If $x_0 \in M$ and $r > 0$, then we call the set $B_r(x) := \{x \in M : \mathrm{d}(x, x_0) < r\}$ the *open ball in M with center* x_0.

(ii) A subset $S \subseteq M$ is called *open* if, for every $x \in S$, there exists a number $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq S$.

(iii) A subset $S \subseteq M$ is called *closed* if the complementary set $M \setminus S$ is open.

Definition A.1.6 (Convergent sequences and Cauchy sequences). Let (M, d) be a metric space and let $(x_k)_{k\in\mathbb{N}}$ be a sequence in M.

(i) The sequence $(x_k)_{k\in\mathbb{N}}$ is said to *converge to an element* $x \in M$ if, for every $\varepsilon > 0$, there exists an index $k_0 \in \mathbb{N}$ such that $x_k \in B_\varepsilon(x)$ for all $k \geq k_0$. In this case, the element x is said to be the *limit* of the sequence $(x_k)_{k\in\mathbb{N}}$ and we write $x = \lim_{k\to\infty} x_k$ or $x_k \overset{k\to\infty}{\to} x$.

The sequence $(x_k)_{k\in\mathbb{N}}$ is called *convergent* if there exists an element $x \in M$ such that $(x_k)_{k\in\mathbb{N}}$ converges to x.

(ii) The sequence $(x_k)_{k\in\mathbb{N}}$ is called a *Cauchy sequence* if, for every $\varepsilon > 0$, there exists an index $k_0 \in \mathbb{N}$ such that $\mathrm{d}(x_j, x_k) < \varepsilon$ for all $j, k \geq k_0$.

Remark A.1.7. Let (M, d) be a metric space. For every $x \in M$ and every sequence $(x_k)_{k\in\mathbb{N}}$, the following assertions are equivalent:

(i) The sequence $(x_k)_{k\in\mathbb{N}}$ converges to x.

(ii) The sequence $\big(\mathrm{d}(x, x_k)\big)_{k\in\mathbb{N}}$ in $[0, \infty)$ converges to 0.

It is easy to see that every convergent sequence is a Cauchy sequence, but the converse implication is not in general true. This gives rise to the following definition:

Definition A.1.8 (Complete metric space). A metric space (M, d) is said to be *complete* if every Cauchy sequence in M converges.

Proposition A.1.9. *Let (M, d) be a complete metric space and let $S \subseteq M$. Then the following assertions are equivalent:*

(i) *The set S is closed.*

(ii) *The metric space $(S, \mathrm{d}|_{S\times S})$ is complete.*

Proposition A.1.10. *Let (M, d) be a metric space. For every subset $S \subseteq M$ the following assertions are equivalent:*

(a) *The set S is closed.*

(b) *For every sequence $(x_k)_{k\in\mathbb{N}}$ in S which converges to an element of M, it follows that $\lim_{k\to\infty} x_k \in S$.*

Definition A.1.11 (Continuous mappings). Let (M, d_M) and (N, d_N) be metric spaces and let $f : M \to N$ be a mapping.

(i) Let $x \in M$. The mapping f is said to be *continuous at* x if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

(ii) The mapping f is said to be *continuous* if f is continuous at every point $x_0 \in M$.

Proposition A.1.12. *Let (M, d_M) and (N, d_N) be metric spaces and let $f : M \to N$ be a mapping.*

(a) *Let $x \in M$. The mapping f is continuous at x_0 if, for every sequence $(x_k)_{k\in\mathbb{N}}$ in M which converges to x, the sequence $(f(x_k))_{k\in\mathbb{N}}$ in N converges to $f(x)$.*

(b) *The mapping f is continuous if, for every convergent sequence $(x_k)_{k\in\mathbb{N}}$ in M, the sequence $(f(x_k))_{k\in\mathbb{N}}$ in N converges to $f(\lim_{k\to\infty} x_k)$.*

Definition A.1.13 (Bounded sets). Let (M, d) be a metric space.

(i) A subset $S \subseteq M$ is called *bounded* if there exists an element $x \in M$ and a number $r > 0$ such that $S \subseteq B_r(x)$.

(ii) A sequence $(x_k)_{k\in\mathbb{N}}$ in M is called *bounded* if the set $\{x_k : k \in \mathbb{N}\} \subseteq M$ is bounded.

Remarks A.1.14. (a) Let (M, d) be a metric space and let $S \subseteq M$ be bounded. It follows from the triangle inequality that, for *every* $x \in M$, there exists a number $r > 0$ such that $S \subseteq B_r(x)$.

(b) It is easy to see that every Cauchy sequence in M is bounded. In particular, every convergent sequence in M is bounded.

A.2 Product metrics

Definition A.2.1 (Product metric). Let (M, d_M) and (N, d_N) be metric spaces. A metric d on $M \times N$ is called a *product metric* of d_M and d_N if it fulfils the following two properties for all $x, x_1, x_2 \in M$ and all $y, y_1, y_2 \in N$:

(P1) Both $\mathrm{d}_M(x_1, x_2)$ and $\mathrm{d}_N(y_1, y_2)$ are no larger than $\mathrm{d}\big((x_1, y_1), (x_2, y_2)\big)$.

(P2) $\mathrm{d}\big((x_1, y), (x_2, y)\big) = \mathrm{d}_M(x_1, x_2)$ and $\mathrm{d}\big((x, y_1), (x, y_2)\big) = \mathrm{d}_N(y_1, y_2)$.

Remark A.2.2. Let (M, d_M) and (N, d_N) be metric spaces. Then there exists at product metric d on $M \times N$. For for instance, we can define

$$\mathrm{d}\big((x_1, y_1), (x_2, y_2)\big) = \mathrm{d}_M(x_1, x_2) + \mathrm{d}_N(y_1, y_2)$$

for all (x_1, y_1) and all (x_2, y_2) in $M \times N$.

Proposition A.2.3. *Let* (M, d_M) *and* (N, d_N) *be metric spaces and let* d *be a product metric on* $M \times N$*. Let* $(x_k)_{k\in\mathbb{N}}$ *be a sequence in* M *and* $(y_k)_{k\in\mathbb{N}}$ *a sequence in* N*. Let* $x \in M$ *and* $y \in N$*. Then the following assertions are equivalent:*

(i) *The sequence* $\big((x_k, y_k)\big)_{k\in\mathbb{N}}$ *in* $M \times N$ *converges to* (x, y) *(with respect to* d*).*

(ii) *The sequence* $(x_k)_{k\in\mathbb{N}}$ *converges to* x *and the sequence* $(y_k)_{k\in\mathbb{N}}$ *converges to* y*.*

Proof. "(i) $\Rightarrow$ (ii)" If (i) holds, then

$$0 \le \mathrm{d}_M(x_k, x) \le \mathrm{d}\Big((x_k, y_k), (x, y)\Big) \to 0$$

as $k \to \infty$, so $x = \lim_{k\to\infty} x_k$. Similarly, one shows that $y = \lim_{k\to\infty} y_k$.

"(ii) $\Rightarrow$ (i)" If (ii) holds, then we have

$$\begin{aligned}\mathrm{d}\Big((x_k, y_k), (x, y)\Big) &\le \mathrm{d}\Big((x_k, y_k), (x, y_k)\Big) + \mathrm{d}\Big((x, y_k), (x, y)\Big)\\ &= \mathrm{d}_M(x_k, x) + \mathrm{d}_N(y_k, y) \to 0\end{aligned}$$

as $k \to \infty$. Hence, $(x, y) = \lim_{k\to\infty}(x_k, y_k)$. $\square$

Remark A.2.4. Let (M, d_M) and (N, d_N) be metric spaces and let d be a product metric on $M \times N$. The above proposition shows that the question whether a sequence in $M \times N$ converges to a given element of $M \times N$ does not depend on the particular choice of the product metric d.

In conjuction with Proposition A.1.12 this shows that the question whether a mapping from another metric space into $M \times N$ (or from $M \times N$ into another metric space) is continuous does not depend on the choice of the given product metric d.

We will always assume tacitly that the product $M \times N$ of two metric spaces is endowed with a product metric which we do not specify further.

Proposition A.2.5. *Let* (M, d) *be a metric space. Then* $\mathrm{d} : M \times M \to [0, \infty)$ *is a continuous mapping.*